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ABSOLUTE CONTINUITY OF POSITIVE LINEAR FUNCTIONALS

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Abstract. The goal of this paper is to present characterizations for absolute continuity of representable positive functionals on general *-algebras. From the results we give a new and very different proof to our recently published Lebesgue decomposition theorem for representable positive functionals. On unital C^*-algebras and measure algebras of compact groups further characterizations are included in the paper. As an application of our results, we answer Gudder's problem on the uniqueness of the Lebesgue decomposition in the case of commutative *-algebras and measure algebras of compact groups. Another application to faithful positive functionals defined on the latter *-algebras is also included.

1. Introduction and preliminaries

Many mathematicians investigated the Lebesgue type decompositions of nonnegative quadratic forms (forms, for short), positive functionals and positive operators. In a very general setting S. Hassi, Z. Sebestyén and H. de Snoo ([10]) studied forms on complex vector spaces. The author in a recent paper ([24]) proved results for representable forms on complex algebras. For positive functionals on a *-algebra (which naturally induce forms) H. Kosaki ([15]), M. Henle ([11]) and S. P. Gudder ([8]) gave noteworthy theorems on σ-finite von Neumann algebras, unital C^*-algebras and Banach *-algebras, respectively. The author proved strong

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connections between these theories in [23] and [24], and the latter article also contains a very general Lebesgue decomposition theorem for representable positive functionals on arbitrary *-algebras (Corollary 3.2). This statement sharpens the decomposition of Hassi, Sebestyén and de Snoo ([10, Theorems 2.11 and 3.9]) for representable positive functionals, and generalizes the similar results of Kosaki ([15, Theorem 3.5]) and Gudder ([8, Corollary 3]), since on unital Banach *-algebras every positive functional is representable. In the paragraph after Theorem 1.16 we will show that Henle's analogous theorem ([11, Theorem 1]) is also a consequence of our decomposition.

A common point of these settings is the two fundamental concepts of the decompositions, namely the singularity and the absolute continuity. For the preceding different terms and seemingly different definitions have appeared in the above mentioned papers. However, in Theorem 3 of [25] the author proved among others that all of these singularity concepts are equivalent for representable positive functionals on arbitrary *-algebras. Hence it is worthy to examine more closely the absolute continuity in the case of representable positive functionals, and to give a number of characterizations, along with corollaries and some applications. We will introduce the definitions and our motivations after some notations.

Notations. For a complex Hilbert space $\mathcal{H}$ the inner product will be denoted by $(\cdot|\cdot)$, and $\mathscr{B}(\mathcal{H})$ stands for the space of bounded linear operators on $\mathcal{H}$.

Let $\mathfrak{t}$ be a form on the complex vector space E. Then the $E \times E \to \mathbb{C}$ semi-inner product and $E \to \mathbb{R}_+$ seminorm uniquely determined by $\mathfrak{t}$ will be denoted by $(\cdot|\cdot)^\bullet_\mathfrak{t}$ and $\|\cdot\|^\bullet_\mathfrak{t}$, respectively. For simplicity we set $\mathfrak{t}[x] := (x|x)^\bullet_\mathfrak{t}$ $(x \in E)$. For the kernels $\ker \|\cdot\|^\bullet_\mathfrak{t}$ will stand, i. e., $\ker \|\cdot\|^\bullet_\mathfrak{t} := \{x \in E | \mathfrak{t}[x] = 0\}$. If $\mathfrak{w}$ is another form on E, then $\mathfrak{t} \leq \mathfrak{w}$ means that $\mathfrak{t}[x] \leq \mathfrak{w}[x]$ holds for all $x \in E$.

A form $\mathfrak{t}$ naturally induces an inner product on the quotient space $E/\ker \|\cdot\|^\bullet_\mathfrak{t}$, that is,

$$(x + \ker \|\cdot\|^\bullet_\mathfrak{t}, y + \ker \|\cdot\|^\bullet_\mathfrak{t}) \mapsto (x|y)^\bullet_\mathfrak{t} \qquad (x, y \in E). \tag{1.1}$$

The pair $(\mathcal{H}_\mathfrak{t}, (\cdot|\cdot)_\mathfrak{t})$ stands for the Hilbert space associated to the form $\mathfrak{t}$, i. e., the completion of the pre-Hilbert space $E/\ker \|\cdot\|^\bullet_\mathfrak{t}$ equipped with the inner product in (1.1).

If A is a *-algebra and f is a positive linear functional on A (i. e., $f(a^*a) \geq 0$ for all $a \in A$), then the gothic letter $\mathfrak{f}$ stands for the form generated by f, namely

$$\mathfrak{f}[a] := f(a^*a) \qquad (a \in A).$$

A positive functional f is said to be *faithful*, if for any $a \in A$ the equation $\mathfrak{f}[a] = 0$ implies $a = 0$. For positive functionals f and g on A, $f \leq g$ denotes the natural ordering, i. e., $\mathfrak{f} \leq \mathfrak{g}$. The notations A^* and $(A^*)_+$ stand for the set of linear functionals and positive linear functionals on A, respectively. If the algebra is normed, then A' denotes the space of continuous linear functionals on A.

A positive functional f is said to be *representable*, if there exists a cyclic *-representation $\pi : A \to \mathscr{B}(\mathcal{H})$ on a Hilbert space $(\mathcal{H}, (\cdot|\cdot))$ with cyclic vector $\xi \in \mathcal{H}$ (i. e., $\{\pi(a)\xi | a \in A\}$ is dense in $\mathcal{H}$) such that for any $a \in A$

$$f(a) = (\pi(a)\xi|\xi)$$

holds true. By the aid of the remarkable Gelfand-Naimark-Segal (GNS) construction there are equivalent conditions to this property (see [6], [17], [21]). The usual notations $(\mathcal{H}_f, (\cdot|\cdot)_f)$, π_f and ξ_f stand for the associated Hilbert space, *-representation and the distinguished cyclic vector from the GNS-construction, respectively. A positive functional f is *faithful*, if for every $a \in A$ the equation $f(a^*a) = 0$ implies that $a = 0$.

The notations A_{sa} and A_+ stand for the selfadjoint and positive elements of A, respectively. If A is unital, then the unit element of A will be denoted by $\mathbf{1}$.

For *-algebras, positive functionals and the GNS construction the reader is referred to [2], [3] and [18].

Our investigations focus on positive functionals, but in the first place we recall some fundamental definitions and theorems from the Lebesgue decomposition theory of forms (see [10]), since these facts are essential to understand our mathematical motivations. These will be discussed at the end of the section, along with our results and main ideas.

Definition 1.1 (Domination)**.** Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex vector space E. We say that $\mathfrak{w}$ *dominates* $\mathfrak{t}$ (or $\mathfrak{t}$ is *dominated by* $\mathfrak{w}$) if there exists an $\alpha \in \mathbb{R}_+$ such that $\mathfrak{t} \leq \alpha\mathfrak{w}$.

We introduce the concept of absolute continuity for forms (see [10, Section 2.5 and Theorem 3.8]).

Theorem 1.2. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex vector space* E*. The following statements are equivalent.*

(i) *For any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* E

$$\Big(\big(\lim_{n\to+\infty} \mathfrak{w}[x_n] = 0\big) \ \wedge \ \big(\lim_{n,m\to+\infty} \mathfrak{t}[x_n - x_m] = 0\big)\Big) \ \Rightarrow \ \lim_{n\to+\infty} \mathfrak{t}[x_n] = 0$$

is true.

(ii) *There exists an increasing sequence* $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ *of forms on* E *and there is a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}_+$ *such that for any* $n \in \mathbb{N}$ *the inequality* $\mathfrak{t}_n \leq \alpha_n\mathfrak{w}$ *holds and* $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$*.*

Definition 1.3 (Absolute continuity)**.** We say that the form $\mathfrak{t}$ is *absolutely continuous (or closable) with respect to* $\mathfrak{w}$, if one (hence all) of the properties of the previous theorem holds.

Remark 1.4. For (ii) of Theorem 1.2 the term $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$ was used in [10]. Furthermore, closability refers to that the densely defined

$$J_{\mathfrak{w},\mathfrak{t}} : \mathcal{H}_{\mathfrak{w}} \to \mathcal{H}_{\mathfrak{t}};\ x + \ker \|\cdot\|_{\mathfrak{w}}^{\bullet} \mapsto x + \ker \|\cdot\|_{\mathfrak{t}}^{\bullet}$$

linear operator is well-defined and closable ((i) of Theorem 1.2). We note that the well-definedness of the operator $J_{\mathfrak{w},\mathfrak{t}}$ is equivalent to the inclusion

$$\ker \|\cdot\|_{\mathfrak{w}}^{\bullet} \subseteq \ker \|\cdot\|_{\mathfrak{t}}^{\bullet}.$$

The following theorem (Proposition 2.2 in [10]) presents a concept which plays a central role in the Lebesgue decomposition of forms.

Theorem 1.5. *Let E be a complex vector space, and let $\mathfrak{t}, \mathfrak{w}$ be forms on E. Then the formula*

$$(\mathfrak{t} : \mathfrak{w})[x] := \inf\{\mathfrak{t}[x - z] + \mathfrak{w}[z] | z \in E\} \qquad (x \in E)$$

defines a form on E.

Definition 1.6. The form $(\mathfrak{t} : \mathfrak{w})$ is the *parallel sum* of the forms $\mathfrak{t}$ and $\mathfrak{w}$.

Remark 1.7. As Theorem 1.11 shows, the parallel sum of two forms is a very important concept in the Lebesgue decomposition theory of forms, since the decomposition itself is defined by means of the parallel sum. The first main theorem of this paper, which characterizes the absolute continuity of representable positive functionals (Theorem 2.15) extremely depends on a result about this concept (Corollary 2.13).

