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LOCAL HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

HONGBIN WANG[1*] AND ZONGGUANG LIU[2]

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Abstract. In this paper, we introduce a certain Herz-type Hardy space with variable exponent and establish the atom decomposition theorem for it. Using this decomposition, we obtain some boundedness on the Herz-type Hardy space with variable exponent for a class of pseudo-differential operators.

1. Introduction

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The L^p spaces with variable exponent are a special case of Musielak-Orlicz spaces.

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* Corresponding author.
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The space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \to [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. There exist some sufficient conditions on $p(\cdot)$ such that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, see [1, 2]. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *([6]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. *([5]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C \frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$ are constants.

Throughout this paper δ_1 and δ_2 are the same as in Lemma 1.2.

Lemma 1.3. *([5]) Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Firstly we give the definition of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{N}$ and $\mathbb{Z}_+$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{A_k} is the characteristic function of A_k.

Definition 1.4. ([5]) Let $\alpha \in \mathbb{R}, 0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

In [10], we gave the definition of Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N f(x) = \sup_{\phi\in\mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.5. ([10]) Let $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the largest integer less than or equal to α.

Definition 1.6. ([10]) Let $n\delta_2 \le \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $s \ge [\alpha - n\delta_2]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \le s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and
(1)$'$ $\operatorname{supp} a \subset B(0,r), r \geq 1$.

Lemma 1.7. *([10]) Let $n\delta_2 \leq \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$(or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left(\text{or} \sum_{k=0}^{\infty} \lambda_k a_k \right), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$(or $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left(\text{or } \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decomposition of f.

In [11], we gave some real-variable characterizations for $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with integral 1. For $t > 0$, set $\phi_t(x) = t^{-n}\phi(x/t)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, define the maximal operator ϕ^*_+, $\phi^*_{\nabla,N}$ (with $N > 1$) and ϕ^{**}_M (with $M \in \mathbb{N}$) by

$$\phi^*_+(f)(x) = \sup_{t>0} |(f * \phi_t)(x)|, \ \phi^*_{\nabla,N}(f)(x) = \sup_{t>0} \sup_{|x-y|<Nt} |(f * \phi_t)(y)|$$

and

$$\phi^{**}_M(f)(x) = \sup_{(y,t)\in\mathbb{R}^{n+1}_+} |(f * \phi_t)(y)| \left(\frac{t}{|x-y|+t} \right)^M.$$

Lemma 1.8. *([11]) Let $0 < \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:*
(i) $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).
(ii) For some $N > 1$, $\phi^*_{\nabla,N}(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).
(iii) $\phi^*_{\nabla}(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).
(iv) $\phi^*_+(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).
Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla,N}(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla}(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_+(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$$

and

$$\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla,N}(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla}(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_+(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

In [9], we establish the block decomposition for the Herz spaces with variable exponent.

Definition 1.9. ([9]) Let $0 < \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-block if

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le Cr^{-\alpha}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-block of restricted type if

(1) $\operatorname{supp} a \subset B(0,r)$ for some $r \ge 1$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le r^{-\alpha}$.

Inspired by [3, 10], we introduce a certain Herz-type Hardy space with variable exponent and establish the atom decomposition theorem for it. Using this decomposition, we obtain some boundedness on the Herz-type Hardy space with variable exponent for a class of pseudo-differential operators.

2. Main results and their proofs

In this section, we will give the definition of local Herz-type Hardy spaces with variable exponent $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ firstly.

Definition 2.1. Let $n\delta_2 \le \alpha < \infty, 0 < p \le \infty, q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(i)A function $f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is said to be in the space $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if for some $N \in \mathbb{N}$ with $N > \alpha/\delta_2 + 1$, the maximal function

$$\tilde{G}_N(f) = \sup_{\phi \in \mathcal{A}_N} \sup_{0<t<1, |x-y|<t} |f * \phi_t(y)|$$

belongs to the space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$; where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $\phi_t(x) = t^{-n}\phi(x/t)$.

(ii)A function f is said to be in the space $hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if $\tilde{G}_N(f) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Moreover, we define that $\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|\tilde{G}_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$ and $\|f\|_{hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|\tilde{G}_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

It should be pointed out that, the spaces $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ also enjoy the characterizations in terms of the maximal operator, which are similar to those of the spaces $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, namely,

Theorem 2.2. *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:*

(i) $f \in h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *(or $hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).*

(ii) $\tilde{\phi}^*_\nabla(f) = \sup_{0<t<1} \sup_{|x-y|<t} |f * \phi_t(y)| \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *(or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).*

(iii) $\tilde{\phi}^*_+(f) = \sup_{0<t<1} |f * \phi_t(x)| \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *(or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).*

Moreover,

$$\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\tilde{\phi}^*_\nabla(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\tilde{\phi}^*_+(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$$

and

$$\|f\|_{hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\tilde{\phi}^*_\nabla(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\tilde{\phi}^*_+(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. The method of proof is similar to Lemma 1.8. Here we omit it. ∎

To give another characterization of $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, we first give a preliminary lemma.

