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STRUCTURAL TRANSITION BETWEEN $L^p(G)$ AND $L^p(G/H)$

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Abstract. Let H be a compact subgroup of a locally compact group G. We consider the homogeneous space G/H equipped with a strongly quasi-invariant Radon measure μ. For $1 \leq p \leq +\infty$, we introduce a norm decreasing linear map from $L^p(G)$ onto $L^p(G/H, \mu)$ and show that $L^p(G/H, \mu)$ may be identified with a quotient space of $L^p(G)$. Also, we prove that $L^p(G/H, \mu)$ is isometrically isomorphic to a closed subspace of $L^p(G)$. These help us study the structure of the classical Banach spaces constructed on a homogeneous space via those created on topological groups.

1. Introduction

The function spaces on a locally compact Hausdorff topological group may possess special structures and properties which may fail to hold on the function spaces related to a locally compact Hausdorff space. For instance, $L^1(G)$ is known as an involutive Banach algebra, where G is a locally compact topological group, whereas $L^1(X)$ is known just as a Banach space, where X is a locally compact Hausdorff space. When G is a locally compact Hausdorff topological group, for every closed subgroup H of G, the space G/H consisting of all left cosets of H in G is a locally compact Hausdorff topological space on which G acts transitively from the left. The term homogeneous space means a transitive G-space which is topologically isomorphic to G/H, for some closed subgroup H of G. It has been shown that if G is σ-compact, then every transitive G-space is homeomorphic

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to the quotient space G/H for some closed subgroup H (cf. [4, Subsection 2.6]). Among all locally compact Hausdorff spaces, it seems valuable to consider homogeneous spaces and investigate the structures and the properties of their function spaces. Some results have been obtained by figuring out some relations between the function spaces on a homogeneous space and the function spaces on the corresponding topological group.

In [5], [6], [9], and [11] it has been introduced and investigated Fourier algebra $A(G/H)$ and Fourier–Stieltjes algebra $B(G/H)$, where H is compact. In [7], it has been tried to find a relation between frames and Bessel sequences of $L^2(G)$ and those of $L^2(G/H)$, where H is compact and G/H has been attached to an invariant Radon measure. Also, in [8] by assuming that H is a closed subgroup of G and G/H has been attached to a relatively invariant Radon measure, a class of continuous wavelet transform is introduced deriving from the generalized quasi–regular representations which obtained by the tensor product of a character of G and the induced representation of G on $L^2(G/H)$. In [3], a connection between the existence of admissible wavelets for G and those of G/H, where H is compact, has been illustrated. Moreover, one may refer to [2] and [10] to find some results on the space of the right cosets of H in G.

In this paper, by taking a compact subgroup H of G, we equip the homogeneous space G/H with a strongly quasi-invariant Radon measure and we demonstrate some structural connections between two function spaces $L^p(G)$ and $L^p(G/H)$, where $1 \leq p \leq +\infty$. In section 3, by introducing a bounded surjective linear map $T_p : L^p(G) \to L^p(G/H)$, we show that $L^p(G/H)$ may be identified as a quotient space of $L^p(G)$, where $1 \leq p \leq +\infty$. Therefore, every left module structure of $L^p(G)$ obviously induces a left module structure on $L^p(G/H)$, provided that $\ker T_p$ is an invariant subspace of $L^p(G)$ under the module action. Next, in section 4, by restricting the domain of T_p to a special closed subspace $L^p(G : H)$ of $L^p(G)$, we show that $L^p(G/H)$ is isometrically isomorphic to $L^p(G : H)$. By using these facts, we can study the structure of the L^p-spaces constructing on a homogeneous space via those created on topological groups. For an example, we show that the mapping T_2 is the orthogonal projection of $L^2(G)$ on $L^2(G/H)$, by considering $L^2(G/H)$ as a closed subspace of $L^2(G)$. Hence, T_2 maps every frame of $L^2(G)$ onto a frame of $L^2(G/H)$. Accordingly, when H is compact, it would be clear how to define a left module action of $L^1(G)$ on $L^p(G/H)$ and also a convolution on $L^1(G/H)$ to construct a Banach algebra.