We recall here some properties of the parallel sum, which we will need later.

- $(\mathfrak{t} : \mathfrak{w}) = (\mathfrak{w} : \mathfrak{t})$;
- $(\mathfrak{t} : \mathfrak{w}) \leq \mathfrak{t}$;
- If $\mathfrak{p}$ is another form on E, then $\mathfrak{p} \leq \mathfrak{t}$ implies that $(\mathfrak{p} : \mathfrak{w}) \leq (\mathfrak{t} : \mathfrak{w})$.

For the proofs, other properties and some historical remarks the reader is referred to [10, Introduction and Lemma 2.3].

We introduce the concept of singularity through the next theorem ([10, Proposition 2.10]).

Theorem 1.8. *Let E be a complex vector space, $\mathfrak{t}$ and $\mathfrak{w}$ forms on E. The following statements are equivalent.*

(i) *For any form $\mathfrak{p}$ on E:*

$$(\mathfrak{p} \leq \mathfrak{w} \ \wedge \ \mathfrak{p} \leq \mathfrak{t}) \ \Rightarrow \ \mathfrak{p} = 0.$$

(ii) $(\mathfrak{t} : \mathfrak{w}) = 0$.

Definition 1.9 (Singularity)**.** We say that the forms $\mathfrak{t}$ and $\mathfrak{w}$ are *singular* (or either of them is *singular (with respect) to the other*), if one (hence all) of the properties of the previous theorem holds.

Remark 1.10. Let $\mathfrak{t}$, $\mathfrak{w}$ and $\mathfrak{p}$ be forms on the complex vector space E. Then one can easily check the following properties from the definitions.

- The definition of singularity is symmetric in $\mathfrak{t}$ and $\mathfrak{w}$.
- For any $\alpha_1, \alpha_2 \in \mathbb{R}^+$ the forms $\mathfrak{t}$ and $\mathfrak{w}$ are singular if and only if $\alpha_1\mathfrak{t}$ and $\alpha_2\mathfrak{w}$ are singular, furthermore $\mathfrak{t}$ is absolutely continuous with respect to $\mathfrak{w}$ if and only if $\alpha_1\mathfrak{t}$ is absolutely continuous with respect to $\alpha_2\mathfrak{w}$.
- If $\mathfrak{t}$ dominates $\mathfrak{p}$ and $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$, then $\mathfrak{p}$ and $\mathfrak{w}$ are also singular.
- The implications ($\mathfrak{w}$ *dominates* $\mathfrak{t}$) $\Rightarrow$ ($\mathfrak{t}$ *is absolutely continuous with respect to* $\mathfrak{w}$) $\Rightarrow$ ($\ker \|\cdot\|_{\mathfrak{w}}^{\bullet} \subseteq \ker \|\cdot\|_{\mathfrak{t}}^{\bullet}$) are true. The reversed implications are not true in general (see [8] for simple examples).
- If $\mathfrak{t}$ is absolutely continuous with respect to $\mathfrak{w}$ (in particular, if $\mathfrak{w}$ dominates $\mathfrak{t}$) and they are simultaneously singular, then $\mathfrak{t} = 0$.

In the following remarkable theorem we summarize the main achievements of Hassi, Sebestyén and de Snoo (Theorems 2.11, 3.6, 3.8 and 3.9 in [10], see also [22]).

Theorem 1.11 (Lebesgue decomposition of forms)**.** *Let $\mathfrak{t}, \mathfrak{w}$ be forms on the complex vector space E. Define a form $\mathfrak{t}_{reg}$ by the equation*

$$\mathfrak{t}_{reg}[x] := \sup_{n\in\mathbb{N}}(\mathfrak{t} : (n\mathfrak{w}))[x] \qquad (x \in E),$$

and let $\mathfrak{t}_{sing} := \mathfrak{t} - \mathfrak{t}_{reg}$. Then in the sum $\mathfrak{t} = \mathfrak{t}_{reg} + \mathfrak{t}_{sing}$ the form $\mathfrak{t}_{reg}$ is absolutely continuous with respect to $\mathfrak{w}$, and the forms $\mathfrak{t}_{sing}$ and $\mathfrak{w}$ are singular. Moreover, the form $\mathfrak{t}_{reg}$ is the greatest among all of the forms $\mathfrak{p}$ such that $\mathfrak{p} \leq \mathfrak{t}$ and $\mathfrak{p}$ is absolutely continuous with respect to $\mathfrak{w}$.

Definition 1.12. The forms $\mathfrak{t}_{reg}$ and $\mathfrak{t}_{sing}$ are called the *regular* and *singular part* of $\mathfrak{t}$, respectively. The sum $\mathfrak{t} = \mathfrak{t}_{reg} + \mathfrak{t}_{sing}$ is the *Lebesgue decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$.

We must note here that positive operators on Hilbert spaces also have a Lebesgue type decomposition ([1], [26]).

We will need the following properties of the regular and singular parts ([10, Theorem 3.6, Corollary 3.11, Proposition 3.13]).

Proposition 1.13. *With the notations of Theorem 1.11 the following hold.*

- *The forms $\mathfrak{t}_{sing}$ and $\mathfrak{w} + \mathfrak{t}_{reg}$ are singular (hence $\mathfrak{t}_{sing}$ and $\mathfrak{t}_{reg}$ are singular, as well).*
- *$\mathfrak{t}$ is absolutely continuous with respect to $\mathfrak{w} \Leftrightarrow \mathfrak{t} = \mathfrak{t}_{reg}$.*
- *$\mathfrak{t}$ is singular to $\mathfrak{w} \Leftrightarrow \mathfrak{t} = \mathfrak{t}_{sing}$.*

Now we turn to positive functionals on *-algebras. The definitions of the fundamental concepts are formulated by means of the induced forms.

Definition 1.14. Let A be a *-algebra, and let f, g be positive linear functionals on A. We say that f is

- dominated by g, if $\mathfrak{f}$ is dominated by $\mathfrak{g}$;
- absolutely continuous with respect to g, if $\mathfrak{f}$ is absolutely continuous with respect to $\mathfrak{g}$;
- singular to g, if $\mathfrak{f}$ is singular to $\mathfrak{g}$.

In the earlier mentioned papers ([8], [11], [15], [24]) for absolute continuity (Theorem 1.2 (i)) the same definitions have appeared, but different terms were used (strongly absolute continuity, almost domination, closability). The singularity concepts were seemingly different, but Theorem 3 in our paper [25] shows that all of the formulations appeared in [8], [11], [15], [24] are equivalent for representable positive functionals. In this paper the following formulations will be used ((*iii*) was called *semisingularity* in Gudder's paper [8]).

Theorem 1.15. *Let A be a *-algebra, and let f, g be representable positive functionals on A, ξ_f and ξ_g are the cyclic vectors in the respective GNS constructions. The following statements are equivalent.*

(i) *f and g are singular.*
(ii) *There exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that*

$$\big(\lim_{n\to+\infty}\pi_g(a_n)\xi_g=0\big)\ \wedge\ \big(\lim_{n\to+\infty}\pi_f(a_n)\xi_f=\xi_f\big),$$

that is,

$$\big(\lim_{n\to+\infty}a_n+\ker\|\cdot\|_g^\bullet=0\big)\ \wedge\ \big(\lim_{n\to+\infty}a_n+\ker\|\cdot\|_f^\bullet=\xi_f\big).$$

(iii) *There exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that for every $a\in A$*

$$\big(\lim_{n\to+\infty}\mathfrak{g}[a_n]=\lim_{n,m\to+\infty}\mathfrak{f}[a_n-a_m]=0\big)\ \wedge\ \big(\lim_{n\to+\infty}f(a_n^*a)=f(a)\big).$$

The following decomposition theorem is due to the author ([24, Theorem 3.1, Corollary 3.2]). It states that if we apply Theorem 1.11 to forms derived from representable positive functionals, then the regular and singular parts also come from positive functionals. The proof in [24] based on another result of ours for representable forms on complex algebras, wherein closed invariant subspaces of representations appeared ([24, Theorem 2.6]). We note that the classical Lebesgue decomposition related to set-measures is an immediately consequence of this theorem ([24, Theorem 4.2]).

Theorem 1.16 (Lebesgue decomposition of representable positive functionals)**.** *Let A be a *-algebra and let f,g be representable positive functionals on A. Then f admits a Lebesgue decomposition $f=f_{reg}+f_{sing}$ into a sum of representable positive functionals such that f_{sing} and g are singular, while f_{reg} is absolutely continuous with respect to g. The form induced by f_{reg} is*

$$\mathfrak{f}_{\mathfrak{reg}}[a]=\sup_{n\in\mathbb{N}}(\mathfrak{f}:(n\mathfrak{g}))[a]\qquad(a\in A).$$

Moreover, the form $\mathfrak{f}_{\mathfrak{reg}}$ is the greatest among all of the forms $\mathfrak{p}$ such that $\mathfrak{p}\le\mathfrak{f}$ and $\mathfrak{p}$ is absolutely continuous with respect to $\mathfrak{g}$.

The similar decompositions of Gudder (for positive functionals on unital Banach *-algebras, Corollary 3 in [8]) and Kosaki (for normal states on σ-finite von Neumann algebras, Theorem 3.5 in [15]) follow from our result ([24, Theorem 3.7, Corollary 3.17]). Furthermore, Henle's theorem for positive functionals on unital C^*-algebras ([11, Theorem 1]) is also a consequence of our statement. Indeed, if f and g are positive functionals on the unital C^*-algebra A, then the regular part f_{reg} in Theorem 1.16 is maximal among the absolutely continuous functionals equal or lower than f. But Henle also proved this for his decomposition in [11, page 90], thus Theorem 1.16 coincides with Henle's result on unital C^*-algebras.