Theorem 2.3. *Let $n\delta_2 \le \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \phi(x)dx = 1, \ \int_{\mathbb{R}^n} x^\beta \phi(x)dx = 0, \ \text{for all}\, \beta \in \mathbb{N}^n, \ |\beta| \le N.$$

Then

$$\|f - \phi * f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Moreover, if $f \in h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, then $f - \phi * f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof. Take $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)dx = 1$. By Lemma 1.8, we see that

$$\begin{aligned}
\|f - \phi * f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= C\|\sup_{t>0} |\psi_t * (f - \phi * f)|\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\|\sup_{1>t>0} |\psi_t * f|\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + C\|\sup_{1>t>0} |\psi_t * \phi * f|\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + C\|\sup_{\infty>t\ge 1} |\psi_t * (f - \phi * f)|\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= U_1 + U_2 + U_3.
\end{aligned}$$

Note that $\sup_{0<t<1} |\psi_t * f| \le C\tilde{G}_N(f)(x)$. So by Definition 2.1 we have

$$U_1 \le C\|\tilde{G}_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = C\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

To estimate U_2, we first claim that there is a constant C_0 independent of $t \le 1$ such that $C_0^{-1}\psi_t * \phi \in \mathcal{A}_N$. In fact, for any γ with $|\gamma| \le n + N$ and any β with $|\beta| \le N$, a trivial computation leads to that

$$\begin{aligned}
|x^\gamma D^\beta(\psi_t * \phi)(x)| &\le |x|^{|\gamma|} \left| \int_{2|y|<|x|} \frac{1}{t^n}\psi\left(\frac{y}{t}\right) D^\beta\phi(x-y)dy \right| \\
&\quad + |x|^{|\gamma|} \left| \int_{2|y|\ge|x|} \frac{1}{t^n}\psi\left(\frac{y}{t}\right) D^\beta\phi(x-y)dy \right| \\
&\le C|x|^{|\gamma|} \int_{2|y|<|x|} \left|\psi(t^{-1}y)\right| \frac{dy}{(1+|x-y|)^{|\gamma|}} \\
&\quad + C|x|^{|\gamma|} t^{-n} \int_{2|y|\ge|x|} \frac{t^{|\gamma|}}{|y|^{|\gamma|}} dy \\
&\le C\|\psi\|_{L^1(\mathbb{R}^n)} + C\||y|^{|\gamma|}\psi\|_{L^1(\mathbb{R}^n)} \le C_0,
\end{aligned}$$

which means that $C_0^{-1}(\psi_t * \phi) \in \mathcal{A}_N(\mathbb{R}^n)$. Thus,

$$|\psi_t * \phi * f(x)| \le C\tilde{G}_N(f)(x)$$

and

$$U_2 \le C\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

On the other hand, a cumbersome but straightforward computation shows that for any fixed $k \geq 1$, any multi-index β and any $t \geq 1$,

$$|D^{\beta}(\psi_t - \phi * \psi_t)| \leq C_{k,\beta}(1+|x|)^{-k},$$

which via a trivial computation leads to that

$$\sup_{1 \leq t < \infty} |\psi_t * (f - \phi * f)(x)| \leq C\tilde{G}_N(f)(x).$$

Thus we get that

$$U_3 \leq C\|\tilde{G}_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = C\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 2.3. □

Next, we will give the atomic decomposition for $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.4. *Let $n\delta_2 \leq \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if*

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where for $k \leq 0$, a_k is a central $(\alpha, q(\cdot))$-atom, while for $k > 0$, a_k is a central $(\alpha, q(\cdot))$-block, and $\{\lambda_k\}_{k=-\infty}^{\infty}$ satisfies $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decomposition of f.

Proof. To prove the necessity, let $f \in h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ be the same as in Theorem 2.3. Then $g = f - \phi * f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, which implies that

$$f = g + \phi * f, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n).$$

For g, we have the central $(\alpha, q(\cdot))$-atom decomposition by Lemma 1.7. Thus, it suffices to decompose $\phi * f$.

Let ψ be a radial smooth function such that $\operatorname{supp}\psi \subset \{x : 1/2 - \varepsilon \leq |x| \leq 1+\varepsilon\}$ with $0 < \varepsilon < 1/4$, and $\psi(x) = 1$ if $1/2 \leq |x| \leq 1$. Set $\psi_k(x) = \phi(2^{-k}x)$ for $k \in \mathbb{Z}$ and $\tilde{A}_{k,\varepsilon} = \{x : 2^{k-1} - 2^k\varepsilon \leq |x| \leq 2^k + 2^k\varepsilon\}$. We know that

$$\operatorname{supp}\psi \subset \tilde{A}_{k,\varepsilon}, \quad \psi_k(x) = 1 \text{ if } x \in A_k = \{x : 2^{k-1} < |x| \leq 2^k\}.$$