2. Preliminaries and notations

In this section, for the reader's convenience, we provide a summary of the mathematical notations and definitions which will be used in the sequel. For details, we refer the reader to the general reference [4], or any other standard book of harmonic analysis.

Throughout this paper, we suppose that G is a locally compact Hausdorff topological group with the left Haar measure dx, H is a compact subgroup of

G with the normalized Haar measure $d\xi$, and $q : G \to G/H$ is the canonical quotient map. Also, for a function f on G and $x \in G$ we mean by $L_x f$ and $R_x f$ the left translation and the right translation of f by x, respectively, which are defined by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$, $y \in G$.

When X is a locally compact Hausdorff space, the space of all continuous complex-valued functions on X with compact support is denoted by $C_c(X)$ and if μ is a positive measure on X, the Banach space of all equivalence classes of μ-measurable complex-valued functions on X whose pth power are integrable, is denoted by $L^p(X, \mu)$, where $1 \leq p < +\infty$. If $p = +\infty$, by $L^\infty(X)$ we mean the Banach spaces of all equivalence classes of locally measurable functions which are locally essentially bounded.

For a closed subgroup H of G, a Radon measure μ on G/H is called strongly quasi-invariant if there is a continuous function $\lambda : G \times (G/H) \to (0, +\infty)$ such that $d\mu_x(yH) = \lambda(x, yH)d\mu(yH)$ for all $x \in G$, where μ_x is defined by $\mu_x(E) = \mu(xE)$ for all Borel subsets E of G/H.

Let Δ_G and Δ_H be the modular functions of G and H respectively. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, +\infty)$ for which $\rho(x\xi) = \big(\Delta_H(\xi)/\Delta_G(\xi)\big)\rho(x)$ for all $x \in G$ and $\xi \in H$. By [4, Proposition 2.54], the pair (G, H) always admits a rho-function. If μ is a strongly quasi-invariant Radon measure on G/H, then there exists a rho-function ρ for the pair (G, H) such that the Weil formula holds for all $f \in C_c(G)$; i.e.,

$$\int_{G/H} \int_H f(x\xi)\, d\xi\, d\mu(xH) = \int_G f(x)\rho(x)\, dx,$$

(cf. [4, Subection 2.6]). Moreover, the Mackey–Bruhat formula

$$\int_{G/H} \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi\, d\mu(xH) = \int_G f(x)\, dx,$$

also holds for all $f \in L^1(G)$ (cf. [12, Subection 8.2]).

Following [12, Subsection 8.2], the mapping $T_{H,\rho} : L^1(G) \to L^1(G/H, \mu)$ defined by $T_{H,\rho} f(xH) = \int_H f(x\xi)/\rho(x\xi) d\xi$ is a well-defined norm decreasing surjective linear map and

$$\|\varphi\|_1 = \inf\{\|f\|_1 : f \in L^1(G),\ T_{H,\rho} f = \varphi\},$$

for all $\varphi \in L^1(G/H, \mu)$. This yields that $L^1(G/H, \mu)$ and the quotient space $L^1(G)/\mathcal{J}^1(G, H)$ are isometrically isomorphic as two Banach spaces, where $L^1(G)/\mathcal{J}^1(G, H)$ is considered with the usual quotient norm and $\mathcal{J}^1(G, H)$ is the closure of $\{f \in C_c(G) : T_{H,\rho} f = 0\}$ in $L^1(G)$.

One of the main tool in the proof of the results above is Fubini's theorem, which allows the order of integration to be changed in an iterated integral. Accordingly, for a closed subgroup H of G, we have

$$\int_{G/H} |\int_H f(x\xi)|\, d\xi \leq \int_{G/H} \int_H |f(x\xi)|\, d\xi \quad (f \in C_c(G)).$$

For $1 < p \leq +\infty$, when H is compact, by using Minkowski's inequality for integrals we show that

$$\int_{G/H} |\int_H f(x\xi)|^p \, d\xi \leq \int_{G/H} \int_H |f(x\xi)|^p \, d\xi \quad (f \in C_c(G)).$$

Moreover, the compactness of H makes $\{\varphi \circ q : \varphi \in C_c(G/H)\}$ a subalgebra of $C_c(G)$. These help us prove the results of [12, Subsection 8.2] for all $1 < p \leq +\infty$, where H is compact. In addition, using the obtained subalgebra of $C_c(G)$ we conclude that $L^p(G/H)$ may be also considered as a closed subspace of $L^p(G)$, where $1 \leq p \leq +\infty$.