Natural questions arise in the context of the absolute continuity of positive functionals. As we have seen in Theorems 1.8 and 1.11 the concept of the parallel sum is very important, but it is not very clear whether or not the parallel sum of two forms derived from positive functionals is induced by a positive functional. Theorem 1.16 shows that the regular part (which is a supremum of parallel sums) in the Lebesgue decomposition for representable positive functionals is a form of this kind. But in Corollary 2.13 we will prove that the parallel sum of two forms derived from a positive functional is always induced by a representable positive functional (the main tool in the proof is Theorem 2.4, which is classical statement in the context of C^*-algebras). This result has two consequences. Absolute continuity can be characterized via dominated increasing sequences of representable positive functionals (Theorem 2.15), just like in Theorem 1.2 (i) (in this statement only forms appeared). This is the main result in Section 2. From this we give a new proof to our decomposition Theorem 1.16, which is very different from the one appeared in our article ([24, Corollary 3.2]). We note that this is a generalization of the decompositions mentioned above, and that the classical set-measure Lebesgue decomposition is an immediately consequence of this theorem ([24, Section 4]).

We study the case of two concrete very important *-algebras, namely unital C^*-algebras and measure algebras of compact groups. Following R. van Handel's idea ([9]) for the preceding, we investigate the connections between the positive functionals defined on the algebra and the probability Radon measures on the algebra's state space. Proposition 4.1 in [9] is a remarkable characterization of absolute continuity via unital commutative C^*-algebras, which was used for quantum statistical investigations. However, the proof uses a nontrivial result of Gudder ([8]), and a very deep theorem from Sakai's work ([20, 3.1]). With the aid of our characterization (Theorem 2.15) this can be proved easier (Theorem 3.8), moreover domination also can be characterized in a similar way (Theorem 3.7). These are the main results in Section 3. The proofs lie on an extension theorem of Ky Fan ([5]) and on a proposition of ours (Lemma 3.3), which shows the strong connection between the concepts (absolute continuity and singularity) of the functional decomposition theory and the set-measure decomposition. We note that a characterization of singularity via Choquet theory can be found in our paper [25, Theorem 7]. For positive functionals on measure algebras of compact groups we will prove that a weak assumption automatically implies absolute continuity (Theorem 4.3, the main result in Section 4), namely the well-definedness of the operator in Remark 1.4. The key tool is the well-known fact that an irreducible representation of the measure algebra of a compact group is always finite dimensional (Lemma 4.1)

We mentioned that all of the Lebesgue decompositions for positive functionals above coincide with the same settings, it is natural to ask whether or not it is unique. In the penultimate section of the paper we discuss this question, and as an application of our results (Lemma 3.3 and Theorem 4.3) we will prove that the decomposition is unique on commutative *-algebras and measure algebras of compact groups (Theorems 5.13 and 5.15).

From the characterization in Section 4 we prove statements on faithful representations of the measure algebra of a compact group.

2. Characterizations of absolute continuity, general *-algebras

The next three sections of the paper introduce characterizations on the absolute continuity of positive linear functionals defined on three different classes of *-algebras. More precisely, the three cases are the following. The first section deals with representable positive functionals on a general *-algebra, including a result similar to Theorem 1.2 (ii) in the context of representable positive functionals (Theorem 2.15), furthermore we examine the parallel sum of forms derived from this kind of functionals (Corollary 2.13). As a consequence, we gain a new proof to our Lebesgue decomposition, Theorem 1.16 (see Corollary 2.17).

The second and the third sections focus on positive functionals on unital C^*-algebras and measure algebras of compact groups. In the first case we characterize domination and absolute continuity via positive functionals on unital *commutative* C^*-algebras (Theorems 3.7 and 3.8). For measure algebras of compact groups we will prove that for absolute continuity it is sufficient that the operator defined in Remark 1.4 is well-defined (closability is a consequence, Theorem 4.3). Some of these results are necessary in the investigations of the uniqueness of the Lebesgue decomposition.

Before we begin to study the absolute continuity of representable positive functionals, we recall some well-known facts from the theories of C^*-algebras and representable positive functionals on *-algebras.

Theorem 2.1. *Let A be a *-algebra and let f be a positive functional on A. Then the following statements are equivalent (*[17, Theorem 9.4.15] *and* [21, Theorem 1]*).*

(i) *f is representable.*
(ii) (ii)' *There exists an $m \in \mathbb{R}_+$ such that for any $a \in A$ the inequality $|f(a)|^2 \le mf(a^*a)$ holds.*
 (ii)" *There exists a $p : A \to \mathbb{C}$ C^*-seminorm and $M \in \mathbb{R}_+$ such that for any $a \in A$ the inequality $|f(a)| \le Mp(a)$ holds.*
 (In fact, $(ii)''$ is sufficient).

Remark 2.2. For a positive functional f on the *-algebra A define the *Hilbert bound* ([17, Definition 9.4.2]) of f by

$$\|f\|_H := \sup\{|f(a)|^2 | a \in A \ \wedge \ f(a^*a) \le 1\}.$$

Point $(ii)''$ in the previous theorem shows that the Hilbert bound of a representable positive functional is finite.

Theorem 2.3. *Let A be a *-algebra and let f, g be positive functionals on A (cf.* [17, Proposition 9.4.22]*).*

(a) *If f and g are representable, then $f + g$ is also representable.*
(b) *If g is representable, $f \le g$ and $\|f\|_H < +\infty$, then f and $g - f$ are also representable, moreover the inequality $\max\{\|f\|_H, \|g-f\|_H\} \le \|g\|_H$ holds true.*

Theorem 2.4. *Let A be a C^*-algebra. Then the following statements are true.*

(a) *There exists an increasing net $(e_i)_{i\in I}$ in A_+ such that $\sup_{i\in I} \|e_i\| \le 1$ and for any $a \in A$*
$$\lim_{i,I} ae_i = a = \lim_{i,I} e_i a$$
holds, i. e., A admits an increasing approximate identity in the positive elements with norm bound one ([18, Theorem 1.4.2]*).*

(b) *A linear functional on A is positive $\Leftrightarrow$ f is continuous, moreover the equation $\|f\| = \lim_{i,I} f(e_i)$ holds for a net $(e_i)_{i\in I}$ with the properties in (a) (*[18, Proposition 3.1.4]*).*

(c) *If f is a positive linear functional on A, then f is representable (see* [18, Theorem 3.3.3]*), hence it is selfadjoint (i. e., $f(a^*) = \overline{f(a)}$ for all $a \in A$).*

Since a positive functional on a Banach *-algebra with a bounded approximate identity is representable ([17, Theorem 11.3.7]), the following lemma is a special case of [3, 2.2.10] and [4, Lemma 2].

Lemma 2.5. *Let A be a Banach *-algebra with isometric involution, and assume that A admits an approximate identity $(e_i)_{i\in I}$ with norm bound 1. Let f be a positive functional on A. Then the net $(\pi_f(e_i))_{i\in I}$ converges to the identity operator $id_{\mathcal{H}_f}$ in the strong operator topology. In particular,*

$$\lim_{i,I} \pi_f(e_i)\xi_f = \lim_{i,I}(e_i + \ker\|\cdot\|_f^\bullet) = \xi_f$$

holds.

The main goal is to give an equivalent condition for the absolute continuity among representable positive functionals similar to the form case in Theorem 1.2 (ii). Our idea will be the following. As we mentioned at the Introduction and preliminaries, a form $\mathfrak{t}$ is absolutely continuous with respect to an other form $\mathfrak{w}$ if and only if $\mathfrak{t} = \mathfrak{t}_{reg}$ holds in the Lebesgue decomposition with respect to $\mathfrak{w}$ by Theorem 1.11 (see Proposition 1.13). But, as Theorem 1.11 states that, the regular part is a supremum of parallel sums strongly connected to the forms $\mathfrak{t}$ and $\mathfrak{w}$. Hence we "only" have to show that the parallel sum of two representable positive functionals is derived from a representable positive functional. Therein the following concept plays a key role, which can be found in our recent paper [24, Section 2].

Definition 2.6. Let A be a complex algebra. A form $\mathfrak{t} : A \to \mathbb{R}_+$ is *representable* if

$$(\forall a \in A)\ (\exists \lambda_a \in [0, +\infty[)\ (\forall b \in A) : \mathfrak{t}[ab] \le \lambda_a \mathfrak{t}[b],$$

that is, for any $a \in A$ the left multiplication operator

$$L_a : A/\ker\|\cdot\|_{\mathfrak{t}}^\bullet \to A/\ker\|\cdot\|_{\mathfrak{t}}^\bullet;\ L_a(b + \ker\|\cdot\|_{\mathfrak{t}}^\bullet) := ab + \ker\|\cdot\|_{\mathfrak{t}}^\bullet$$

is well-defined and continuous with respect to the Hilbert norm $\|\cdot\|_{\mathfrak{t}}$.

If $a \in A$, then denote by $\pi_{\mathfrak{t}}(a)$ the unique continuous linear extension of L_a to $\mathcal{H}_{\mathfrak{t}}$.

Proposition 2.7. *The mapping $\pi_{\mathfrak{t}} : A \to \mathscr{B}(\mathcal{H}_{\mathfrak{t}})$; $a \mapsto \pi_{\mathfrak{t}}(a)$ is an algebra representation on the Hilbert space $\mathcal{H}_{\mathfrak{t}}$, what we call the induced representation by $\mathfrak{t}$.*

The next result for representable forms is also from [24, Lemma 2.4].