Obviously, $1 \leq \sum_{k=-\infty}^{\infty} \psi_k(x) \leq 2$, $|x| > 0$. Let

$$\Phi_k(x) = \begin{cases} \psi_k(x) \Big/ \sum_{l=-\infty}^{\infty} \psi_l(x), & x \neq 0, \\ 0, \quad x = 0, \end{cases}$$

then $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$ for $x \neq 0$. For some $m \in \mathbb{Z}_+$, we denote by $\mathcal{P}_m$ the class of all the real polynomials with the degree less than m. Let $P_k(x) = P_{\tilde{A}_{k,\varepsilon}}((f * \phi)\Phi_k)(x)\chi_{\tilde{A}_{k,\varepsilon}}(x) \in \mathcal{P}_m(\mathbb{R}^n)$ be the unique polynomial such that

$$\int_{\tilde{A}_{k,\varepsilon}} ((f * \phi)(x)\Phi_k(x) - P_k(x))\, x^\beta dx = 0, \ \ |\beta| \le m = [\alpha - n\delta_2].$$

Write

$$f*\phi(x) = \sum_{k=-\infty}^{\infty} (f*\phi(x)\Phi_k(x) - P_k(x)) + \sum_{k=-\infty}^{\infty} P_k(x) = \sum_{k=-\infty}^{\infty} H_k^1(x) + \sum_{k=-\infty}^{\infty} H_k^2(x).$$

For the term $\sum_{k=-\infty}^{\infty} H_k^1(x)$, let $\{\phi_l^k : |l| \le m\}$ be the orthogonal polynomials limited on $\tilde{A}_{k,\varepsilon}$ with respect to the weight $|\tilde{A}_{k,\varepsilon}|^{-1}$, which are obtained from $\{x^\beta : |\beta| \le m\}$ by the Gram-Schmidt method, that is

$$\langle \phi_\nu^k, \phi_\mu^k \rangle = \frac{1}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x) dx = \delta_{\nu\mu}.$$

It is easy to see that

$$P_k(x) = \sum_{|l| \le m} \langle (f * \phi)\Phi_k, \phi_l^k \rangle \phi_l^k(x), \quad x \in \tilde{A}_{k,\varepsilon}.$$

On the other hand, from

$$\frac{1}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x) dx = \delta_{\nu\mu}$$

we infer that

$$\frac{1}{|\tilde{A}_{1,\varepsilon}|} \int_{\tilde{A}_{1,\varepsilon}} \phi_\nu^k(2^{k-1}y)\phi_\mu^k(2^{k-1}y) dy = \delta_{\nu\mu}.$$

It then follows directly that $\phi_\nu^k(2^{k-1}y) = \phi_\nu^1(y), y \in \tilde{A}_{1,\varepsilon}$. That is $\phi_\nu^k(x) = \phi_\nu^1(2^{-(k-1)}x)$ for $x \in \tilde{A}_{k,\varepsilon}$. Thus $|\phi_\nu^k(x)| \le C$, and for $x \in \tilde{A}_{k,\varepsilon}$, by the generalized Hölder inequality we have

$$\begin{aligned} |P_k(x)| &\le \frac{C}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} |(f * \phi)(x)\Phi_k(x)| dx \\ &\le \frac{C}{|\tilde{A}_{k,\varepsilon}|} \|(f * \phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Therefore, by Lemma 1.3 we have that

$$\begin{aligned}\|H_k^1\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le \|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|P_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \frac{C}{|\tilde{A}_{k,\varepsilon}|}\|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C\|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\|(f*\phi)\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{j=k-1}^{k+1}\|\tilde{G}_N(f)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Set $\lambda_k^1 = 2^{(k+1)\alpha}\|H_k^1\|_{L^{q(\cdot)}(\mathbb{R}^n)}$ and $b_k(x) = 2^{-(k+1)\alpha}\|H_k^1\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}H_k^1(x)$. It is easy to verify that each b_k is central $(\alpha, q(\cdot))$-atom and $\sum_{k=-\infty}^{\infty} H_k^1(x) = \sum_{k=-\infty}^{\infty}\lambda_k a_k(x)$. Moreover,

$$\begin{aligned}\sum_{k=-\infty}^{\infty}|\lambda_k|^p &= \sum_{k=-\infty}^{\infty}2^{(k+1)\alpha p}\|H_k^1\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|\tilde{G}_N(x)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &= C\|f\|^p_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

We now turn our attention to $\sum_{k=-\infty}^{\infty} H_k^2(x)$. Let $\{\psi_l^k : |l| \le m\}$ be the dual basis of $\{x^\beta : |\beta| \le m\}$ limited on $\tilde{A}_k$ with respected to the weight $|\tilde{A}_{k,\varepsilon}|^{-1}$, that is,

$$\langle\psi_l^k,\, x^\beta\rangle = \frac{1}{\tilde{A}_{k,\varepsilon}}\int_{\tilde{A}_{k,\varepsilon}} x^\beta\psi_l^k(x)dx = \delta_{\beta l}.$$

