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ON SOME NEW CONVERSES OF CONVEX INEQUALITIES IN HILBERT SPACE

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ABSTRACT. In this paper we study some new converses of the operator version for the Jensen inequality and the scalar Lah–Ribarič inequality for continuous convex functions and we will give applications of those results to quasi-arithmetic means and power means for selfadjoint operators. Furthermore, we will also give improvements of the obtained results.

1. INTRODUCTION AND PRELIMINARIES

The Jensen inequality for convex functions plays a very important role in the Theory of Inequalities due to the fact that it implies the whole series of the other classical inequalities such as the quasi-arithmetic mean and arithmetic mean inequalities, Hölder and Minkowski inequalities, Ky Fan’s inequality etc.

One of the most famous ones amongst them is the so called Lah–Ribarič inequality, which we state in the following theorem.

Theorem 1.1. ([10]) *Let $f: [m, M] \rightarrow \mathbb{R}$ be a convex function, $y_1, \dots, y_n \in [m, M]$ real numbers and let p_1, \dots, p_n be nonnegative real numbers such that $P_n = \sum_{i=1}^n p_i > 0$. Then*

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(y_i) \leq \frac{M - \bar{y}}{M - m} f(m) + \frac{\bar{y} - m}{M - m} f(M), \quad (1.1)$$

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where $\bar{y} = \frac{1}{P_n} \sum_{i=1}^n p_i y_i$.

It was obtained in 1973. by P. Lah and M. Ribarić in their paper [10]. Since then, there have been many papers written on the subject of it's generalizations and converses.

In this paper we refer to a general form of the Jensen inequality for selfadjoint operators. In order to present our results, we first need to introduce the appropriate settings.

Let H be a Hilbert space and let $\mathcal{B}(H)$ be the C^* -algebra of all bounded (i.e., continuous) linear operators on H . A bounded linear operator A on a Hilbert space H is said to be selfadjoint if $A = A^*$. An operator $A \in \mathcal{B}(H)$ is selfadjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for every $x \in H$. We denote by $\mathcal{B}_h(H)$ a semi-space of all selfadjoint operators in $\mathcal{B}(H)$. If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for every $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H . Equivalently, if both f and g are real valued continuous function on $Sp(A)$, then the following property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \quad (1.2)$$

in the operator order of $\mathcal{B}(H)$.

The result stated below provides an operator version for Jensen's inequality and it is due to Mond and Pečarić [15] (see also [3]):

Theorem 1.2. ([15]) (Mond – Pečarić) *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle \quad (1.3)$$

for each $x \in H$ with $\|x\| = 1$.

Jensen's type inequalities for selfadjoint operators in Hilbert spaces have been an object of study for many different papers. For more on the Mond–Pečarić inequality, i.e. Jensen's inequality for selfadjoint operators and it's refinements, converses and applications, see [1], [4], [5], [6], [11], [12] and [13].

R. Jakšić and J. Pečarić in their paper [7] gave the following result, which concerns the difference between the right and the left side of the Mond–Pečarić inequality (1.3).

Theorem 1.3. ([7]) *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I , then*

$$\begin{aligned} 0 &\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)) \end{aligned} \quad (1.4)$$

holds for each unit vector x in H . If f is concave, the inequalities in (1.4) are reversed.

We also need the following converse for the Mond–Pečarić inequality that provides a generalization of the Lah–Ribarič inequality for convex functions stated earlier:

Theorem 1.4. ([15]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m and M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A)x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} f(m) + \frac{\langle Ax, x \rangle - m}{M - m} f(M) \quad (1.5)$$

for each $x \in H$ with $\|x\| = 1$.

Remark 1.5. One can see that if in Theorem 1.4 we choose $A = \text{diag}(y_1, \dots, y_n)$ and $x = \frac{1}{\sqrt{P_n}}(\sqrt{p_1}, \dots, \sqrt{p_n})$, where y_i, p_i ($i = 1, \dots, n$) and P_n are defined in Theorem 1.1, then the inequality (1.5) becomes exactly (1.1).

The authors in [7] also gave a series of inequalities that concerns with the difference between the right and the left side of the inequality (1.5), which we state below.

Theorem 1.6. ([7]) *Let the assumptions of Theorem 1.3 hold. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I , then*

$$\begin{aligned} 0 &\leq \frac{M - \langle Ax, x \rangle}{M - m} f(m) + \frac{\langle Ax, x \rangle - m}{M - m} f(M) - \langle f(A)x, x \rangle \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m} \langle (M1_H - A)(A - m1_H)x, x \rangle \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m} (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)) \end{aligned} \quad (1.6)$$

holds for each unit vector x in H . If f is concave, the inequalities in (1.6) are reversed.

The main aim of this paper is to give refinements and improvements of converses of the Jensen inequality (1.4) and the scalar Lah–Ribarič inequality (1.6) for selfadjoint operators obtained in [7].

In the next section, in Theorem 2.1 we shall give a refinement of the inequality (1.4) and in Theorem 2.4 we shall give two similar series of inequalities that refine the inequality (1.6). Also, we shall give applications of the obtained results to quasi-arithmetic means and power means of selfadjoint operators.