Lemma 2.8. *Let A be a complex algebra and let $\mathfrak{t}, \mathfrak{w}$ be representable forms on A. Then the parallel sum $(\mathfrak{t} : \mathfrak{w})$ is also a representable form, moreover for any $a \in A$ the inequality*

$$\|\pi_{(\mathfrak{t}:\mathfrak{w})}(a)\| \le \max\{\|\pi_{\mathfrak{t}}(a)\|, \|\pi_{\mathfrak{w}}(a)\|\}$$

holds.

As an immediately corollary we gain the following fact for the parallel sum of two representable positive functionals.

Lemma 2.9. *Let A be a *-algebra, let f and g be representable positive functionals on A. Then the forms $\mathfrak{f}$ and $\mathfrak{g}$ are representable, moreover $\pi_{\mathfrak{f}} = \pi_f$ and $\pi_{\mathfrak{g}} = \pi_g$ hold. Thus the parallel sum $(\mathfrak{f} : \mathfrak{g})$ is a representable form, and for any $a \in A$ we have*

$$\|\pi_{(\mathfrak{f}:\mathfrak{g})}(a)\| \leq \max\{\|\pi_f(a)\|, \|\pi_g(a)\|\}. \tag{2.1}$$

Proof. Since f and g are representable positive functionals, then for every $a, b \in A$ we get

$$(\mathfrak{f}[ab] \leq \|\pi_f(a)\|^2 \mathfrak{f}[b]) \ \wedge \ (\mathfrak{g}[ab] \leq \|\pi_g(a)\|^2 \mathfrak{g}[b]),$$

i. e. the forms $\mathfrak{f}$ and $\mathfrak{g}$ are representable. The equality of the morphisms follows from the definitions, and Lemma 2.8 implies the representability of $(\mathfrak{f} : \mathfrak{g})$ and the estimate (2.1). □

Thus $\pi_{(\mathfrak{f}:\mathfrak{g})}$ is an algebra-representation of A, but its *-preserving and cyclicity are far from the evidence. To see these properties we will prove two lemmas.

Our first lemma for normed *-algebras characterizes the three fundamental concepts of the Lebesgue decomposition theory by means of dense *-subalgebras. We also use it in the subsequent parts of the paper (Lemmas 3.3 and 4.1).

Lemma 2.10. *Let A be a normed *-algebra with continuous involution. Let $\alpha \in \mathbb{R}_+$ be an arbitrary number. If B is a dense *-subalgebra of A, f and g are continuous positive functionals on A, then*

(a) *f is absolutely continuous with respect to $g \Leftrightarrow f|_B$ is absolutely continuous with respect to $g|_B$.*
(b) *f is singular to $g \Leftrightarrow f|_B$ is singular to $g|_B$.*
(c) *$f \leq \alpha g \Leftrightarrow f|_B \leq \alpha g|_B$.*

Proof. (*a*): $\Rightarrow$: It is clear from Definition 1.3.

$\Leftarrow$: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A such that

$$(\lim_{n\to+\infty} \mathfrak{g}[a_n] = 0) \ \wedge \ (\lim_{n,m\to+\infty} \mathfrak{f}[a_n - a_m] = 0).$$

Since B is dense in A, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in B such that $\|b_n - a_n\| \to 0$. From this we have that

$$\mathfrak{g}[a_n - b_n] \leq \|g\| \|(a_n - b_n)^*(a_n - b_n)\| \to 0$$

and that

$$\mathfrak{f}[a_n - b_n] \leq \|f\| \|(a_n - b_n)^*(a_n - b_n)\| \to 0, \tag{2.2}$$

so

$$(\lim_{n\to+\infty} (\mathfrak{g}|_{\mathfrak{B}})[b_n] = 0) \ \wedge \ (\lim_{n,m\to+\infty} (\mathfrak{f}|_{\mathfrak{B}})[b_n - b_m] = 0)$$

hold true. Thus the absolute continuity of $f|_B$ implies $(\mathfrak{f}|_B)[b_n] \to 0$, hence by (2.2)

$$\lim_{n\to+\infty} \mathfrak{f}[a_n] = 0$$

follows.

(b): $\Rightarrow$: We will use Theorem 1.15 (iii). Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A such that

$$(\lim_{n\to+\infty} \mathfrak{g}[a_n] = 0 \ \wedge \ \lim_{n,m\to+\infty} \mathfrak{f}[a_n - a_m] = 0) \ \wedge \ (\forall a \in A : \ \lim_{n\to+\infty} f(a_n^* a) = f(a))$$

Since B is dense in A, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in A with the property $\|a_n - b_n\| \to 0$. We state that

$$(\lim_{n\to+\infty} \mathfrak{g}[b_n] = 0 \ \wedge \ \lim_{n,m\to+\infty} \mathfrak{f}[b_n - b_m] = 0) \ \wedge \ (\forall b \in \ B : \ \lim_{n\to+\infty} f(b_n^* b) = f(b)),$$

i. e., the desired result comes true.

Let $n \in \mathbb{N}$ be arbitrary. If $b \in B$, then

$$|f(a_n^* b) - f(b_n^* b)| = |f((a_n - b_n)^* b)| \leq \|f\| \|(a_n - b_n)^* b\| \to 0,$$

since f and the involution are continuous. From this it follows that $f(b_n^* b) \to f(b)$.

Since g determines a semi-inner product, then for any $a, b \in A$ we infer that

$$|\sqrt{g(a^* a)} - \sqrt{g(b^* b)}| \leq \sqrt{g((a-b)^*(a-b))} \leq \sqrt{\|g\|}\sqrt{\|(a-b)^*(a-b)\|}$$

Thus, if $n \in \mathbb{N}$, then

$$|\sqrt{g(a_n^* a_n)} - \sqrt{g(b_n^* b_n)}| \leq \sqrt{\|g\|}\sqrt{\|(a_n - b_n)^*(a_n - b_n)\|} \to 0,$$

that is $g(b_n^* b_n) \to 0$. The equality $\lim_{n,m\to+\infty} \mathfrak{f}[b_n - b_m] = 0$ follows from the very same argument.

$\Leftarrow$: Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in B such that

$$\begin{aligned} &(\lim_{n\to+\infty} \mathfrak{g}[b_n] = 0 \ \wedge \ \lim_{n,m\to+\infty} \mathfrak{f}[b_n - b_m] = 0) \ \textit{and} \\ &(\forall b \in \ B : \ \lim_{n\to+\infty} f(b_n^* b) = f(b)). \end{aligned} \tag{2.3}$$

We state that

$$(\lim_{n\to+\infty} \mathfrak{g}[b_n] = 0 \ \wedge \ \lim_{n,m\to+\infty} \mathfrak{f}[b_n - b_m] = 0) \ \wedge \ (\forall a \in \ A : \ \lim_{n\to+\infty} f(b_n^* a) = f(a)),$$

i. e., the desired results come true. We only need the last statement. Let $\varepsilon > 0$ be an arbitrary number. From the second property in (2.3) we conclude the existence of a number $M \geq 0$ such that for every $n \in \mathbb{N}$ $\sqrt{\mathfrak{f}[b_n]} \leq M$ holds true. If $a \in A$, then for ε from the density of B and from the continuity of the involution and f we may fix an element $b \in B$ such that

$$\max\{\|a - b\|, \sqrt{\|(a-b)^*(a-b)\|}\} \leq \frac{1}{3}\min\{\frac{\varepsilon}{M\sqrt{\|f\|}+1}, \frac{\varepsilon}{\|f\|+1}\}.$$

From the third property in (2.3) there is an $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$ the inequality

$$|f(b_n^* b) - f(b)| \leq \frac{\varepsilon}{3}$$

is true. Hence for $n \in \mathbb{N}$, $n \geq n_0$ from the continuity of f and the Cauchy-Schwarz inequality we conclude that

$$\begin{aligned}|f(b_n^*a) - f(a)| &= |f(b_n^*a) - f(b_n^*b) + f(b_n^*b) - f(b) + f(b) - f(a)| \\ &\leq \sqrt{\mathfrak{f}[b_n]}\sqrt{\mathfrak{f}[a-b]} + \frac{\varepsilon}{3} + \|f\|\|a-b\| \\ &\leq M\sqrt{\|f\|}\sqrt{\|(a-b)^*(a-b)\|} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon,\end{aligned}$$

i. e., $f(b_n^*a) \to f(a)$.

(c): It is obvious from the continuity. □

Our second lemma will be very important in the investigations of the parallel sum.

Lemma 2.11. *Let A be a C^*-algebra, $\mathcal{H}$ Hilbert space and let $\pi : A \to \mathscr{B}(\mathcal{H})$ be a continuous algebra-representation such that $\|\pi\| \leq 1$. Assume that π is nondegenerate, that is, the set $\{\pi(a)\eta | (a \in A) \ \wedge \ (\eta \in \mathcal{H})\}$ is dense in $\mathcal{H}$. Then π preserves the involution, i. e., π is a *-representation of the *-algebra A.*

Proof. First we show that for any $b \in A$ and $\eta \in \mathcal{H}$ the linear functional

$$h_{b,\eta} : A \to \mathbb{C};\ a \mapsto (\pi(a)(\pi(b)\eta)|\pi(b)\eta)$$

is positive. By Theorem 2.4 (a) there is an increasing approximate identity in $A_+ \cap \{a \in A | \|a\| \leq 1\}$. Moreover, by Theorem 2.4 (b) a linear functional f on A is positive if and only if it is continuous and $\|f\| = \lim_{i,I} f(e_i)$ holds for an approximate identity with the properties above.