We can prove that if $\phi_l^k(x) = \sum_{|\nu|\le m}\beta_{l\nu}^k x^\nu$, then $\psi_l^k(x) = \sum_{|\nu|\le m}\beta_{\nu l}^k\phi_\nu^k(x)$. In fact, let $\psi_l^k(x) = \sum_{|\nu|\le m} C_{\nu l}^k\phi_\nu^k(x)$, then

$$C_{\nu l}^k = \langle\psi_l^k,\, \phi_\nu^k\rangle = \langle\psi_l^k,\, \sum_{|\gamma|\le m}\beta_{\nu\gamma}^k x^\gamma\rangle = \sum_{|\gamma|\le m}\beta_{\nu\gamma}^k\langle\psi_l^k,\, x^\gamma\rangle = \beta_{\nu l}^k.$$

Thus, for $x \in \tilde{A}_{k,\varepsilon}$,

$$\begin{aligned} P_k(x) &= \sum_{|\nu|\le m} \langle (f*\phi)\Phi_k,\, \phi_\nu^k\rangle \phi_\nu^k(x) \\ &= \sum_{|\nu|\le m} \langle (f*\phi)\Phi_k,\, \sum_{|l|\le m} \beta_{\nu l}^k y^l\rangle \phi_\nu^k(x) \\ &= \sum_{|l|\le m} \langle (f*\phi)\Phi_k,\, y^l\rangle \sum_{|\nu|\le m} \beta_{\nu l}^k \phi_\nu^k(x) \\ &= \sum_{|l|\le m} \langle (f*\phi)\Phi_k,\, y^l\rangle \psi_l^k(x). \end{aligned}$$

It is easy to prove that if $x \in \tilde{A}_{k,\varepsilon}$, then $|\psi_l^k(x)| \le C2^{-k|l|}$(see [7, Theorem 2.1]). It follows that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} H_k^2(x) &= \sum_{k=-\infty}^{\infty} P_k(x) \\ &= \sum_{k=-\infty}^{\infty} \sum_{|l|\le m} \langle (f*\phi)\Phi_k,\, y^l\rangle \psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x) \\ &= \sum_{|l|\le m} \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} (f*\phi)(y)\Phi_k(y) y^l dy\right) \frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} \\ &= \sum_{|l|\le m} \sum_{k=-\infty}^{0} \left(\sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} (f*\phi)(y)\Phi_j(y) y^l dy\right) \\ &\qquad \times \left(\frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_l^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\right) \\ &\quad + \sum_{|l|\le m} \left\{ \left(\sum_{j=-\infty}^{1} \int_{\mathbb{R}^n} (f*\phi)(y)\Phi_j(y) y^l dy\right) \frac{\psi_l^1(x)\chi_{\tilde{A}_{1,\varepsilon}}(x)}{|\tilde{A}_{1,\varepsilon}|} \right. \\ &\quad \left. + \sum_{k=2}^{\infty} \left(\int_{\mathbb{R}^n} (f*\phi)(y)\Phi_k(y) y^l dy\right) \frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} \right\} \\ &= I(x) + J(x). \end{aligned}$$

We first decompose $I(x)$. From $\displaystyle\sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} \Phi_j(y) y^l dy \le C2^{k(n+|l|)}$, it is easy to deduce that

$$\left| \sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} (f*\phi)(y)\Phi_j(y) y^l dy \right| \chi_{B(0,2^{k+2})}(x) \le C2^{k(n+|l|)} \tilde{G}(f)(x) \chi_{B(0,2^{k+2})}(x).$$

In addition, we have that

$$\left| \frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_l^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{\tilde{A}_{k+1,\varepsilon}} \right| \le C2^{-k(n+|l|)} \sum_{j=k-1}^{k+1} \chi_j(x).$$

Set

$$h^1_{k,l}(x) = \left(\sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} (f*\phi)(y)\Phi_j(y)y^l dy\right)\left(\frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_l^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\right)$$

and

$$a^1_{k,l} = 2^{-(k+2)\alpha}\|h_{k,l}\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Then $a^1_{k,l}$ is a central $(\alpha, q(\cdot))$-atom with support B_{k+2}.
Let $\tau^1_{k,l} = 2^{(k+2)\alpha}\|h_{k,l}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$. Then we have that

$$I(x) = \sum_{|l|\le m}\sum_{k=-\infty}^{0} \tau^1_{k,l}a^1_{k,l}(x)$$

and

$$\begin{aligned}\sum_{|l|\le m}\sum_{k=-\infty}^{0} |\tau^1_{k,l}|^p &\le C\sum_{|l|\le m}\sum_{k=-\infty}^{0} 2^{(k+2)\alpha p}\left(\sum_{j=k-1}^{k+2}\|\tilde{G}_N(f)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C\sum_{|l|\le m}\sum_{k=-\infty}^{2} 2^{(k+2)\alpha p}\|\tilde{G}_N(f)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\|f\|^p_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now we decompose J. A trivial computation as above gives us that

$$\left|\sum_{j=-\infty}^{1}\int_{\mathbb{R}^n}(f*\phi)(y)\Phi_j(y)y^l dy\right|\chi_{B_2}(x) \le C\tilde{G}_N(f)(x)\chi_{B_2}(x).$$