In the last section, in Theorem 5.2 we shall improve the results from Theorem 2.1, and show that we have obtained more accurate upper bounds for the difference between left and right side of the scalar Jensen inequality for selfadjoint operators (1.3). We shall also obtain an improvement of the scalar Lah–Ribarič inequality for selfadjoint operators (1.5), and in Theorem 5.7 we shall improve the results from Theorem 2.4, and show that we have obtained more accurate lower

bound for the difference between left and right side of the scalar Lah–Ribarič inequality for selfadjoint operators (1.5)

2. MAIN RESULTS

Our first result is a refinement of the converse of Jensen’s inequality (1.4) for selfadjoint operators obtained in [7].

Theorem 2.1. *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I and $x \in H$ such that $\|x\| = 1$, then*

$$\begin{aligned} 0 &\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)). \end{aligned} \quad (2.1)$$

We also have the inequalities

$$\begin{aligned} 0 &\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \frac{1}{4}(M - m)^2 \Psi_f(\langle Ax, x \rangle; m, M) \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)), \end{aligned} \quad (2.2)$$

where $\Psi_f(\cdot; m, M): \langle m, M \rangle \rightarrow \mathbb{R}$ is defined by

$$\Psi_f(t; m, M) = \frac{1}{M - m} \left(\frac{f(M) - f(t)}{M - t} - \frac{f(t) - f(m)}{t - m} \right). \quad (2.3)$$

If f is concave, the inequalities are reversed.

Proof. First we assume that f is convex. If $\langle Ax, x \rangle = m$ or $\langle Ax, x \rangle = M$, the inequalities are clear. Let us suppose that $\langle Ax, x \rangle \in \langle m, M \rangle$.

The first inequality follows directly from (1.3). By Theorem 1.4, we have

$$\begin{aligned} \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) &\leq \frac{M - \langle Ax, x \rangle}{M - m} f(m) + \frac{\langle Ax, x \rangle - m}{M - m} f(M) - f(\langle Ax, x \rangle) \\ &= \frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M - m} \left\{ \frac{f(M) - f(\langle Ax, x \rangle)}{M - \langle Ax, x \rangle} - \frac{f(\langle Ax, x \rangle) - f(m)}{\langle Ax, x \rangle - m} \right\} \\ &= (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \Psi_f(\langle Ax, x \rangle; m, M) \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M), \end{aligned}$$

so we have proved the second inequality in (2.1).

$$\begin{aligned}
 \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) &= \frac{1}{M - m} \sup_{t \in \langle m, M \rangle} \left\{ \frac{f(M) - f(t)}{M - t} - \frac{f(t) - f(m)}{t - m} \right\} \\
 &\leq \frac{1}{M - m} \left(\sup_{t \in \langle m, M \rangle} \frac{f(M) - f(t)}{M - t} + \sup_{t \in \langle m, M \rangle} \frac{-(f(t) - f(m))}{t - m} \right) \\
 &= \frac{1}{M - m} \left(\sup_{t \in \langle m, M \rangle} \frac{f(M) - f(t)}{M - t} - \inf_{t \in \langle m, M \rangle} \frac{f(t) - f(m)}{t - m} \right) \\
 &= \frac{f'_-(M) - f'_+(m)}{M - m},
 \end{aligned}$$

which proves the third inequality in (2.1). To prove the last inequality in (2.1), we notice that for every $t \in [m, M]$, the inequality $\frac{(M - t)(t - m)}{M - m} \leq \frac{1}{4}(M - m)$ is valid. Since $\langle Ax, x \rangle \in [m, M]$, we can replace $t \leftrightarrow \langle Ax, x \rangle$ and the proof is completed.

The proof of the inequalities (2.2) is clear from the proof of the inequalities (2.1). If f is concave, then $-f$ is convex, so we can apply (2.1) and (2.2) to function $-f$ and obtain the reversed inequalities for f . \square

Remark 2.2. Observe that the function $\Psi_f(\cdot; m, M): \langle m, M \rangle \rightarrow \mathbb{R}$ defined in (2.3) is actually the second order divided difference of the function f at the points m , t and M for any $t \in \langle m, M \rangle$.

In order to obtain a converse of the Lah–Ribarič inequality for convex functions (1.6), we need the following result found in [7]:

Lemma 2.3. ([7]) Let f be a continuous convex function on an interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I . Then for any $t \in [m, M]$ the following inequalities are valid:

$$\begin{aligned}
 \Delta_f(t; m, M) &= \frac{(t - m)f(M) + (M - t)f(m)}{M - m} - f(t) \\
 &\leq (M - t)(t - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\
 &\leq \frac{(M - t)(t - m)}{M - m} (f'_-(M) - f'_+(m)) \tag{2.4} \\
 &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)).
 \end{aligned}$$

Also we have

$$\Delta_f(t; m, M) \leq \frac{1}{4}(M - m)^2 \Psi_f(t; m, M) \leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)),$$

where $\Psi_f(\cdot; m, M): \langle m, M \rangle \rightarrow \mathbb{R}$ is defined in (2.3). If f is concave, the inequalities are reversed.