Let $(e_i)_{i\in I}$ be an approximate identity with these properties and let $b \in A$, $\eta \in \mathcal{H}$ be fixed elements. By the Cauchy-Schwarz inequality and the assumption for π we have that for every $a \in A$

$$|h_{b,\eta}(a)| = |(\pi(a)(\pi(b)\eta)|\pi(b)\eta)| \leq \|a\|\|\pi(b)\eta\|^2,$$

hence $h_{b,\eta}$ is continuous and its norm is lower or equal to $\|\pi(b)\eta\|^2$. On the other hand, from the continuity of π we have

$$\begin{aligned}\|\pi(b)\eta\|^2 &= (\pi(b)\eta|\pi(b)\eta) \\ &= \lim_{i,I}(\pi(e_ib)\eta|\pi(b)\eta) = \lim_{i,I}(\pi(e_i)(\pi(b)\eta)|\pi(b)\eta) = \lim_{i,I} h_{b,\eta}(e_i),\end{aligned}$$

thus $\|e_i\| \leq 1$ $(i \in I)$ implies that

$$\|h_{b,\eta}\| = \|\pi(b)\eta\|^2 = \lim_{i,I} h_{b,\eta}(e_i),$$

hence $h_{b,\eta}$ is a positive linear functional.

By Theorem 2.4 (c) we infer that $h_{b,\eta}$ is self-adjoint. From this we show that $\pi(a^*) = \pi(a)^*$ holds for any $a \in A$:

$$\begin{aligned}(\pi(a^*)(\pi(b)\eta)|\pi(b)\eta) &= h_{b,\eta}(a^*) = \overline{h_{b,\eta}(a)} = (\pi(b)\eta|\pi(a)(\pi(b)\eta)) \\ &= (\pi(a)^*(\pi(b)\eta)|\pi(b)\eta),\end{aligned}$$

that is $((\pi(a^*) - \pi(a)^*)(\pi(b)\eta)|\pi(b)\eta) = 0$ holds true for any $b \in A$ and $\eta \in \mathcal{H}$. But π is a nondegenerate algebra-representation, hence for any $\eta \in \mathcal{H}$ the equation $((\pi(a^*) - \pi(a)^*)\eta|\eta) = 0$ holds, which implies $\pi(a^*) = \pi(a)^*$. □

Later on we will use often the following important construction.

Remark 2.12. Let A be a *-algebra and let f be a representable positive functional on A. Then by Theorem 2.1 there exist a $p : A \to \mathbb{R}_+$ C^*-seminorm and $M \in \mathbb{R}_+$ such that

$$|f(a)| \leq Mp(a) \tag{2.4}$$

holds for any $a \in A$. Denote by $(B, \|\cdot\|_p)$ the completion of the pre-C^*-algebra $A/\ker p$ (equipped with the factor-norm derived from p). Furthermore let

$$f_0' : A/\ker p \to \mathbb{C};\ \ f_0'(a + \ker p) := f(a). \tag{2.5}$$

By (2.4) and (2.5) this mapping is a well-defined and continuous (with respect to $\|\cdot\|_p$) positive linear functional on a dense subspace of B. Hence there exists a unique $f' : B \to \mathbb{C}$ (continuous and representable) positive linear functional which extends f_0'.

Now we are in position to prove that the parallel sum of representable positive functionals is *always* induced by a representable positive functional.

Corollary 2.13. *Let A be a *-algebra and let f, g be representable positive functionals on A. Then there exists a representable positive functional h on A such that for every $a \in A$ the equation $h(a^*a) = (\mathfrak{f} : \mathfrak{g})[a]$ holds.*

Proof. (I): First assume that A to be a C^*-algebra. By Lemma 2.9 $(\mathfrak{f} : \mathfrak{g})$ is a representable form on A, and for every $a \in A$ we have

$$\|\pi_{(\mathfrak{f}:\mathfrak{g})}(a)\| \leq \max\{\|\pi_f(a)\|, \|\pi_g(a)\|\}.$$

Since the domains of π_f and π_g is a C^*-algebra and the ranges of both *-representation is included in a C^*-algebra, from a well-known Theorem ([3, Proposition on page 9]) $\|\pi_{(\mathfrak{f}:\mathfrak{g})}\| \leq 1$ holds. We show that the algebra-representation $\pi_{(\mathfrak{f}:\mathfrak{g})}$ is nondegenerate, thus the previous Lemma implies that $\pi_{(\mathfrak{f}:\mathfrak{g})}$ is also a *-representation. To see this, it is enough to prove that the set $\{\pi_{(\mathfrak{f}:\mathfrak{g})}(a)(b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})}) | a, b \in A\}$ is dense in $\{b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})} | b \in A\}$, since it is a dense subspace of $\mathcal{H}_{(\mathfrak{f}:\mathfrak{g})}$.

Let $b \in A$ arbitrary and fix an $(e_i)_{i\in I}$ approximate identity in A by Theorem 2.4 (a). By Remark 1.7 and the C^*-algebra properties for any $i \in I$ we conclude

$$\begin{aligned}&\|\pi_{(\mathfrak{f}:\mathfrak{g})}(e_i)(b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})}) - (b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = (\mathfrak{f} : \mathfrak{g})[e_i b - b]\\ &\leq \mathfrak{f}[e_i b - b] = f((e_i b - b)^*(e_i b - b)) \leq \|f\|\|e_i b - b\|^2.\end{aligned}$$

Since $\lim_{i,I} \|e_i b - b\|^2 = 0$, then

$$\lim_{i,I} \|\pi_{(\mathfrak{f}:\mathfrak{g})}(e_i)(b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})}) - (b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = 0,$$

hence $\pi_{(\mathfrak{f}:\mathfrak{g})}$ is nondegenerate, thus $\pi_{(\mathfrak{f}:\mathfrak{g})}$ is a *-representation of A.

We note here that from similar argument it follows that

$$\lim_{i,I} \|\pi_{(\mathfrak{f}:\mathfrak{g})}(b)(e_i + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})}) - (b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = 0, \tag{2.6}$$

since for $i \in I$

$$\begin{aligned}&\|\pi_{(\mathfrak{f}:\mathfrak{g})}(b)(e_i + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})}) - (b + \ker \|\cdot\|^\bullet_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = (\mathfrak{f} : \mathfrak{g})[be_i - b]\\ &\leq \mathfrak{f}[be_i - b] = \|(be_i - b) + \ker \|\cdot\|^\bullet_f\|^2_f = f((be_i - b)^*(be_i - b)) \leq \|f\|\|be_i - b\|^2\end{aligned}$$

holds, thus $\lim_{i,I}\|be_i - b\|^2 = 0$ implies (2.6).

We prove the existence of a vector $\xi_{(\mathfrak{f}:\mathfrak{g})} \in \mathcal{H}_{(\mathfrak{f}:\mathfrak{g})}$ such that for every $a \in A$

$$(\mathfrak{f}:\mathfrak{g})[a] = (\pi_{(\mathfrak{f}:\mathfrak{g})}(a^*a)\xi_{(\mathfrak{f}:\mathfrak{g})}|\xi_{(\mathfrak{f}:\mathfrak{g})})_{(\mathfrak{f}:\mathfrak{g})}$$

holds true, hence by the *-preserving the mapping

$$h : A \to \mathbb{C};\ a \mapsto (\pi_{(\mathfrak{f}:\mathfrak{g})}(a)\xi_{(\mathfrak{f}:\mathfrak{g})}|\xi_{(\mathfrak{f}:\mathfrak{g})})_{(\mathfrak{f}:\mathfrak{g})}$$

is a positive functional on A, moreover the parallel sum of f and g is derived from h (and, of course, the distinguished cyclic vector for π_h is $\xi_{(\mathfrak{f}:\mathfrak{g})}$).

First we show that the net $(e_i + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})})_{i\in I}$ is Cauchy in the Hilbert space $\mathcal{H}_{(\mathfrak{f}:\mathfrak{g})}$, hence convergent and $\lim_{i,I}(e_i + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})}) =: \xi_{(\mathfrak{f}:\mathfrak{g})}$ is the desired vector. By Lemma 2.5 $(\pi_f(e_i)\xi_f)_{i\in I} = (e_i + \ker\|\cdot\|^{\bullet}_f)_{i\in I}$ is a Cauchy net. Since for any $i, j \in I$

$$\|(e_i + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})}) - (e_j + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = (\mathfrak{f}:\mathfrak{g})[e_i - e_j] \le \mathfrak{f}[e_i - e_j]$$

holds, from this it follows that $(e_i + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})})_{i\in I}$ is also a Cauchy net, hence convergent. Denote by $\xi_{(\mathfrak{f}:\mathfrak{g})}$ the limit, moreover let

$$h : A \to \mathbb{C};\ a \mapsto (\pi_{(\mathfrak{f}:\mathfrak{g})}(a)\xi_{(\mathfrak{f}:\mathfrak{g})}|\xi_{(\mathfrak{f}:\mathfrak{g})})_{(\mathfrak{f}:\mathfrak{g})}.$$

We show that for this positive functional $h(a^*a) = (\mathfrak{f}:\mathfrak{g})[a]$ is true for arbitrary $a \in A$. Indeed, the properties of $\pi_{(\mathfrak{f}:\mathfrak{g})}$, the continuity of the operators $\pi_{(\mathfrak{f}:\mathfrak{g})}(a)$ and (2.6) imply

$$\begin{aligned} h(a^*a) &= (\pi_{(\mathfrak{f}:\mathfrak{g})}(a^*a)\xi_{(\mathfrak{f}:\mathfrak{g})}|\xi_{(\mathfrak{f}:\mathfrak{g})})_{(\mathfrak{f}:\mathfrak{g})} = \|\pi_{(\mathfrak{f}:\mathfrak{g})}(a)\xi_{(\mathfrak{f}:\mathfrak{g})}\|^2_{(\mathfrak{f}:\mathfrak{g})} \\ &= \lim_{i,I}\|\pi_{(\mathfrak{f}:\mathfrak{g})}(a)(e_i + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})})\|^2_{(\mathfrak{f}:\mathfrak{g})} = \|a + \ker\|\cdot\|^{\bullet}_{(\mathfrak{f}:\mathfrak{g})}\|^2_{(\mathfrak{f}:\mathfrak{g})} = (\mathfrak{f}:\mathfrak{g})[a]. \end{aligned}$$

We finished that part when A is a C^*-algebra.