Set

$$h^2_{1,l}(x) = \sum_{j=-\infty}^{1}\left(\int_{\mathbb{R}^n}(f*\phi)(y)\Phi_j(y)y^l dy\right)\frac{\psi_l^1(x)\chi_{\tilde{A}_{1,\varepsilon}}(x)}{|\tilde{A}_{1,\varepsilon}|}$$

and

$$\lambda^2_{1,l}(x) = 2^{\alpha}\|h^2_{1,l}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

We can verify that

$$\|h^2_{1,l}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\sum_{j=0}^{2}\|\tilde{G}_N(f)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Moreover, $a^2_{1,l}(x) = (\lambda^2_{1,l})^{-1}h^2_{1,l}(x)$ is a central $(\alpha, q(\cdot))$-block supported on B_2 and $h^2_{1,l}(x) = (\lambda^2_{1,l})a^2_{1,l}(x)$. For integer $k \ge 2$, let

$$h^2_{k,l}(x) = \left(\int_{\mathbb{R}^n}(f*\phi)(y)\Phi_k(y)y^l dy\right)\frac{\psi_l^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}.$$

By Lemma 1.3 and the generalized Hölder inequality we have

$$\begin{aligned}\|h^2_{k,l}\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\|\tilde{G}_N(f)\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{j=k-1}^{k+1}\|\tilde{G}_N(f)\chi_{A_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

This implies that $a_{k,l}^2(x) = \left(\|h_{k,l}^2\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{k\alpha}\right)^{-1} h_{k,l}^2(x)$ is a central $(\alpha, q(\cdot))$-block which is supported on B_{k+1}. Set $\lambda_{k,l}^2 = \|h_{k,l}^2\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{k\alpha}$. It then follows that

$$J(x) = \sum_{|l|\le m} \sum_{k=1}^{\infty} \lambda_{k,l}^2 a_{k,l}^2(x)$$

and

$$\sum_{|l|\le m} \sum_{k=1}^{\infty} |\lambda_{k,l}^2|^p \le C \sum_{k=0}^{\infty} 2^{k\alpha p} \|\tilde{G}_N(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \le C\|f\|_{h\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}^p.$$

Similar to the method of [10, Theorem 2.1], we can get

$$f * \phi = \sum_{k=1}^{\infty} \lambda_k a_k \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n).$$

Now we prove the sufficiency. Let $\{\lambda_k\}_{k=-\infty}^{\infty}$ be a sequence of number such that $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$, $\{a_k\}_{k\le 0}$ be a sequence of central $(\alpha, q(\cdot))$-atom supported on B_k and $\{a_k\}_{k>0}$ be a sequence of central $(\alpha, q(\cdot))$-block, and

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n).$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\operatorname{supp}\phi \subset B(0,1)$. Our goal is to prove that

$$\tilde{\phi}_+^*(f) \in \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n).$$

To this aim, we consider the following two cases.

When $0 < p \le 1$. In this case, we just need to show that there exists a constant C such that for any a_k,

$$\|\tilde{\phi}_+^*(a_k)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C.$$

If $k \le 0$, write

$$\begin{aligned}\|\tilde{\phi}_+^*(a_k)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}^p &= \sum_{j=-\infty}^{k+3} 2^{j\alpha p} \|\tilde{\phi}_+^*(a_k)\chi_{A_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p + \sum_{j=k+4}^{\infty} 2^{j\alpha p} \|\tilde{\phi}_+^*(a_k)\chi_{A_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \\ &= L_1^1 + L_2^1.\end{aligned}$$

Applying the trivial estimate that

$$\tilde{\phi}_+^*(a_k)(x) \le C\mathcal{M}a_k(x) \tag{2.1}$$

and the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, we see that

$$L_1^1 \le \sum_{j=-\infty}^{k+3} 2^{j\alpha p} \|\mathcal{M}a_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \le C \sum_{j=-\infty}^{k+3} 2^{j\alpha p} \|a_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \le C.$$