3. $L^p(G/H)$ as a quotient space of $L^p(G)$

In this section, when H is a compact subgroup of G with the normalized Haar measure $d\xi$ and $1 \leq p \leq +\infty$, we introduce a bounded linear map T_p of $L^p(G)$ onto $L^p(G/H, \mu)$ and show that $L^p(G/H, \mu)$ may be considered as a quotient space of $L^p(G)$.

Suppose that H is a closed subgroup of G. By [4, Subection 2.6], there is a surjective linear map $P : C_c(G) \to C_c(G/H)$ such that

$$Pf(xH) = \int_H f(x\xi) \, d\xi, \quad (f \in C_c(G), \, x \in G).$$

If $1 \leq p < +\infty$, $f \in C_c(G)$, and H is compact, then $|Pf(xH)|^p \leq \int_H |f(x\xi)|^p d\xi$ for all $x \in G$. According to Weil's formula, we get

$$\begin{aligned} \|Pf\|_p &= \Big(\int_{G/H} |Pf(xH)|^p \, d\mu(xH) \Big)^{1/p} \\ &\leq \Big(\int_{G/H} \int_H |f(x\xi)|^p \, d\xi \, d\mu(xH) \Big)^{1/p} \\ &= \Big(\int_G |f(x)|^p \, \rho(x) \, dx \Big)^{1/p} \\ &\leq \|f \, \rho^{1/p}\|_p. \end{aligned}$$

Therefore, the surjective linear map $T_p : C_c(G) \to C_c(G/H)$ defined by

$$T_p f(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}} \, d\xi \quad (x \in G)$$

is norm decreasing with respect to $\| \, \|_p$.

Proposition 3.1. *Let H be a compact subgroup of G and $1 \leq p < +\infty$. Then*

$$\|\varphi\|_p = \inf\{\|f\|_p : \, f \in C_c(G), \, \varphi = T_p f\},$$

for all $\varphi \in C_c(G/H)$.

Proof. Let $\varphi \in C_c(G/H)$. Since, T_p is norm decreasing,

$$\|\varphi\|_p \leq \inf\{\|f\|_p : \, f \in C_c(G), \, \varphi = T_p f\}.$$

By taking $f = \rho^{1/p}\,(\varphi \circ q)$, we have $f \in C_c(G)$, $T_p f = \varphi$, and by Weil's formula,

$$\begin{aligned}\|f\|_p^p &= \int_G \rho(x)\,|(\varphi\circ q)(x)|^p\,dx\\ &= \int_{G/H} P\big(|\varphi|^p \circ q\big)(xH)\,d\mu(xH)\\ &= \int_{G/H} |\varphi(xH)|^p\,d\mu(xH)\\ &= \|\varphi\|_p^p.\end{aligned}$$

This implies that

$$\|\varphi\|_p = \inf\{\|f\|_p :\, f \in C_c(G),\, \varphi = T_p f\}.$$

□

It is easy to check that if X and Y are dense subspaces of Banach spaces $\bar{X}$ and $\bar{Y}$, respectively, then every linear map $T : X \to Y$ with

$$\|T(x)\| = \inf\{\|z\| :\, z \in X, T(z) = T(x)\} \quad (x \in X)$$

has a unique extension $\bar{T} : \bar{X} \to \bar{Y}$ such that $\overline{\ker T} = \ker \bar{T}$ and

$$\|\bar{T}(x)\| = \inf\{\|z\| :\, z \in \bar{X}, \bar{T}(z) = \bar{T}(x)\} \quad (x \in \bar{X}).$$