(II): Let A be an arbitrary *-algebra, and let f, g be representable positive functionals on A. By Theorem 2.1 there exist C^*-seminorms $p_f, p_g : A \to \mathbb{R}_+$ such that f (resp. g) is continuous with respect to p_f (resp. p_g). From this it follows that there exists an $M \in \mathbb{R}_+$ such that for the C^*-seminorm $p := \sup\{p_f, p_g\}$ and for every $a \in A$

$$(|f(a)| \le Mp(a)) \ \wedge\ (|g(a)| \le Mp(a))$$

holds true. Denote by $(B, \|\cdot\|_p)$ the completion of the pre-C^*-algebra $A/\ker p$ (equipped with the factor-norm derived from p). Furthermore let

$$f_0', g_0' : A/\ker p \to \mathbb{C};\ f_0'(a + \ker p) := f(a),\ g_0'(a + \ker p) := g(a).$$

From Remark 2.12 these mappings are continuous positive functionals; denote by f' and g' the unique continuous positive linear extensions to B.

By (I) there exists an h' positive linear functional on B such that for any $b \in B$ $h'(b^*b) = (\mathfrak{f}':\mathfrak{g}')[b]$ is true. We state that the mapping

$$h : A \to \mathbb{C};\ h(a) := h'(a + \ker p)$$

is a well-defined representable positive functional on A, furthermore the equation $h(a^*a) = (\mathfrak{f}:\mathfrak{g})[a]$ holds for any $a \in A$.

From the definition it is obvious that h is a well-defined positive functional. From Theorem 2.4 (c) and (b) we conclude that h' continuous and representable on B, hence for every $a \in A$ we infer

$$|h(a)| = |h'(a + \ker p)| \leq \|h'\| \|a + \ker p\|_p = \|h'\| p(a),$$

that is, h is continuous with respect to the C^*-seminorm p. Moreover, the representability of h' implies the existence of an $m_{h'} \in \mathbb{R}_+$ such that for any $b \in B$ the inequality $|h'(b)|^2 \leq m_{h'} h'(b^*b)$ holds, hence for arbitrary $a \in A$ we conclude

$$|h(a)|^2 = |h'(a + \ker p)|^2 \leq m_{h'} h'(a^*a + \ker p) = m_{h'} h(a^*a),$$

thus by Theorem 2.1 we have that h is representable.

In the end let $a \in A$ be an arbitrary element. Then

$$\begin{aligned}
&(\mathfrak{f} : \mathfrak{g})[a] \\
&= \inf\{f((a-c)^*(a-c)) + g(c^*c) | c \in A\} \\
&= \inf\{f'((a-c)^*(a-c) + \ker p) + g'(c^*c + \ker p) | c \in A\} \\
&= \inf\{f'(((a + \ker p) - (c + \ker p))^*((a + \ker p) - (c + \ker p)))+ \\
&\qquad + g'((c + \ker p)^*(c + \ker p)) | c \in A\} \\
&\geq \inf\{f'(((a + \ker p) - b)^*((a + \ker p) - b)) + g'(b^*b) | b \in B\} \\
&= (\mathfrak{f}' : \mathfrak{g}')[a + \ker p] = h'((a + \ker p)^*(a + \ker p)) = h(a^*a),
\end{aligned}$$

that is $(\mathfrak{f} : \mathfrak{g})[a] \geq h(a^*a)$.

For the reversed inequality let $\varepsilon > 0$ be an arbitrary number. Then there exists $b \in B$ such that

$$h(a^*a) + \frac{\varepsilon}{2} = (\mathfrak{f}' : \mathfrak{g}')[a + \ker p] + \frac{\varepsilon}{2} \geq f'(((a + \ker p) - b)^*((a + \ker p) - b)) + g'(b^*b)$$

holds. Since $A/\ker p$ is dense in B, then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $c_n + \ker p \to b$. The functionals f' and g' are continuous, so there is an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&|f'(((a + \ker p) - b)^*((a + \ker p) - b))- \\
&\quad - f'(((a + \ker p) - (c_{n_0} + \ker p))^*((a + \ker p) - (c_{n_0} + \ker p)))| \leq \frac{\varepsilon}{4}
\end{aligned}$$

and

$$|g'(b^*b) - g'((c_{n_0} + \ker p)^*(c_{n_0} + \ker p))| \leq \frac{\varepsilon}{4}$$

thus

$$\begin{aligned}
h(a^*a) + \varepsilon &\geq f'(((a + \ker p) - (c_{n_0} + \ker p))^*((a + \ker p) - (c_{n_0} + \ker p)))+ \\
&\qquad + g'((c_{n_0} + \ker p)^*(c_{n_0} + \ker p)) = \\
&= f((a - c_{n_0})^*(a - c_{n_0})) + g(c_{n_0}^* c_{n_0}) \geq (\mathfrak{f} : \mathfrak{g})[a],
\end{aligned}$$

$h(a^*a) \geq (\mathfrak{f} : \mathfrak{g})[a]$ follows. □

Before our characterization we recall the following construction.

Remark 2.14. We recall that if A is a *-algebra and $(f_n)_{n\in\mathbb{N}}$ is sequence of increasing representable positive functionals on A such that there is a representable positive functional g on A with the property $f_n \le g$ $(\forall n \in \mathbb{N})$, then $\sup_{n\in\mathbb{N}} \mathfrak{f}_\mathfrak{n}$ is an $A \to \mathbb{R}_+$ form, moreover there exists a representable positive functional f on A such that for any $a \in A$ the equality $f(a^*a) = \sup_{n\in\mathbb{N}} \mathfrak{f}_\mathfrak{n}[a]$ is true. Indeed, by Theorem 2.3 (b) for any $n, m \in \mathbb{N}$, $n \ge m$ the functional $f_n - f_m$ is positive and representable, moreover $\|f_n - f_m\|_H \le \|g\|_H$. Hence for arbitrary $a \in A$

$$\begin{aligned} |f_n(a) - f_m(a)|^2 &= |(f_n - f_m)(a)|^2 \\ &\le \|f_n - f_m\|_H (f_n - f_m)(a^*a) \le \|g\|_H (f_n - f_m)(a^*a), \end{aligned}$$

so the convergence of $(f_n(a^*a))_{n\in\mathbb{N}}$ implies that $(f_n(a))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$. Thus the mapping $A \ni a \mapsto \lim_{n\to+\infty} f_n(a)$ determines an $f : A \to \mathbb{C}$ linear functional, which has the property $f(a^*a) = \sup_{n\in\mathbb{N}} f_n(a^*a)$ $(a \in A)$, hence f is positive. The representability of f follows from that for every $a \in A$

$$|f(a)|^2 = \lim_{n\to+\infty} |f_n(a)|^2 \le \|g\|_H \lim_{n\to+\infty} f_n(a^*a) = \|g\|_H f(a^*a),$$

hence $\|f\|_H \le \|g\|_H$, that is, by $f \le g$ and Theorem 2.3 (b) f is representable.

Denote by $\sup_{n\in\mathbb{N}} f_n$ the functional f.

The next statement is our main result in this section. By the aid of this theorem on unital C^*-algebras and measure algebras of compact groups we give further characterizations in the paper.

Theorem 2.15 (Characterization of absolute continuity)**.** *Let A be a *-algebra and let f, g be representable positive functionals on A. The following statements are equivalent.*

(i) *f is absolutely continuous with respect to g.*

(ii) *There exist an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of representable positive functionals on A and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of nonnegative numbers such that $f = \sup_{n\in\mathbb{N}} f_n$ and $f_n \le \alpha_n g$ $(\forall n \in \mathbb{N})$.*

(iii) *If p is a C^*-seminorm on A such that f and g are continuous with respect to p, the positive functional f' defined in Remark 2.12 is absolutely continuous with respect to g'.*

Proof. $(i) \Rightarrow (ii)$: By Proposition 1.13 the equation $\mathfrak{f} = \mathfrak{f}_{reg}$ holds in the Lebesgue decomposition of the form $\mathfrak{f}$ with respect to $\mathfrak{g}$ (Theorem 1.11), hence the identity $\mathfrak{f} = \sup_{n\in\mathbb{N}}(\mathfrak{f} : (n\mathfrak{g}))$ is true. On the other hand, by Corollary 2.13 for any $n \in \mathbb{N}$ there exists a representable positive functional f_n on A such that $\mathfrak{f}_\mathfrak{n} = (\mathfrak{f} : (n\mathfrak{g}))$. Moreover, from Remark 1.7 it follows that the inequalities $f_n \le ng$ and $f_n \le f_{n+1}$ are true, that is the sequences $(f_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$ have the desired properties.

$(ii) \Rightarrow (i)$: It is obvious from the definition.