Theorem 2.4. Let the assumptions of Theorem 2.1 hold. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is

the interior of I and $x \in H$ such that $\|x\| = 1$, then the following inequalities are valid

(i)

$$\begin{aligned}
0 &\leq \frac{(\langle Ax, x \rangle - m)f(M) + (M - \langle Ax, x \rangle)f(m)}{M - m} - \langle f(A)x, x \rangle \\
&\leq \langle (M1_H - A)(A - m1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\
&\leq \frac{\langle (M1_H - A)(A - m1_H)x, x \rangle}{M - m} (f'_-(M) - f'_+(m)) \\
&\leq \frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M - m} (f'_-(M) - f'_+(m)) \\
&\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m))
\end{aligned} \tag{2.5}$$

(ii)

$$\begin{aligned}
0 &\leq \frac{(\langle Ax, x \rangle - m)f(M) + (M - \langle Ax, x \rangle)f(m)}{M - m} - \langle f(A)x, x \rangle \\
&\leq \langle (M1_H - A)(A - m1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\
&\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\
&\leq \frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M - m} (f'_-(M) - f'_+(m)) \\
&\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m))
\end{aligned} \tag{2.6}$$

where $\Psi_f(\cdot; m, M) : \langle m, M \rangle \rightarrow \mathbb{R}$ is defined in (2.3). If f is concave, the inequalities are reversed.

Proof. Let us assume that f is convex. If $\langle Ax, x \rangle = m$ or $\langle Ax, x \rangle = M$, the inequalities are clear. Let us suppose that $\langle Ax, x \rangle \in \langle m, M \rangle$.

The first inequalities follow directly from Theorem 1.4. Due to property (1.2), we can replace t with operator A in the inequalities (2.4) from Lemma 2.3 and obtain

$$\begin{aligned}
&\frac{1}{M - m} ((A - m1_H)f(M) + (M1_H - A)f(m)) - f(A) \\
&\leq (M1_H - A)(A - m1_H) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\
&\leq \frac{1}{M - m} (M1_H - A)(A - m1_H)(f'_-(M) - f'_+(m)) \\
&\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m))1_H
\end{aligned} \tag{2.7}$$

in the operator order of $\mathcal{B}(H)$.

Because scalar product is linear in the first argument and $\|x\| = 1$, by applying

it to (2.7) we have

$$\begin{aligned} & \frac{\langle Ax, x \rangle - m}{M - m} f(M) + \frac{M - \langle Ax, x \rangle}{M - m} f(m) - \langle f(A)x, x \rangle \\ & \leq \langle (M1_H - A)(A - m1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\ & \leq \frac{1}{M - m} \langle (M1_H - A)(A - m1_H)x, x \rangle (f'_-(M) - f'_+(m)). \end{aligned}$$

Since $h(t) = (M - t)(t - m)$ is concave, from the Jensen inequality we have

$$\langle (M1_H - A)(A - m1_H)x, x \rangle \leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m).$$

To prove the last inequality in (2.5), we notice that for every $t \in [m, M]$, $\frac{(M - t)(t - m)}{M - m} \leq \frac{1}{4}(M - m)$ is valid, and since $\langle Ax, x \rangle \in [m, M]$ for every $x \in H$ such that $\|x\| = 1$, we can replace t with $\langle Ax, x \rangle$.

The first and second inequalities in (2.6) are just the first and second inequalities in (2.5). Again, the function $g(t) = (M - t)(t - m)$ is concave, so from Jensen's inequality it follows that $\langle (M1_H - A)(A - m1_H)x, x \rangle \leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)$, which proves the third inequality in (2.6). In the proof of Theorem 2.1 we have

shown that the inequality $\sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \leq \frac{f'_-(M) - f'_+(m)}{M - m}$ is valid, so

the fourth inequality in (2.6) directly follows. To prove the last inequality in (2.6), we notice that for every $t \in [m, M]$, the inequality $\frac{(M - t)(t - m)}{M - m} \leq \frac{1}{4}(M - m)$ is

valid. Since $\langle Ax, x \rangle \in [m, M]$, we have $\frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M - m} \leq \frac{1}{4}(M - m)$

and thus the proof is complete.

If f is concave, then $-f$ is convex, so we can apply (2.5) and (2.6) to function $-f$ and obtain reversed inequalities for f . \square

Remark 2.5. For an extensive list of results in inequalities for functions of self-adjoint operators related to Mond–Pečarić method and applications, see [2] and the references therein.

3. APPLICATIONS TO QUASI-ARITHMETIC MEANS

Let A be a positive invertible operator on a Hilbert space such that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$ and x a unit vector in H . Let f be a strictly monotone continuous function on $[m, M]$. Quasi-arithmetic mean of the operator A with respect to f is defined by

$$M_f(A, x) = f^{-1}(\langle f(A)x, x \rangle). \tag{3.1}$$

In this section, the objective is to apply the results from the previous section to some strictly monotone functions, and in that way get two series of inequalities involving quasi-arithmetic means.