So, by Proposition 3.1, there is a surjective norm decreasing linear map $T_p : L^p(G) \to L^p(G/H)$ such that for all $\varphi \in L^p(G/H)$

$$\|\varphi\|_p = \inf\{\|f\|_p :\, f \in L^p(G),\, \varphi = T_p f\}.$$

It is worthwhile to mention that T_p induces an isometry isomorphism between $L^p(G)/\ker T_p$ and $L^p(G/H)$, where $L^p(G)/\ker T_p$ is equipped with the usual quotient norm. At following, we show that for all $f \in L^p(G)$,

$$T_p f(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi \quad (\mu\text{-almost every } xH \in G/H).$$

Theorem 3.2. *Let H be a compact subgroup of G and $1 \le p < +\infty$. Then for all $f \in L^p(G)$ and μ-almost all $xH \in G/H$, $T_p f(xH) = \int_H f(x\xi)/\rho(x\xi)^{1/p} d\xi$.*

Proof. Let $f \in L^p(G)$. We choose $\{f_n\}_{n\in\mathbb{N}} \subseteq C_c(G)$, such that $\|f_n - f\|_p^p < 2^{-n}$ and $T_p f_n(xH) \to T_p f(xH)$ for μ-almost all $xH \in G/H$. Since $|f|^p \in L^1(G)$, the function defined μ-almost everywhere on G/H by $xH \mapsto \int_H |f(x\xi)|^p/\rho(x\xi)\,d\xi$ belongs to $L^1(G/H)$. Considering the normalized Haar measure on H, we have $L^p(H) \subseteq L^1(H)$ with $\|\ \|_1 \le \|\ \|_p$, for all $1 \le p \le +\infty$, and hence we get

$$\Big|\int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi\Big|^p \le \int_H \frac{|f(x\xi)|^p}{\rho(x\xi)}\,d\xi,$$

for μ-almost all $xH \in G/H$. Therefore, by the Mackey–Bruhat formula, we get

$$\int_{G/H} \Big|\int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi\Big|^p d\mu(xH) \le \|f\|_p^p$$

and so

$$\sum_{n=1}^{+\infty}\int_{G/H}|\int_H \frac{f(x\xi)-f_n(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi|^p\,d\mu(xH)\leq\sum_{n=1}^{+\infty}\|f-f_n\|_p^p<\sum_{n=1}^{+\infty}2^{-n}=1.$$

This implies that, for μ-almost all $xH\in G/H$, integral $\int_H f(x\xi)/\rho(x\xi)^{1/p}d\xi$ must be exist and moreover

$$|\int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi-\int_H \frac{f_n(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi|\to 0$$

as n tends to infinity, i.e.; $\lim_{n\to+\infty}T_pf_n(xH)=\int_H f(x\xi)/\rho(x\xi)^{1/p}\,d\xi$. Therefore,

$$T_pf(xH)=\int_H \frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi \qquad (\mu\text{-almost every } xH\in G/H).$$

□

We should mention that Theorem 3.2 also holds for $p=+\infty$, by assuming $1/+\infty=0$. To show that we need the following lemma.

Lemma 3.3. *Let H be a compact subgroup of G, $1\leq p<+\infty$, and $f\in L^p(G)$. Then $\rho^{1/p}(T_pf\circ q)\in L^p(G)$ and $\|\rho^{1/p}(T_pf\circ q)\|_p=\|T_pf\|_p$.*

Proof. One may easily check that, when H is compact, a subset E of G/H is null if and only if $q^{-1}(E)$ is a null subset of G. Now, for all $f\in L^p(G)$, according to Theorem 3.2, $T_pf(xH)=\int_H(f/\rho^{1/p})(x\xi)d\xi$ for μ-almost all $xH\in G/H$ and so for almost all $x\in G$. Moreover, by Minkowski's inequality for integrals, we can write

$$\begin{aligned}(\int_G|\rho(x)^{1/p}(T_pf\circ q)(x)|^p\,dx)^{1/p}&=(\int_G\rho(x)|\int_H\frac{f(x\xi)}{\rho(x\xi)^{1/p}}\,d\xi|^p\,dx)^{1/p}\\&\leq\int_H(\int_G|f(x\xi)|^p\,dx)^{1/p}\,d\xi\\&=\|f\|_p.\end{aligned}$$