Now let $m \in \mathbb{Z}_+$ such that $\alpha - n\delta_2 < m+1$, and $\mathcal{J}_m$ be the m-order Taylor expansion for ϕ at x/t. For $x \in A_j$ with $j \geq k+4$, a straightforward computation gives us that

$$\begin{aligned}|a_k * \phi_t(x)| &= t^{-n}\left|\int_{\mathbb{R}^n} a_k(y)\phi\left(\frac{x-y}{t})\right) dy\right| \\ &= t^{-n}\left|\int_{\mathbb{R}^n} a_k(y)\left(\phi\left(\frac{x-y}{t}\right) - \mathcal{J}_m\left(\frac{y}{t}\right)\right) dy\right| \\ &\leq C\int_{\mathbb{R}^n} |a_k(y)||y|^{m+1}(t+|x-\theta y|)^{-(n+m+1)}dy \\ &\leq C\frac{2^{k(m+1)}}{|x|^{n+m+1}}\int_{\mathbb{R}^n}|a_k(y)|dy,\end{aligned}$$

where $0 < \theta < 1$. Therefore, for $x \in A_j$ with $j \geq k+4$, we have

$$\tilde{\phi}_+^*(a_k)(x) \leq C2^{k(m+1)}|x|^{-(n+m+1)}|B_k|^{-\alpha/n}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So by Lemma 1.2 and Lemma 1.3 we have

$$\begin{aligned}L_2^1 &\leq C\sum_{j=k+4}^{\infty} 2^{p[k(m+1)-j(n+m+1)]}\left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n}\|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C\sum_{j=k+4}^{\infty} 2^{p[k(m+1)-j(n+m+1)]}\left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n}\|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(|B_j|\|\chi_{B_j}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)^p \\ &= C\sum_{j=k+4}^{\infty} 2^{p(k-j)(m+1-\alpha)}\left(\frac{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\ &\leq C\sum_{j=k+4}^{\infty} 2^{p(k-j)(m+1-\alpha+n\delta_2)} \leq C.\end{aligned}$$

If $k \geq 1$, by (2.1), the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$ and $\operatorname{supp}\tilde{\phi}_+^*(a_k) \subset B_{k+1}$, we have

$$\begin{aligned}\|\tilde{\phi}_+^*(a_k)\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|\tilde{\phi}_+^*(a_k)\chi_{A_j}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|Ma_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C\sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p} \leq C.\end{aligned}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Write

$$\begin{aligned}
\|\tilde{\phi}_+^*(f)\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{\alpha kp} \|\tilde{\phi}_+^*(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq \sum_{k=-\infty}^{\infty} 2^{\alpha kp} \left(\sum_{j=-\infty}^{\infty} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{\alpha kp} \left(\sum_{j=-\infty}^{0} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\quad + C \sum_{k=-\infty}^{0} 2^{\alpha kp} \left(\sum_{j=1}^{\infty} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\quad + C \sum_{k=1}^{\infty} 2^{\alpha kp} \left(\sum_{j=1}^{\infty} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&= C(U_1 + U_2 + U_3).
\end{aligned}$$

For the term U_3, it follows from the Hölder inequality that

$$\begin{aligned}
U_3 &= \sum_{k=1}^{\infty} 2^{\alpha kp} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\leq C \sum_{k=1}^{\infty} 2^{\alpha kp} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{-\alpha j} \right)^p \\
&\leq C \sum_{k=1}^{\infty} 2^{\alpha kp} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p 2^{-\alpha pj/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{-\alpha p' j/2} \right)^{p/p'} \\
&\leq C \sum_{j=0}^{\infty} |\lambda_j|^p < \infty.
\end{aligned}$$

On the other hand, a straightforward computation leads to that

$$\begin{aligned}
U_2 &= \sum_{k=-\infty}^{0} 2^{\alpha kp} \left(\sum_{j=1}^{\infty} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\leq C \sum_{k=-\infty}^{0} 2^{\alpha kp} \left(\sum_{j=1}^{\infty} |\lambda_j| 2^{-\alpha j} \right)^p \\
&\leq C \sum_{j=1}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{0} 2^{\alpha kp} \left(\sum_{j=1}^{\infty} 2^{-\alpha p' j} \right)^{p/p'} \\
&\leq C \sum_{j=0}^{\infty} |\lambda_j|^p < \infty.
\end{aligned}$$

To estimate U_1, write

$$\begin{aligned} U_1 &= \sum_{k=2}^{\infty} 2^{\alpha kp} \left(\sum_{j=-\infty}^{0} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\quad + \sum_{k=-\infty}^{1} 2^{\alpha kp} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\quad + \sum_{k=-\infty}^{1} 2^{\alpha kp} \left(\sum_{j=k-1}^{0} |\lambda_j| \|\tilde{\phi}_+^*(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &= U_1^1 + U_1^2 + U_1^3. \end{aligned}$$

The Hölder inequality, along with the estimate (2.1), gives us that

$$\begin{aligned} U_1^3 &\leq C \sum_{k=-\infty}^{1} 2^{\alpha kp} \left(\sum_{j=k-1}^{0} |\lambda_j| \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\leq C \sum_{k=-\infty}^{1} 2^{\alpha kp} \left(\sum_{j=k-1}^{0} |\lambda_j| 2^{-\alpha j} \right)^p \\ &\leq C \sum_{j=-\infty}^{0} |\lambda_j|^p < \infty. \end{aligned}$$

On the other hand, for $x \in A_k$ with $k \geq j+2$ and $k \leq 1$, similar to the first case we have

$$\tilde{\phi}_+^*(a_j)(x) \leq C 2^{j(m+1)} |x|^{-(n+m+1)} |B_j|^{-\alpha/n} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