Theorem 3.1. *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$, let f, g be strictly monotone continuous functions on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I ,*

and let us denote $[m_g, M_g] = g([m, M])$. If the function $f \circ g^{-1}$ is convex, then for each $x \in H$ such that $\|x\| = 1$ we have

$$\begin{aligned}
0 &\leq f(M_f(A, x)) - f(M_g(A, x)) \\
&\leq (M_g - \langle g(A)x, x \rangle)(\langle g(A)x, x \rangle - m_g) \sup_{t \in \langle m, M \rangle} \Psi_{f \circ g^{-1}}(g(t); m_g, M_g) \\
&\leq (M_g - \langle g(A)x, x \rangle)(\langle g(A)x, x \rangle - m_g) \frac{(f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)}{M_g - m_g} \quad (3.2) \\
&\leq \frac{1}{4}(M_g - m_g)((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)).
\end{aligned}$$

We also have the inequalities

$$\begin{aligned}
0 &\leq f(M_f(A, x)) - f(M_g(A, x)) \leq \frac{1}{4}(M_g - m_g)^2 \Psi_{f \circ g^{-1}}(\langle g(A)x, x \rangle; m_g, M_g) \\
&\leq \frac{1}{4}(M_g - m_g)((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)). \quad (3.3)
\end{aligned}$$

If $f \circ g^{-1}$ is concave, then the inequalities are reversed.

Proof. Function $f \circ g^{-1}$ is obviously continuous. Let us assume that $f \circ g^{-1}$ is convex. If g is strictly increasing, then $m_g = g(m)$, $M_g = g(M)$; if g is strictly decreasing, then $m_g = g(M)$, $M_g = g(m)$. The conditions of Theorem 2.1 are satisfied, so we can obtain (3.2) and (3.3) by substituting $f \leftrightarrow f \circ g^{-1}$ and $A \leftrightarrow g(A)$ in (2.1) and (2.2).

Now let us assume that $f \circ g^{-1}$ is concave. Then the function $-f \circ g^{-1}$ is convex, so we can obtain reversed inequalities by replacing $f \circ g^{-1}$ with $-f \circ g^{-1}$. \square

Theorem 3.2. *Under the assumptions from Theorem 3.1, if the function $f \circ g^{-1}$ is convex, then the following inequalities are valid:*

$$\begin{aligned}
&(i) \\
0 &\leq \frac{M_g - \langle g(A)x, x \rangle}{M_g - m_g} f(m) + \frac{\langle g(A)x, x \rangle - m_g}{M_g - m_g} f(M) - f(M_f(A, x)) \\
&\leq \langle (M_g 1_H - g(A))(g(A) - m_g 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{f \circ g^{-1}}(g(t); m_g, M_g) \\
&\leq \frac{\langle (M_g 1_H - g(A))(g(A) - m_g 1_H)x, x \rangle}{M_g - m_g} ((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)) \\
&\leq \frac{(M_g - \langle g(A)x, x \rangle)(\langle g(A)x, x \rangle - m_g)}{M_g - m_g} ((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)) \\
&\leq \frac{1}{4}(M_g - m_g)((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)) \quad (3.4)
\end{aligned}$$

$$\begin{aligned}
 & \text{(ii)} \\
 0 & \leq \frac{M_g - \langle g(A)x, x \rangle}{M_g - m_g} f(m) + \frac{\langle g(A)x, x \rangle - m_g}{M_g - m_g} f(M) - f(M_f(A, x)) \\
 & \leq \langle (M_g 1_H - g(A))(g(A) - m_g 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{f \circ g^{-1}}(g(t); m_g, M_g) \\
 & \leq (M_g - \langle g(A)x, x \rangle) (\langle g(A)x, x \rangle - m_g) \sup_{t \in \langle m, M \rangle} \Psi_{f \circ g^{-1}}(g(t); m_g, M_g) \\
 & \leq \frac{(M_g - \langle g(A)x, x \rangle) (\langle g(A)x, x \rangle - m_g)}{M_g - m_g} ((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)) \\
 & \leq \frac{1}{4} (M_g - m_g) ((f \circ g^{-1})'_-(M_g) - (f \circ g^{-1})'_+(m_g)) \tag{3.5}
 \end{aligned}$$

for each $x \in H$ such that $\|x\| = 1$, where $[m_g, M_g] = g([m, M])$. If $f \circ g^{-1}$ is concave, the inequalities are reversed.

Proof. Similarly to the proof of the previous theorem, let us assume that $f \circ g^{-1}$ is convex (obviously, $f \circ g^{-1}$ is continuous).

If g is strictly increasing, then $m_g = g(m)$, $M_g = g(M)$; if g is strictly decreasing, then $m_g = g(M)$, $M_g = g(m)$. The conditions of Theorem 2.4 are satisfied. We obtain (3.4) and (3.5) by substituting $f \leftrightarrow f \circ g^{-1}$ and $A \leftrightarrow g(A)$ in (2.5) and (2.6).

Now let us assume that $f \circ g^{-1}$ is concave. Then the function $-f \circ g^{-1}$ is convex, so we can obtain reversed inequalities by replacing $f \circ g^{-1}$ with $-f \circ g^{-1}$. \square

Remark 3.3. Let f, g, x and A be as in Theorem 3.2. We can rewrite the first inequality in (3.4) as:

$$(g(M) - g(m))f(M_f(A, x)) - (f(M) - f(m))g(M_g(A, x)) \leq g(M)f(m) - g(m)f(M)$$

if $f \circ g^{-1}$ is convex and g is strictly increasing. If $f \circ g^{-1}$ is concave, the inequality is reversed.

4. APPLICATIONS TO POWER MEANS

Let A be a positive invertible operator on a Hilbert space and x a unit vector in H . For $r \in \mathbb{R}$, the power mean $M_r(A, x)$ is defined by

$$M_r(A, x) = (\langle A^r x, x \rangle)^{1/r}.$$

In [15, p.69] it has been shown that if $r \rightarrow 0$, then $(\langle A^r x, x \rangle)^{1/r}$ converges monotone to $\exp\langle \log Ax, x \rangle$, so we can extend the definition of the power mean to the case $r = 0$.