Therefore, $\rho^{1/p}(T_pf\circ q)\in L^p(G)$ and

$$\begin{aligned}\|\rho^{1/p}(T_pf\circ q)\|_p^p&=\int_G|\rho(x)^{1/p}(T_pf\circ q)(x)|^p\,dx\\&=\int_{G/H}\int_H\frac{\rho(x\xi)|(T_pf\circ q)(x\xi)|^p}{\rho(x\xi)}\,d\xi\,d\mu(xH)\\&=\|T_pf\|_p^p.\end{aligned}$$

□

Theorem 3.4. *Let H be a compact subgroup of G. Then there is a surjective norm decreasing linear map $T_\infty:L^\infty(G)\to L^\infty(G/H)$ such that for all $f\in L^\infty(G)$,*

$$T_\infty f(xH)=\int_H f(x\xi)\,d\xi \qquad (\mu\textit{-locally almost every } xH\in G/H)$$

and for all $\varphi \in L^\infty(G/H)$,

$$\|\varphi\|_\infty = \inf\{\|f\|_\infty :\ f \in L^\infty(G), \varphi = T_\infty f\}.$$

Proof. Let $f \in L^\infty(G)$. Then, by the duality between $L^\infty(G/H)$ and $L^1(G/H)$ and using Lemma 3.3, there is an element $\psi_f \in L^\infty(G/H)$ such that

$$\int_{G/H} \varphi(xH)\psi_f(xH)\,d\mu(xH) = \int_G \varphi_\rho(x)f(x)\,dx,$$

for all $\varphi \in L^1(G/H)$, where $\varphi_\rho = \rho\,(\varphi \circ q)$. Hence, we can write

$$\begin{aligned}\int_{G/H} \varphi(xH)\psi_f(xH)\,d\mu(xH) &= \int_G \varphi_\rho(x)f(x)\,dx \\ &= \int_{G/H}\int_H \frac{\varphi_\rho(x\xi)f(x\xi)}{\rho(x\xi)}\,d\xi\,d\mu(xH) \\ &= \int_{G/H} \varphi(xH)\int_H f(x\xi)\,d\xi\,d\mu(xH),\end{aligned}$$

for all $\varphi \in L^1(G/H)$. So, $\psi_f(xH) = \int_H f(x\xi)d\xi$ for μ-locally almost every $xH \in G/H$. Now, we can define a linear map $T_\infty : L^\infty(G) \to L^\infty(G/H)$ by

$$T_\infty f(xH) = \int_H f(x\xi)d\xi \qquad (\mu\text{-locally almost every } xH \in G/H).$$

Moreover, for all $f \in L^\infty(G)$,

$$\begin{aligned}\|f\|_\infty &= \sup\{|\langle f, g\rangle| :\ g \in L^1(G), \|g\|_1 \le 1\} \\ &\ge \sup\{|\langle f, \varphi_\rho\rangle| :\ \varphi \in L^1(G/H), \|\varphi\|_1 \le 1\} \\ &= \sup\{|\langle T_\infty f, \varphi\rangle| :\ \varphi \in L^1(G/H), \|\varphi\|_1 \le 1\} \\ &= \|T_\infty f\|_\infty.\end{aligned}$$

Also, if $\varphi \in L^\infty(G/H)$, then $\varphi\circ q \in L^\infty(G)$, $T_\infty(\varphi\circ q) = \varphi$, and $\|\varphi\circ q\|_\infty = \|\varphi\|_\infty$. Hence T_∞ is onto and $\|\varphi\|_\infty = \inf\{\|f\|_\infty :\ f \in L^\infty(G), \varphi = T_\infty f\}$. □

The theorem above asserts that when $L^\infty(G)/\ker T_\infty$ has been attached to the usual quotient norm, it is isometrically isomorphic to $L^\infty(G/H)$. Accordingly, for all $1 \le p \le +\infty$, every left module structure of $L^p(G)$ obviously induces a left module structure on $L^p(G/H)$, by identifying $L^p(G/H)$ with the quotient space $L^p(G)/\ker T_p$, provided that $\ker T_p$ is an invariant subspace of $L^p(G)$ under the module action. Specially, $L^p(G)$ is known as a Banach left $L^1(G)$-module, $1 \le p \le +\infty$, using the convolution of function as the left action. Also, if $f \in L^1(G)$ and $g \in \ker T_p$, then $\rho^{1/p}(T_p g \circ q) = 0$ in $L^p(G)$. So, for almost all