Note that $\alpha - n\delta_2 < m+1$, so we have

$$\begin{aligned} &U_1^2 \\ &\leq C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{j(m+1)-k(n+m+1)} \left(\frac{|B_k|}{|B_j|} \right)^{\alpha/n} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\leq C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(m+1-\alpha+n\delta_2)} \right)^p \\ &\leq C \sum_{j=-\infty}^{-1} |\lambda_j|^p < \infty. \end{aligned}$$

Similarly, we have

$$\begin{aligned} U_1^1 &\leq C \sum_{k=2}^{\infty} \left(\sum_{j=-\infty}^{0} |\lambda_j| 2^{(j-k)(m+1-\alpha+n\delta_2)} \right)^p \\ &\leq C \sum_{j=-\infty}^{0} |\lambda_j|^p < \infty. \end{aligned}$$

Combining the estimates above, it will show that if f has the decomposition $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, then

$$\|\tilde{\phi}_+^*(f)\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty.$$

This completes the proof of Theorem 2.4.

□

Similar to the proof of Theorem 2.4, we are easy to get the following conclusion, which gives another characterization of $hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.5. *Let $n\delta_2\le\alpha<\infty, 0<p<\infty$ and $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then $f\in hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if $f\in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. That is,*

$$f(x)=\sum_{k=0}^{\infty}\lambda_k a_k(x),\quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha,q(\cdot))$-block of restricted type, and $\{\lambda_k\}_{k=0}^{\infty}$ satisfies that $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$. Moreover,

$$\|f\|_{hK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\approx\inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decomposition of f.

The boundedness of pseudo-differential operators on Herz-type spaces was studied by many authors (see [3, 8]). In the following part, as an application of Theorem 2.4, we will prove the boundedness of pseudo-differential operators of order zero on the space $h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, which generalizes the result in [3].

Theorem 2.6. *Let $n\delta_2\le\alpha<\infty, 0<p<\infty$ and $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. If $Tf(x)=\int_{\mathbb{R}^n}\hat{f}(x)\sigma(x,\xi)e^{2\pi ix\cdot\xi}d\xi$ with $\sigma\in\mathbb{S}^0$, that is, $\sigma\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and $|D_x^\alpha D_\xi^\beta\sigma(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|)^{-|\beta|}$, then $\|Tf\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.*

Proof. Let $f\in h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Theorem 2.4, we have $f(x)=\sum_{k=-\infty}^{\infty}\lambda_k a_k(x)$ in distributional sense. Then we consider two cases with $0<p\le 1$ and $1<p<\infty$.

When $0<p\le1$. In this case, we only need to show that $\|Ta_k\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\le C$ and C is independent of a_k. If $k\le 0$, then

$$\begin{aligned}\|Ta_k\|^p_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}&=\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|\tilde{G}_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\&=\sum_{j=-\infty}^{k+2}2^{j\alpha p}\|\tilde{G}_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}+\sum_{j=k+3}^{\infty}2^{j\alpha p}\|\tilde{G}_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\&=I_1+I_2.\end{aligned}$$

For I_1, using the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, we have

$$\begin{aligned} I_1 &= \sum_{j=-\infty}^{k+2} 2^{j\alpha p} \|\tilde{G}_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{j=-\infty}^{k+2} 2^{j\alpha p} \|\mathcal{M}(Ta_k)\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{j=-\infty}^{k+2} 2^{j\alpha p} \|a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{j=-\infty}^{k+2} 2^{(j-k)\alpha p} \le C. \end{aligned}$$

To estimate I_2, by Theorem 4 in [4], we can write that

$$\phi_t * (Ta_k)(x) = \int_{\mathbb{R}^n} K_t(x, x-z) a_k(z) dz.$$

Then we expand $K_t(x, x-z)$ in a Taylor series about $z = 0$. By the vanishing moments of a_k, we get that

$$\phi_t * (Ta_k)(x) = \sum_{|\alpha|=N+1} \int_{\mathbb{R}^n} D^\alpha_z K_t(x, x-\theta z) z^\alpha a_k(z) dz,$$

where $\theta \in (0,1)$ and $N \in \mathbb{Z}_+$ satisfying that $\alpha - n\delta_2 < N+1$. Noting that $x \in A_j$ with $j \ge k+3$, by Theorem 4 in [4], we can obtain that

$$\begin{aligned} |\phi_t * (Ta_k)(x)| &\le \frac{C}{|x|^{n+N+1}} \int_{\mathbb{R}^n} |z|^{N+1} a_k(z) dz \\ &\le \frac{C 2^{k(N+1)}}{|x|^{n+N+1}} \int_{\mathbb{R}^n} a_k(z) dz \\ &\le \frac{C 2^{k(N+1)}}{|x|^{n+N+1}} \|a_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le \frac{C 2^{k(N+1)}}{2^{j(n+N+1)}} |B_k|^{-\alpha/n} \|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