Power means are a special case of quasi-arithmetic means defined in (3.1), so series of inequalities involving arithmetics means analogue to those from the previous section easily follow from Theorem 3.1 and Theorem 3.2 respectively.

Theorem 4.1. *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $0 < m < M < \infty$. Let x be a unit vector in H , $r, s \in \mathbb{R}$ such*

that $r < s$ and let

$$\phi_{s,t}(t) = \begin{cases} t^{s/r} & : r \neq 0, s \neq 0, \\ \frac{1}{r} \log t & : r \neq 0, s = 0, \\ e^{st} & : r = 0, s \neq 0. \end{cases}$$

If $r < 0 < s$ or $0 < r < s$, then

$$\begin{aligned} 0 &\leq (M_s(A, x))^s - (M_r(A, x))^s \\ &\leq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\ &\leq \frac{s}{r} (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\leq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}), \end{aligned} \quad (4.1)$$

and we also have

$$\begin{aligned} 0 &\leq (M_s(A, x))^s - (M_r(A, x))^s \leq \frac{1}{4} (M^r - m^r)^2 \Psi_{\phi_{s,t}}(\langle A^r x, x \rangle; m^r, M^r) \\ &\leq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \quad (4.2)$$

If $r < s < 0$, then

$$\begin{aligned} 0 &\geq (M_s(A, x))^s - (M_r(A, x))^s \\ &\geq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\ &\geq \frac{s}{r} (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\geq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}), \end{aligned} \quad (4.3)$$

and we also have

$$\begin{aligned} 0 &\geq (M_s(A, x))^s - (M_r(A, x))^s \geq \frac{1}{4} (M^r - m^r)^2 \Psi_{\phi_{s,t}}(\langle A^r x, x \rangle; m^r, M^r) \\ &\geq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \quad (4.4)$$

If $r = 0$ and $s > 0$, then

$$\begin{aligned} 0 &\leq (M_s(A, x))^s - (M_0(A, x))^s \\ &\leq (\log M - \langle \log Ax, x \rangle)(\langle \log Ax, x \rangle - \log m) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(\log t; \log m, \log M) \\ &\leq s(\log M - \langle \log Ax, x \rangle)(\langle \log Ax, x \rangle - \log m) \frac{M^s - m^s}{\log M - \log m} \\ &\leq \frac{s}{4} (M^s - m^s) \log \frac{M}{m}, \end{aligned} \quad (4.5)$$

and we also have

$$\begin{aligned} 0 &\leq (M_s(A, x))^s - (M_0(A, x))^s \leq \frac{1}{4} \left(\log \frac{M}{m} \right)^2 \Psi_{\phi_{s,t}}(\langle \log Ax, x \rangle; \log m, \log M) \\ &\leq \frac{s}{4} (M^s - m^s) \log \frac{M}{m}. \end{aligned} \quad (4.6)$$

If $r < 0$ or $s = 0$, then

$$\begin{aligned} 0 &\leq \log(M_0(A, x)) - \log(M_r(A, x)) \\ &\leq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\ &\leq -\frac{1}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r m^r} \\ &\leq \frac{1}{4r} (M^r - m^r)(M^{-r} - m^{-r}), \end{aligned} \quad (4.7)$$

and we also have

$$\begin{aligned} 0 &\leq \log(M_0(A, x)) - \log(M_r(A, x)) \leq \frac{1}{4} (M^r - m^r)^2 \Psi_{\phi_{s,t}}(\langle A^r x, x \rangle; m^r, M^r) \\ &\leq \frac{1}{4r} (M^r - m^r)(M^{-r} - m^{-r}). \end{aligned} \quad (4.8)$$

Proof. This theorem follows directly from Theorem 3.1 in four steps.

Put $f(t) = t^s$ and $g(t) = t^r$ in Theorem 3.1. Then $f \circ g^{-1}$ is convex and $\phi_{s,t} = f \circ g^{-1}$ whenever $0 < r < s$ or $r < 0 < s$. Therefore (4.1) and (4.2) follow easily from (3.2) and (3.3), respectively.

Put $f(t) = -t^s$ and $g(t) = t^r$ in Theorem 3.1. Then $f \circ g^{-1}$ is convex and $\phi_{s,t} = -f \circ g^{-1}$ whenever $r < s < 0$. Therefore (4.3) and (4.4) follow easily from (3.2) and (3.3), respectively.

Put $f(t) = t^s$ and $g(t) = \log t$ in Theorem 3.1. Then $f \circ g^{-1}$ is convex and $\phi_{s,t} = f \circ g^{-1}$ whenever $r = 0$ and $s > 0$. Therefore (4.5) and (4.6) follow easily from (3.2) and (3.3), respectively.