$x \in G$ and hence for μ-almost all $xH \in G/H$ we have

$$\begin{aligned} T_p(f * g)(xH) &= \int_H \int_G \frac{f(y)\, g(y^{-1}x\xi)}{\rho(x)^{1/p}}\, dy\, d\xi \\ &= \frac{1}{\rho(x)^{1/p}} \int_G f(y) \left(\rho^{1/p}(T_p g \circ q)\right)(y^{-1}x)\, dy \\ &= \frac{1}{\rho(x)^{1/p}} \left(f * \rho^{1/p}(T_p g \circ q)\right)(x) \\ &= 0, \end{aligned}$$

which shows that $\ker T_p$ is invariant under the module action of $L^1(G)$ on $L^p(G)$. This makes $L^p(G/H)$ into a Banach left $L^1(G)$-module, where the action is defined by

$$\begin{cases} L^1(G) \times L^p(G/H) \to L^p(G/H) \\ \qquad (f, \psi) \mapsto T_p(f * g) \end{cases}$$

in which $g \in L^p(G)$ and $\psi = T_p g$.

4. $L^p(G/H)$ as a closed subspace of $L^p(G)$

In this section, for all $1 \leq p \leq +\infty$, we show that $L^p(G/H)$ is isometrically isomorphic to a closed subspace of $L^p(G)$. Also, if $1 \leq p < +\infty$, we show that ${T_p}^*$, the adjoint operator of T_p, is given by $T_p^*(\psi) = \rho^{1/p'}(\psi \circ q)$, for all $\psi \in L^{p'}(G/H)$, where p' is the conjugate exponent of p and $1/+\infty = 0$.

For a closed subgroup H of G we define

$$C_c(G : H) = \{f \in C_c(G) :\ R_\xi f = f, \xi \in H\}$$

which is a subalgebra of $C_c(G)$. For all $1 \leq p < +\infty$, we denote by $L^p(G : H)$ the closure of $C_c(G : H)$ in $L^p(G)$ and we set

$$L^\infty(G : H) = \{f \in L^\infty(G) :\ R_\xi f = f, \xi \in H\}.$$

If H is compact and $1 \leq p < +\infty$, then $T_p f \in C_c(G/H)$ and

$$\rho(x)^{1/p}(T_p f \circ q)(x) = \int_H f(x\xi)\, d\xi = f(x),$$

for all $f \in C_c(G : H)$. Therefore,

$$C_c(G : H) = \{\rho^{1/p}(\psi \circ q) :\ \psi \in C_c(G/H)\}.$$

Proposition 4.1. *If H is compact and $1 \leq p \leq +\infty$, then*

$$L^p(G : H) = \{\rho^{1/p}(\psi \circ q) :\ \psi \in L^p(G/H)\}.$$

In particular, $f = \rho^{1/p}(T_p f \circ q)$ for all $f \in L^p(G : H)$.

Proof. For $p = +\infty$, the conclusion follows from the compactness of H. Now, let $1 \le p < +\infty$ and $f \in L^p(G:H)$. Choose a sequence $\{f_n\} \subseteq C_c(G:H)$ such that $\|f_n - f\|_p \to 0$. Then, by Lemma 3.3, we can write

$$\begin{aligned}\|f - \rho^{1/p}(T_p f \circ q)\|_p &\le \|f - f_n\|_p + \|\rho^{1/p} T_p(f_n - f) \circ q\|_p \\ &= \|f - f_n\|_p + \|T_p(f_n - f)\|_p \\ &\le 2\|f - f_n\|_p.\end{aligned}$$

So, we have $f = \rho^{1/p}(T_p f \circ q)$ in $L^p(G)$. The density of $C_c(G/H)$ in $L^p(G/H)$ complete the proof. □

Theorem 4.2. *Let H be a compact subgroup of G and $1 \le p \le +\infty$. Then $L^p(G/H)$ is isometrically isomorphic to $L^p(G:H)$. More precisely, the restriction of T_p on $L^p(G:H)$ is an isometry isomorphism.*