So by Lemma 1.2 and Lemma 1.3 we have

$$\begin{aligned} I_2 &\le C \sum_{j=k+3}^{\infty} 2^{p[k(N+1)-j(n+N+1)]} \left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n} \|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{j=k+3}^{\infty} 2^{p[(k-j)(N+1)-jn]} \left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n} \|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)} \left(|B_j| \|\chi_{B_j}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)^p \\ &= C \sum_{j=k+3}^{\infty} 2^{p(k-j)(N+1-\alpha)} \left(\frac{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\ &\le C \sum_{j=k+3}^{\infty} 2^{p(k-j)(N+1-\alpha+n\delta_2)} \le C. \end{aligned}$$

If $k > 0$, we choose a radial smooth function η such that $\operatorname{supp}\eta \subset B(0,1)$ and η equals 1 near the origin. We split $T = T_1 + T_2$ by decomposing $K(x,z) = K_1(x,z) + K_2(x,z) = \eta(z)K(x,z) + (1-\eta(z))K(x,z)$. Then T_1 and T_2 are of order zero. Noting that $\operatorname{supp}\tilde{\phi}_+^*(T_1a_k) \subset B_{k+1}$ and $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, we get that

$$\begin{aligned}
\|T_1a_k\|^p_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|\tilde{\phi}_+^*(T_1a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|\mathcal{M}(T_1a_k)\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|T_1a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p} \leq C.
\end{aligned}$$

To estimate $T_2a_k(x)$, we have

$$|(K_2)_t(x,z)| \leq C_M(1+|z|)^{-M} \tag{2.2}$$

for any $M \geq n$ (see [4, Theorem 4]). Then we write that

$$\begin{aligned}
\|T_2a_k\|^p_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|\tilde{\phi}_+^*(T_2a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \sum_{j=-\infty}^{k+2} 2^{j\alpha p}\|\tilde{\phi}_+^*(T_2a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{j=k+3}^{\infty} 2^{j\alpha p}\|\tilde{\phi}_+^*(T_2a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= II_1 + II_2.
\end{aligned}$$

About II_1, we can obtain the desirable estimate by a similar method to I_1. For II_2, noting that $x \in A_j$ and $l \geq k+3$, by (2.2) we can obtain that

$$\begin{aligned}
|\phi_t * (T_2a_k)(x)| &= |\int_{\mathbb{R}^n} (K_2)_t(x, x-z)a_k(z)dz| \\
&\leq C_M \int_{\mathbb{R}^n} \frac{1}{(1+|x-z|)^M}|a_k(z)|dz \\
&\leq \frac{C}{|x|^{n+N+1}}\int_{\mathbb{R}^n}|a_k(z)|dz \\
&\leq \frac{C2^{k(N+1)}}{|x|^{n+N+1}}|B_k|^{-\alpha/n}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where we take $N \in \mathbb{Z}_+$ satisfying that $\alpha - n\delta_2 < N+1$. So it is readily to follow that $II_2 \leq C$.

When $1<p<\infty$. In this case, we write that

$$\begin{aligned}\|Tf\|_{h\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\tilde{\phi}^*_+(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &\le \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\|\tilde{\phi}^*_+(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\le \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|\tilde{\phi}^*_+(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\quad + \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}|\lambda_j|\|\tilde{\phi}^*_+(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &= III_1 + III_2.\end{aligned}$$

Similar to I_1, we can get the estimates of III_2. For III_1, we continue to decompose it as follows.

$$\begin{aligned}III_1 &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|\tilde{\phi}^*_+(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\le \left\{\sum_{k=-\infty}^{0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|\tilde{\phi}^*_+(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\quad + \left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|\tilde{\phi}^*_+(T_1a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\quad + \left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|\tilde{\phi}^*_+(T_2a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &= III_1^1 + III_1^2 + III_1^3.\end{aligned}$$

For III_1^2, it is easy to get the estimate by a similar method to I_1. Using the vanishing moments for III_1^1 and (2.2) for III_1^3, it is readily to follow that if $x \in A_k$ and $k \ge j+1$, then

$$|\tilde{\phi}^*_+(Ta_j)(x)|, |\tilde{\phi}^*_+(T_2a_j)(x)| \le \frac{C2^{j(N+1)}}{|x|^{n+N+1}}|B_j|^{-\alpha/n}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where we choose $N \in \mathbb{Z}_+$ such that $\alpha - n\delta_2 < N+1$. From this, it is readily to follow that $III_1^1 + III_1^3 \le C$.

This completes the proof of Theorem 2.6. □

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[1] Department of Mathematical and Physical Science, Zibo Normal College, Zibo 255130, Shandong, P. R. China.

E-mail address: hbwang_2006@163.com

[2] Department of Mathematics, China University of Mining and Technology(Beijing), Beijing 100083, P. R. China.

E-mail address: liuzg@cumtb.edu.cn