Put $f(t) = \log t$ and $g(t) = t^r$ in Theorem 3.1. Then $f \circ g^{-1}$ is convex and $\phi_{s,t} = f \circ g^{-1}$ whenever $r < 0$ and $s = 0$. Therefore (4.7) and (4.8) follow easily from (3.2) and (3.3), respectively. \square

Theorem 4.2. *Under the same hypothesis as in the previous theorem, if $0 < r < s$ or $r < 0 < s$, then:*

(i)

$$\begin{aligned}
0 &\leq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} m^s + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} M^s - (M_s(A, x))^s \\
&\leq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq \frac{s}{r} \frac{\langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\leq \frac{s}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\leq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r})
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\leq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} m^s + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} M^s - (M_s(A, x))^s \\
&\leq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq \frac{s}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\leq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}).
\end{aligned}$$

If $r < s < 0$ then:

(i)

$$\begin{aligned}
0 &\geq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} m^s + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} M^s - (M_s(A, x))^s \\
&\geq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\geq \frac{s}{r} \frac{\langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\geq \frac{s}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\geq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r})
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\geq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} m^s + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} M^s - (M_s(A, x))^s \\
&\geq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\geq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\geq \frac{s}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\geq \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}).
\end{aligned}$$

If $s = 0$ and $r < 0$, then:

(i)

$$\begin{aligned}
0 &\leq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} \log m + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} \log M - \log(M_0(A, x)) \\
&\leq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq -\frac{1}{r} \frac{\langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle}{M^r m^r} \\
&\leq -\frac{1}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r m^r} \\
&\leq \frac{1}{4r} (M^r - m^r)(M^{-r} - m^{-r})
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\leq \frac{M^r - \langle A^r x, x \rangle}{M^r - m^r} \log m + \frac{\langle A^r x, x \rangle - m^r}{M^r - m^r} \log M - \log(M_0(A, x)) \\
&\leq \langle (M^r 1_H - A^r)(A^r - m^r 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq (M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_{s,t}}(t^r; m^r, M^r) \\
&\leq -\frac{1}{r} \frac{(M^r - \langle A^r x, x \rangle)(\langle A^r x, x \rangle - m^r)}{M^r m^r} \\
&\leq \frac{1}{4r} (M^r - m^r)(M^{-r} - m^{-r}).
\end{aligned}$$

If $r = 0$ and $s > 0$, then:

$$\begin{aligned}
& \text{(i)} \\
0 & \leq \frac{\log M - \langle \log Ax, x \rangle}{\log M - \log m} m^s + \frac{\langle \log Ax, x \rangle - \log m}{\log M - \log m} M^s - (M_s(A, x))^s \\
& \leq \langle (\log M 1_H - \log A)(\log A - \log m 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_s, t}(\log t; \log m, \log M) \\
& \leq s \frac{\langle (\log M 1_H - \log A)(\log A - \log m 1_H)x, x \rangle}{\log M - \log m} (M^s - m^s) \\
& \leq s \frac{(\log M - \langle \log Ax, x \rangle)(\langle \log Ax, x \rangle - \log m)}{\log M - \log m} (M^s - m^s) \\
& \leq \frac{s}{4} (M^s - m^s) \log \frac{M}{m}
\end{aligned}$$

$$\begin{aligned}
& \text{(ii)} \\
0 & \leq \frac{\log M - \langle \log Ax, x \rangle}{\log M - \log m} m^s + \frac{\langle \log Ax, x \rangle - \log m}{\log M - \log m} M^s - (M_s(A, x))^s \\
& \leq \langle (\log M 1_H - \log A)(\log A - \log m 1_H)x, x \rangle \sup_{t \in \langle m, M \rangle} \Psi_{\phi_s, t}(\log t; \log m, \log M) \\
& \leq (\log M - \langle \log Ax, x \rangle)(\langle \log Ax, x \rangle - \log m) \sup_{t \in \langle m, M \rangle} \Psi_{\phi_s, t}(\log t; \log m, \log M) \\
& \leq s \frac{(\log M - \langle \log Ax, x \rangle)(\langle \log Ax, x \rangle - \log m)}{\log M - \log m} (M^s - m^s) \\
& \leq \frac{s}{4} (M^s - m^s) \log \frac{M}{m}.
\end{aligned}$$

Proof. All the inequalities can be obtained directly from (3.4) and (3.5) by applying the same methods as in the proof of the previous theorem. \square

Remark 4.3. The following inequalities are obtained directly from Theorem 4.2:

$$(M^r - m^r)(M_s(A, x))^s - (M^s - m^s)(M_r(A, x))^r \leq M^r m^s - m^r M^s$$

if $0 < r < s$ or $r < 0 < s$. In case $r < s < 0$, the inequality is reversed. Similarly, for $r = 0$ we have:

$$(M_s(A, x))^s \log \frac{M}{m} - (M^s - m^s) \log(M_0(A, x)) \leq m^s \log M - M^s \log m$$

Remark 4.4. It is easy to see that $M_r(A, x) = (M_{-r}(A, x))^{-1}$ holds for every unit vector $x \in H$, selfadjoint operator A and $r \in \mathbb{R}$. Using that result, we can obtain analogue sequences of inequalities from Theorem 4.1 and Theorem 4.2 by replacing $A \leftrightarrow A^{-1}$, $-r \leftrightarrow s$ and $-s \leftrightarrow r$.

Remark 4.5. For more results concerning quasi-arithmetic mean and power mean, and order relations amongst them and arithmetic mean, see for instance [16] and [17].

5. IMPROVEMENTS

In this section we will give improvements of Theorem 2.1 and Theorem 2.4. In order to do so, we first need to state a result found in [9].