Proof. Let $1 \le p < +\infty$. By the definition of $L^p(G:H)$ and the density of $C_c(G/H)$ in $L^p(G/H)$ it is enough to show that the mapping $T_p : C_c(G:H) \to C_c(G/H)$ is an isometry isomorphism, where $C_c(G:H)$ and $C_c(G/H)$ are equipped with $\|\,\|_p$. To show this, first note that if $\psi \in C_c(G/H)$, then $\rho^{1/p}(\psi \circ q) \in C_c(G:H)$ and $T_p(\rho^{1/p}(\psi \circ q)) = \psi$. Now, if $f \in C_c(G:H)$, then

$$\begin{aligned}\|T_p f\|_p^p &= \int_{G/H} |T_p f(xH)|^p \, d\mu(xH) \\ &= \int_{G/H} \frac{|f(x)|^p}{\rho(x)} \, d\mu(xH) \\ &= \int_{G/H} \int_H \frac{|f(x\xi)|^p}{\rho(x\xi)} \, d\xi \, d\mu(xH) \\ &= \int_G |f(x)|^p \, dx \\ &= \|f\|_p^p.\end{aligned}$$

The conclusion for the case $p = +\infty$ follows from Proposition 4.1

□

We note that, for each $1 \le p < +\infty$ with conjugate exponent p', by using the well-known transform $L^{p'}(G/H) \to L^p(G/H)^*$ which indicates the duality between $L^{p'}(G/H)$ and $L^p(G/H)$ we may find an isometry isomorphism $L^{p'}(G:H) \to L^p(G:H)^*$ by creating the commutative diagram below:

$$\begin{array}{ccc} L^{p'}(G:H) & \longrightarrow & L^p(G:H)^* \\ {\scriptstyle T_{p'}}\big\downarrow & & \big\uparrow{\scriptstyle T_p^*} \\ L^{p'}(G/H) & \longrightarrow & L^p(G/H)^* \end{array}$$

In other words, this duality between $L^p(G:H)$ and $L^{p'}(G:H)$ is such that for all $f \in L^p(G:H)$ and $g \in L^{p'}(G:H)$ we have

$$\langle f,\, g\rangle = \langle T_p f,\, T_{p'} g\rangle.$$

Specially, $T_2 : L^2(G : H) \to L^2(G/H)$ is a unitary operator by the following proposition.

Proposition 4.3. *Let H be a compact subgroup of G and p' be the conjugate exponent of $1 \leq p < +\infty$. If $T_p^* : L^{p'}(G/H) \to L^{p'}(G)$ is the adjoint operator of T_p, then $T_p^*(\psi) = \rho^{1/p'}(\psi \circ q)$, for all $\psi \in L^{p'}(G/H)$.*

Proof. Let $\psi \in L^{p'}(G/H)$. Then, by Lemma 3.3, for all $f \in L^p(G)$ we have

$$\begin{aligned}
\langle T_p^*(\psi), f\rangle &= \int_{G/H} \psi(xH)\, T_p f(xH)\, d\mu(xH) \\
&= \int_{G/H}\int_H \frac{(\psi\circ q)(x\xi)\, f(x\xi)}{\rho(x\xi)^{1/p}}\, d\xi\, d\mu(xH) \\
&= \int_{G/H}\int_H \frac{\rho(x)^{1/p'}(\psi\circ q)(x\xi)\, f(x\xi)}{\rho(x\xi)}\, d\xi\, d\mu(xH) \\
&= \int_G \rho^{1/p'}(x)\,(\psi\circ q)(x)\, f(x)\, dx \\
&= \langle \rho^{1/p'}(\psi\circ q), f\rangle.
\end{aligned}$$

Hence $T_p^*(\psi) = \rho^{1/p'}(\psi \circ q)$.