$(i) \Leftrightarrow (iii)$: By Lemma 2.10 (a) it is enough to prove that f is absolutely continuous with respect to g if and only if $f'|_{A/\ker p}$ is absolutely continuous with respect to $g'|_{A/\ker p}$, since $A/\ker p$ is a dense *-subalgebra in B. But this is an immediately consequence of the definitions and the following properties:

$$(f'(a + \ker p) = f(a)) \;\wedge\; (g'(a + \ker p) = g(a)) \qquad (a \in A).$$

This finishes the proof. □

Remark 2.16. Kosaki proved a similar result for normal states on σ-finite von Neumann algebras in [15], but he substantially made use of the properties of the states and the von Neumann algebra.

In the end of this section we present a new proof to our general Lebesgue decomposition of representable positive functionals, Theorem 1.16 (Corollary 3.2 in [24]), as a consequence of the previous theorem. The argument here is very different from the one introduced in [24, Theorem 3.1], since the latter is based on a decomposition theorem of ours for representable forms on complex algebras ([24, Theorem 2.6]), wherein invariant subspaces of representations appeared as main tools.

Corollary 2.17 (Lebesgue decomposition of representable positive functionals)**.** *Let A be a *-algebra and let f, g be representable positive functionals on A. Then f admits a Lebesgue decomposition $f = f_{reg} + f_{sing}$ to a sum of representable positive functionals such that f_{sing} and g are singular, while f_{reg} is absolutely continuous with respect to g. The form induced by f_{reg} is*

$$\mathfrak{f}_{\mathfrak{reg}}[a] = \sup_{n\in\mathbb{N}}(\mathfrak{f} : (n\mathfrak{g}))[a] \qquad (a \in A).$$

Moreover, the form $\mathfrak{f}_{\mathfrak{reg}}$ is the greatest among all of the forms $\mathfrak{p}$ (in particular, forms induced by positive functionals) such that $\mathfrak{p} \le \mathfrak{f}$ and $\mathfrak{p}$ is absolutely continuous with respect to $\mathfrak{g}$.

Proof. According to Theorem 1.11, the Lebesgue decomposition of the form $\mathfrak{f}$ (derived from f) is $\mathfrak{f} = \mathfrak{f}_{reg} + \mathfrak{f}_{sing}$, where $\mathfrak{f}_{reg} := \sup_{n\in\mathbb{N}}(\mathfrak{f} : (n\mathfrak{g}))$. By Corollary 2.13 for every $n \in \mathbb{N}$ there exists a representable positive functional f_n on A such that $\mathfrak{f}_{\mathfrak{n}} = (\mathfrak{f} : (n\mathfrak{g}))$. Hence for the positive functional $f_{reg} := \sup_{n\in\mathbb{N}} f_n$ we conclude

$$\mathfrak{f}_{\mathfrak{reg}} = \sup_{n\in\mathbb{N}}(\mathfrak{f} : (n\mathfrak{g})) = \mathfrak{f}_{reg}.$$

By Remark 1.7, $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence, thus the property $f_n \le f$ ($n \in \mathbb{N}$) and Remark 2.14 implies that f_{reg} is representable. From this it follows that $f_{sing} := f - f_{reg}$ is also a positive functional, which is representable by Theorem 2.3 (b), and $\mathfrak{f}_{\mathfrak{sing}} = \mathfrak{f}_{sing}$ holds. The singularity of the functional f_{sing} with respect to g and the absolute continuity of the functional f_{reg} with respect to g, as well as the property for f_{reg} follow from Theorem 1.11. □

We must note here that the classical Lebesgue decomposition related to set-measures is a direct consequence of this result, see Section 4 in our paper [24].

The question of the uniqueness will be discussed in the penultimate section of the paper.

3. Characterizations of absolute continuity, unital C^*-algebras

Let A be a unital C^*-algebra, and denote by $\mathrm{E}(A)$ the set of the states on A, i. e.,

$$\mathrm{E}(A) := \{f \in (A^*)_+ | \|f\| = 1\}.$$

This is a convex subset in the vector space of continuous functionals A' (Theorem 2.4). It is well-known that if $\sigma(A', A)$ stands for the weakest topology on A' such that all of the linear functionals

$$\widehat{a} : A' \to \mathbb{C};\ h \mapsto h(a) \qquad (a \in A)$$

are continuous, then $\mathrm{E}(A)$ is a compact Hausdorff space with the restriction of $\sigma(A', A)$ ([2], [3], [17], [18]). We call this compact and convex set *the state space of* A.

In Proposition 4.1 of [9] R. van Handel proved a noteworthy theorem on absolute continuity of states. He observed that this concept has a very close connection with the absolute continuity of the probability Radon measures (Definition 3.1) on the state space $\mathrm{E}(A)$. The key tool in this was the so-called *Choquet theory* of compact convex sets (see [2, Chapter 4]). Following van Handel's idea, in [25] we gave a characterization of singularity by means of probability Radon measures on the state space. In this section of the paper we gave a similar characterization for domination (Theorem 3.7) and absolute continuity (Theorem 3.8) of the states on a unital C^*-algebra (including a very different and simpler proof to van Handel's result by the aid of our statement Theorem 2.15). The significance of these theorems is that the conditions formulated in them are referring to positive functionals on unital *commutative* C^*-algebras.

First of all we set up notations. For the rest of the paper we use the convention that *every locally compact space is assumed to be Hausdorff.* If T is a locally compact space, then $\mathscr{K}(T;\mathbb{C})$ (resp. $\overline{\mathscr{K}}(T;\mathbb{C})$) denotes the space of $T \to \mathbb{C}$ continuous functions with compact support (resp. vanishing at infinity). The notation $\mathscr{C}(T;\mathbb{C})$ (resp. $\mathscr{C}(T;\mathbb{R})$) stands for the set of $T \to \mathbb{C}$ (resp. $T \to \mathbb{R}$) continuous functions. The $T \to \mathbb{C}$ constant 1 function will be denoted by 1_T.

Definition 3.1. Let T be a locally compact space. For a positive linear functional $\mu : \mathscr{K}(T;\mathbb{C}) \to \mathbb{C}$ we say that μ is a *Radon measure* on T. If μ is continuous with respect to the supremum-norm $|||\cdot|||_T$ on $\mathscr{K}(T;\mathbb{C})$, then we say that μ is *bounded*. If T is compact, then μ is *probability Radon measure*, when $\mu(1_T) = 1$. The set of these functionals will be denoted by $\mathscr{M}^1_+(T)$.

It is obvious that when T is compact, then $\mathscr{K}(T;\mathbb{C}) = \mathscr{C}(T;\mathbb{C})$, hence it is a C^*-algebra (with the usual operations), moreover by Theorem 2.4 (b) it follows that every positive Radon measure is bounded, and their norms attain at 1_T.

We will need the following remarkable Riesz representation theorem ([19]) to make connections between absolute continuity, singularity and domination of positive Radon measures and the same concepts related to (positive) set measures. For a locally compact space T denote by $\mathscr{B}_0(T)$ the σ-algebra of the *Baire sets*, i. e., the σ-algebra generated by the compact G_δ-sets. If $\mu^* : \mathscr{B}_0(T) \to \mathbb{R}_+$ is a Baire measure, then $\mathscr{L}^1_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$ (resp. $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$) stands for the space of μ^*-integrable (resp. square-integrable) $T \to \mathbb{C}$ functions, equipped with the standard seminorm.

Theorem 3.2. *Let μ be a bounded positive Radon measure on the locally compact space T. Then there exists exactly one*

$$\mu^* : \mathscr{B}_0(T) \to \mathbb{R}_+$$

bounded Baire measure such that $\overline{\mathscr{K}}(T;\mathbb{C}) \subseteq \mathscr{L}^1_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)$*, and for any* $\varphi \in \overline{\mathscr{K}}(T;\mathbb{C})$

$$\widetilde{\mu}(\varphi) = \int_T \varphi \, \mathrm{d}\mu^*,$$

where $\widetilde{\mu}$ *denotes the unique continuous linear extension of* μ *to* $\overline{\mathscr{K}}(T;\mathbb{C})$*.*

The following lemma, our first result in this context, clarifies the connection between the fundamental concepts of the Lebesgue decomposition theories related to positive Radon measures and Baire measures. This will be crucial in our characterizations of domination and absolute continuity (Theorems 3.7 and 3.8), as well as in the commutative case of the uniqueness (Theorem 5.13).

Lemma 3.3. *Let* T *be a locally compact space, furthermore let* μ^*, ν^* *be bounded Baire measures on* T *and* $\alpha \in \mathbb{R}_+$*. For any* $\varphi \in \mathscr{K}(T;\mathbb{C})$ *let*

$$\mu(\varphi) := \int_T \varphi \, \mathrm{d}\mu^*; \;\; \nu(\varphi) := \int_T \varphi \, \mathrm{d}\nu^*.$$

Then μ *and* ν *are bounded positive Radon measures on* T*.*

Denote by $\widetilde{\mu}$ *(resp.* $\widetilde{\nu}$*) the unique continuous linear extension of* μ *(resp.* ν*) to* $\overline{\mathscr{K}}(T;\mathbb{C})$*. Then*

(a) $\widetilde{\mu}$ *is absolutely continuous with respect to* $\widetilde{\nu}$ $\Leftrightarrow \mu$ *is absolutely continuous with respect to* ν $\Leftrightarrow$ μ^* *is absolutely continuous with respect to* ν^**.*
(b) $\widetilde{\mu}$ *is singular to* $\widetilde{\nu}$ $\Leftrightarrow$ μ *is singular to* ν $\Leftrightarrow$ μ^* *is singular to* ν^**.*
(c) $\widetilde{\mu} \leq \alpha\widetilde{\nu} \Leftrightarrow \mu \leq \alpha\nu \Leftrightarrow \mu^* \leq \alpha\nu^*$*.*

Proof. It is clear that μ and ν are bounded positive Radon measures on T.