Lemma 5.1. ([9]) Let f be a convex function on D_f , $x, y \in D_f$ and $p, q \in [0, 1]$ such that $p + q = 1$. Then

$$\begin{aligned} \min\{p, q\} \left[f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right] &\leq pf(x) + qf(y) - f(px + qy) \\ &\leq \max\{p, q\} \left[f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right]. \end{aligned} \quad (5.1)$$

The following result is an improvement of Theorem 2.1.

Theorem 5.2. Let $A \in \mathcal{B}_n(H)$ be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I and $x \in H$ such that $\|x\| = 1$, then

$$\begin{aligned} 0 &\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \\ &\quad - \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f \\ &\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \frac{f'_-(M) - f'_+(m)}{M - m} - \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)) - \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f. \end{aligned} \quad (5.2)$$

We also have the inequalities

$$\begin{aligned} 0 &\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\ &\leq \frac{1}{4}(M - m)^2 \Psi_f(\langle Ax, x \rangle; m, M) - \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f \\ &\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m)) - \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f, \end{aligned} \quad (5.3)$$

where $\Psi_f(\cdot; m, M): \langle m, M \rangle \rightarrow \mathbb{R}$ is defined in (2.3) and

$$\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right), \quad \tilde{A} = \left| A - \frac{m+M}{2} 1_H \right|. \quad (5.4)$$

If f is concave, the inequalities are reversed.

Proof. First we assume that f is convex. Let $t \in [m, M]$. Since $\frac{t-m}{M-m}, \frac{M-t}{M-m} \in [0, 1]$ and $\frac{t-m}{M-m} + \frac{M-t}{M-m} = 1$, from (5.1) it follows that

$$\begin{aligned}
f(t) &= f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) \\
&\leq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\
&\quad - \min\left\{\frac{t-m}{M-m}, \frac{M-t}{M-m}\right\} \left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \\
&= \frac{Mf(m) - mf(M)}{M-m} + \frac{f(M) - f(m)}{M-m}t - \min\left\{\frac{t-m}{M-m}, \frac{M-t}{M-m}\right\} \delta_f \\
&= \frac{Mf(m) - mf(M)}{M-m} + \frac{f(M) - f(m)}{M-m}t - \left(\frac{1}{2} - \frac{1}{M-m} \left|t - \frac{m+M}{2}\right|\right) \delta_f
\end{aligned} \tag{5.5}$$

Due to property (1.2) we can replace t with the operator A in (5.5) and obtain:

$$\begin{aligned}
f(A) &\leq \frac{Mf(m) - mf(M)}{M-m} 1_H + \frac{f(M) - f(m)}{M-m} A \\
&\quad - \left(\frac{1}{2} 1_H - \frac{1}{M-m} \left|A - \frac{m+M}{2} 1_H\right|\right) \delta_f
\end{aligned} \tag{5.6}$$

Since scalar product is linear in the first argument, if we apply it on (5.6) for $x \in H$ such that $\|x\| = 1$ we get

$$\begin{aligned}
\langle f(A)x, x \rangle &\leq \frac{M - \langle Ax, x \rangle}{M-m} f(m) + \frac{\langle Ax, x \rangle - f(m)}{M-m} f(M) \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m} \left\langle \left|A - \frac{m+M}{2} 1_H\right| x, x \right\rangle\right) \delta_f.
\end{aligned} \tag{5.7}$$

The first inequality in (5.2) follows directly from the operator version of Jensen's inequality (1.3).

Now if we subtract $f(\langle Ax, x \rangle)$ from both sides of the inequality (5.7) we have

$$\begin{aligned}
&\langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\
&\leq \frac{M - \langle Ax, x \rangle}{M-m} f(m) + \frac{\langle Ax, x \rangle - m}{M-m} f(M) \\
&\quad - f(\langle Ax, x \rangle) - \left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle\right) \delta_f \\
&= \frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M-m} \left\{ \frac{f(M) - f(\langle Ax, x \rangle)}{M - \langle Ax, x \rangle} - \frac{f(\langle Ax, x \rangle) - f(m)}{\langle Ax, x \rangle - m} \right\} \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle\right) \delta_f \\
&= (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \Psi_f(\langle Ax, x \rangle; m, M) - \left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle\right) \delta_f \\
&\leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) - \left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle\right) \delta_f,
\end{aligned} \tag{5.8}$$

so we have proved the second inequality in (5.2).
 In the proof of Theorem 2.1 we have shown that

$$\sup_{t \in \langle m, M \rangle} \Psi_f(t; m, M) \leq \frac{f'_-(M) - f'_+(m)}{M - m},$$

which in combination with (5.8) proves the third inequality in (5.2). To prove the last inequality in (5.2), we notice that for every $t \in [m, M]$, the inequality $\frac{(M-t)(t-m)}{M-m} \leq \frac{1}{4}(M-m)$ is valid. Since $\langle Ax, x \rangle \in [m, M]$, we can replace $t \leftrightarrow \langle Ax, x \rangle$ and the proof is completed.

The proof of the inequalities (5.3) is clear from the proof of the inequalities (5.2). If f is concave, then $-f$ is convex, so we can apply (5.2) and (5.3) to function $-f$ and obtain the reversed inequalities. \square

Remark 5.3. The inequality (5.6) obtained in the proof of the previous theorem is a known result (see for instance [14]), and the proof is given for the sake of completeness.