□

For a given Hilbert space, it is often useful to find a basis, a Riesz basis, or a frame as generalization of a basis, to get a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that any vector f can be written as $f = \Sigma_{n=1}^{+\infty} c_n g_n$ for some scalers c_n, $n \in \mathbb{N}$ (cf. [1]). To construct a frame of Hilbert space $L^2(G/H)$, it is instrumental to assert that the mapping $T_2 : L^2(G) \to L^2(G/H)$ is the orthogonal projection of $L^2(G)$ on $L^2(G/H)$, by considering $L^2(G/H)$ as a closed subspace of $L^2(G)$. In fact, Proposition 4.1 shows that for all $1 \leq p \leq +\infty$, the mapping $f \mapsto \rho^{1/p}(T_p f \circ q)$ is a projection on $L^p(G)$. Particularly, one may easily check that

$$\left\{\begin{array}{c} L^2(G) \to L^2(G : H) \subseteq L^2(G) \\ f \mapsto \rho^{1/2}(T_2 f \circ q) \end{array}\right.$$

is the orthogonal projection of $L^2(G)$ on $L^2(G : H)$. This helps us study the structure of the L^p-spaces constructing on a homogeneous space via those created on topological groups. For instance, T_2 maps every frame of $L^2(G)$ onto a frame of $L^2(G/H)$, and if $\{\psi_n\}_{n\in\mathbb{N}}$ is a frame for $L^2(G/H)$, then $\{\rho^{1/2}(\psi_n \circ q)\}_{n\in\mathbb{N}}$ is a frame sequence in $L^2(G)$.

Following we give another characterization of $L^p(G : H)$.

Proposition 4.4. *Let H be a compact subgroup of G. Then for all $1 \leq p \leq +\infty$ we have*

$$L^p(G : H) = \{f \in L^p(G) :\ R_\xi f = f \text{ in } L^p(G),\ \xi \in H\}.$$

Proof. First, suppose that $f \in L^p(G:H)$. Then, by using Proposition 4.1, we can write

$$\begin{aligned} R_\xi f(x) &= R_\xi\big(\rho^{1/p}\,(T_p f\circ q)\big)(x) \\ &= \rho^{1/p}(x\xi)\,(T_p f\circ q)(x\xi) \\ &= \rho^{1/p}(x)\,(T_p f\circ q)(x) \\ &= f(x), \end{aligned}$$

for μ-almost all $xH \in G/H$ and so for almost all $x\in G$. So, $R_\xi f = f$ as elements of $L^p(G)$. Now, assume that $f\in L^p(G)$ and $R_\xi f = f$ for all $\xi \in H$. According to the duality of $L^p(G/H)$ and $L^{p'}(G/H)$, for all $g\in L^{p'}(G)$ we have

$$\begin{aligned} \langle \rho^{1/p}\,(T_p f\circ q),\, g\rangle &= \int_G \rho(x)^{1/p}\,T_p f(xH)\,g(x)\,dx \\ &= \int_G\int_H f(x\xi)\,d\xi\,g(x)\,dx \\ &= \int_H\int_G R_\xi f(x)\,g(x)\,dx\,d\xi \\ &= \int_H \langle R_\xi f,\, g\rangle\,d\xi \\ &= \int_H \langle f,\, g\rangle\,d\xi \\ &= \langle f,\, g\rangle, \end{aligned}$$

where p and p' are conjugate exponents. So, $f = \rho^{1/p}\,(T_p f\circ q)\in L^p(G:H)$. $\square$

By using Proposition 4.4, one may easily show that $L^1(G:H)$ is a closed left ideal and hence a closed subalgebra of $L^1(G)$. So, by using the isometry isomorphism of Banach spaces $T_1 : L^1(G:H)\to L^1(G/H)$, we can transfer the multiplication of $L^1(G:H)$ to $L^1(G/H)$. In other words, $L^1(G/H)$ is a Banach algebra with respect to the multiplication

$$\begin{cases} \qquad L^1(G/H)\times L^1(G/H)\to L^1(G/H) \\ (\varphi,\,\psi)\mapsto \varphi*\psi = T_1\big(\rho(\varphi\circ q)*\rho(\psi\circ q)\big) \end{cases}$$

Moreover, the Banach algebra $L^1(G/H)$ is isometrically isomorphic to a subalgebra of $L^1(G)$. Hence, every $L^1(G)$-module may be considered as an $L^1(G/H)$-module.

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