Remark 5.4. In Theorem 5.2 with $|A|$ is denoted the absolute value of a Hilbert space operator A , that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint operator of A . Since A is a selfadjoint operator we have $A^* = A$, so

$$\begin{aligned} \tilde{A} &= \left| A - \frac{m+M}{2} 1_H \right| = \left(\left(A - \frac{m+M}{2} 1_H \right)^* \left(A - \frac{m+M}{2} 1_H \right) \right)^{\frac{1}{2}} \\ &= \left(\left(A - \frac{m+M}{2} 1_H \right) \left(A - \frac{m+M}{2} 1_H \right) \right)^{\frac{1}{2}} \\ &= \left(\left(A - \frac{m+M}{2} 1_H \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Remark 5.5. From $Sp(A) \subseteq [m, M]$, that is $m1_H \leq A \leq M1_H$, we easily obtain

$$-\frac{M-m}{2} 1_H \leq A - \frac{m+M}{2} 1_H \leq \frac{M-m}{2},$$

which is

$$\left(\left(A - \frac{m+M}{2} 1_H \right)^2 \right)^{\frac{1}{2}} \leq \frac{M-m}{2},$$

so we have $\tilde{A} \leq \frac{M-m}{2}$, and therefore $\langle \tilde{A}x, x \rangle \leq \frac{M-m}{2}$ for any $x \in H$ such that $\|x\| = 1$. Now,

$$\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle \geq \frac{1}{2} - \frac{1}{M-m} \frac{M-m}{2} = 0.$$

Since for a convex function f we have

$$\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) \geq 0,$$

the following inequality is valid

$$\left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle \right) \delta_f \geq 0. \quad (5.9)$$

Remark 5.6. Due to inequality (5.9), one can see that under the assumptions from the previous theorem, the inequality (5.7) which we state again

$$\begin{aligned} \langle f(A)x, x \rangle &\leq \frac{M - \langle Ax, x \rangle}{M - m} f(m) + \frac{\langle Ax, x \rangle - f(m)}{M - m} f(M) \\ &\quad - \left(\frac{1}{2} - \frac{1}{M - m} \left\langle \left| A - \frac{m + M}{2} 1_H \right| x, x \right\rangle \right) \delta_f \end{aligned}$$

is an improvement of the scalar Lah–Ribarić inequality for selfadjoint operators (1.5). We can see as well that Theorem 5.2 is really an improvement of Theorem 2.1 obtained in Section 2.

Our next result gives us an improvement of the lower bound from Theorem 2.4 and it is proved in a similar way as the previous theorem by using inequalities (5.7), (5.9) and the operator version of Jensen’s inequality (1.3).

Theorem 5.7. *Let the assumptions of Theorem 5.2 hold. If f is a continuous convex function on an interval of real numbers I such that $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I and $x \in H$ such that $\|x\| = 1$, then the following inequalities are valid*

(i)

$$\begin{aligned} 0 &\leq \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f \\ &\leq \frac{\langle Ax, x \rangle - m}{M - m} f(M) + \frac{M - \langle Ax, x \rangle}{M - m} f(m) - \langle f(A)x, x \rangle \\ &\leq \langle (M1_H - A)(A - m1_H)x, x \rangle \sup_{t \in (m, M)} \Psi_f(t; m, M) \\ &\leq \langle (M1_H - A)(A - m1_H)x, x \rangle \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq (M - \langle Ax, x \rangle) (\langle Ax, x \rangle - m) \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq \frac{1}{4} (M - m) (f'_-(M) - f'_+(m)) \end{aligned} \tag{5.10}$$

(ii)

$$\begin{aligned} 0 &\leq \left(\frac{1}{2} - \frac{1}{M - m} \langle \tilde{A}x, x \rangle \right) \delta_f \\ &\leq \frac{\langle Ax, x \rangle - m}{M - m} f(M) + \frac{M - \langle Ax, x \rangle}{M - m} f(m) - \langle f(A)x, x \rangle \\ &\leq \langle (M1_H - A)(A - m1_H)x, x \rangle \sup_{t \in (m, M)} \Psi_f(t; m, M) \\ &\leq (M - \langle Ax, x \rangle) (\langle Ax, x \rangle - m) \sup_{t \in (m, M)} \Psi_f(t; m, M) \\ &\leq (M - \langle Ax, x \rangle) (\langle Ax, x \rangle - m) \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq \frac{1}{4} (M - m) (f'_-(M) - f'_+(m)) \end{aligned} \tag{5.11}$$

where $\Psi_f(\cdot; m, M): \langle m, M \rangle \rightarrow \mathbb{R}$ is defined in (2.3) and \tilde{A} and δ_f are defined in (5.4). If f is concave, the inequalities are reversed.

Remark 5.8. One can see that by subtracting $\left(\frac{1}{2} - \frac{1}{M-m} \langle \tilde{A}x, x \rangle\right) \delta_f$ from all of the inequalities in (5.10) and (5.11) we can obtain an estimate for the difference between left and right side in inequality (5.7).

Theorem 5.2 and Theorem 5.7 can be applied to quasi-arithmetic means and power means in an analogous way as we did with Theorem 2.1 and Theorem 2.4. In that way we can get improvements of the results obtained in Sections 3 and 4.

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