Since $\overline{\mathscr{K}}(T;\mathbb{C})$ is a C^*-algebra, hence the involution is continuous, furthermore every positive linear functional is continuous on it (Theorem 2.4 (*b*)). The *-subalgebra $\mathscr{K}(T;\mathbb{C})$ is dense in $\overline{\mathscr{K}}(T;\mathbb{C})$, thus by Lemma 2.10 the first equivalences of (*a*), (*b*) and (*c*) are true.

To prove the other implications, first we show that if $S \in \mathscr{B}_0(T)$ is an arbitrary Baire set, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathscr{K}(T;\mathbb{C})$ such that for every $n \in \mathbb{N}$

$$(0 \leq \varphi_n \leq 1) \;\wedge\; (\lim_{n\to+\infty} \varphi_n = \chi_S) \tag{3.1}$$

hold in $\mathscr{L}^2_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)$ and in $\mathscr{L}^2_{\mathbb{C}}(T,\mathscr{B}_0(T),\nu^*)$, as well.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to+\infty} \alpha_n = 0$. Since every Baire measure is regular ([19]), then the boundedness of μ^* and ν^* implies that for a fixed $n \in \mathbb{N}$ there exist compact subsets $K_{\mu^*,n}, K_{\nu^*,n} \in \mathscr{B}_0(T)$ of S and open sets $G_{\mu^*,n}, G_{\nu^*,n} \in \mathscr{B}_0(T)$ contain S such that

$$(\mu^*(G_{\mu^*,n} \setminus K_{\mu^*,n}) \leq \alpha_n) \;\wedge\; (\nu^*(G_{\nu^*,n} \setminus K_{\nu^*,n}) \leq \alpha_n).$$

Then for the compact Baire set $K := K_{\mu^*,n} \cup K_{\nu^*,n} \subseteq S$ and for the open Baire set $G := G_{\mu^*,n} \cap G_{\nu^*,n} \supseteq S$ we have

$$(\mu^*(G \setminus K) \leq \alpha_n) \;\wedge\; (\nu^*(G \setminus K) \leq \alpha_n).$$

Since $K \subseteq G$, then by Urysohn's Lemma ([19]) there exists $\varphi_n \in \mathscr{K}(T;\mathbb{C})$ such that

$$(0 \leq \varphi_n \leq 1) \ \wedge \ (K \subseteq \{t \in T | \varphi_n(t) = 1\} =: [\varphi_n = 1]) \ \wedge \ (\operatorname{supp}(\varphi_n) \subseteq G).$$

Hence for any $n \in \mathbb{N}$ it follows that

$$\int\limits_T |\varphi_n - \chi_S|^2 \,\mathrm{d}\mu^* = \int\limits_K |\varphi_n - \chi_S|^2 \,\mathrm{d}\mu^* + \int\limits_{G\setminus K} |\varphi_n - \chi_S|^2 \,\mathrm{d}\mu^* + \int\limits_{T\setminus G} |\varphi_n - \chi_S|^2 \,\mathrm{d}\mu^*$$
$$= 0 + \int\limits_{G\setminus K} |\varphi_n - \chi_S|^2 \,\mathrm{d}\mu^* + 0 \leq \mu^*(G \setminus K) \leq \alpha_n$$

and

$$\int\limits_T |\varphi_n - \chi_S|^2 \,\mathrm{d}\nu^* = \int\limits_K |\varphi_n - \chi_S|^2 \,\mathrm{d}\nu^* + \int\limits_{G\setminus K} |\varphi_n - \chi_S|^2 \,\mathrm{d}\nu^* + \int\limits_{T\setminus G} |\varphi_n - \chi_S|^2 \,\mathrm{d}\nu^*$$
$$= 0 + \int\limits_{G\setminus K} |\varphi_n - \chi_S|^2 \,\mathrm{d}\nu^* + 0 \leq \nu^*(G \setminus K) \leq \alpha_n.$$

Thus we showed the existence of a sequence with the properties (3.1).

Now we are ready to prove the second equivalences.

(a): $\Rightarrow$: Assume that μ is absolutely continuous with respect to ν. Let S be a Baire set such that $\nu^*(S) = 0$; we show the equality $\mu^*(S) = 0$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence with the properties (3.1). From this it follows that

$$0 = \nu^*(S) = \int\limits_T |\chi_S|^2 \,\mathrm{d}\nu^* = \lim_{n\to+\infty} \int\limits_T |\varphi_n|^2 \,\mathrm{d}\nu^* = \lim_{n\to+\infty} \nu(\varphi_n^*\varphi_n)$$

and

$$0 = \lim_{n,m\to+\infty} \int\limits_T |\varphi_n - \varphi_m|^2 \,\mathrm{d}\mu^* = \lim_{n,m\to+\infty} \mu((\varphi_n - \varphi_m)^*(\varphi_n - \varphi_m)).$$

From the assumption we have $\mu(\varphi_n^*\varphi_n) \to 0$. But

$$0 = \lim_{n\to+\infty} \mu(\varphi_n^*\varphi_n) = \lim_{n\to+\infty} \int\limits_T |\varphi_n|^2 \,\mathrm{d}\mu^* = \int\limits_T |\chi_S|^2 \,\mathrm{d}\mu^* = \mu^*(S),$$

i.e., μ^* is absolutely continuous with respect to ν^*.

$\Leftarrow$: Assume that μ^* is absolutely continuous with respect to ν^*. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{K}(T;\mathbb{C})$ such that

$$(\lim_{n\to+\infty} \nu(\psi_n^*\psi_n) = 0) \ \wedge \ (\lim_{n,m\to+\infty} \mu((\psi_n - \psi_m)^*(\psi_n - \psi_m)) = 0),$$

that is

$$(\lim_{n\to+\infty} \int\limits_T |\psi_n|^2 \,\mathrm{d}\nu^* = 0) \ \wedge \ (\lim_{n,m\to+\infty} \int\limits_T |\psi_n - \psi_m|^2 \,\mathrm{d}\mu^* = 0),$$

This means that the sequence $(\psi_n)_{n\in\mathbb{N}}$ converges to zero in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \nu^*)$ meanwhile Cauchy (hence convergent) in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$. By the well-known theorem of Riesz on $\mathscr{L}_2$-convergence there exists a subsequence $(\psi_{\sigma(n)})_{n\in\mathbb{N}}$ of $(\psi_n)_{n\in\mathbb{N}}$ such that $\psi_{\sigma(n)} \to 0$ ν^*-almost everywhere. Where the sequence not converges, that is a ν^*-nullset, hence by the hypothesis that is a μ^*-nullset, as well. Thus $\psi_{\sigma(n)} \to 0$ μ^*-almost everywhere comes true. As a consequence, the convergence holds true also in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$, thus one limit of $(\psi_n)_{n\in\mathbb{N}}$ is the constant 0 function in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$, that is

$$\lim_{n\to+\infty} \mu(\psi_n^*\psi_n) = \lim_{n\to+\infty} \int_T |\psi_n|^2 \,\mathrm{d}\mu^* = 0.$$

(b) $\Rightarrow$: Suppose that μ and ν are singular. By Theorem 1.15 (iii) there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{K}(T;\mathbb{C})$ such that for any $\varphi \in \mathscr{K}(T;\mathbb{C})$

$$\begin{aligned} &\lim_{n\to+\infty} \nu(\psi_n^*\psi_n) = \lim_{n,m\to+\infty} \mu((\psi_n-\psi_m)^*(\psi_n-\psi_m)) = 0;\\ &\lim_{n\to+\infty} \mu(\psi_n^*\varphi) = \mu(\varphi) \end{aligned} \tag{3.2}$$

hold. The first property means that $(\psi_n)_{n\in\mathbb{N}}$ tends to zero in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \nu^*)$ meanwhile Cauchy in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$. Denote by ψ one of the limits in the latter space. By Riesz's Theorem there exists a subsequence of $(\psi_n)_{n\in\mathbb{N}}$ such that converges to zero ν^*-almost everywhere. From this subsequence we may choose a subsequence such that converges to ψ μ^*-almost everywhere. For the simplicity of the notations assume that $(\psi_n)_{n\in\mathbb{N}}$ already has these properties (this will not cause any problems).

Let $S := \{t \in T \mid \lim_{n\to+\infty} \psi_n(t) = 0\}$. The sets S and $T \setminus S$ are Baire sets, and $T = S \cup^* (T \setminus S)$. From the definition of S we have $\nu^*(T\setminus S) = 0$. We show that $\mu^*(S) = 0$, and this proves the singularity.

First we note that for any subset $E \in \mathscr{B}_0(T)$ of S $\mu^*(E \cap [\psi \neq 0]) = 0$ holds, otherwise $(\psi_n)_{n\in\mathbb{N}}$ does not converge to ψ μ^*-almost everywhere. Every Baire measure is regular ([19]) and μ^* is bounded, hence for $\varepsilon > 0$ there exist $K_0 \subseteq S$ compact and $G_0 \supseteq S$ open Baire sets such that $\mu^*(G_0 \setminus K_0) \leq \varepsilon$. By Urysohn's Lemma there exists $\varphi \in \mathscr{K}(T;\mathbb{C})$ such that

$$0 \leq \varphi \leq 1, \;\; K_0 \subseteq [\varphi = 1] \; \textit{and} \; \operatorname{supp}(\varphi) \subseteq G_0.$$

Thus these properties, the positivity of μ, the second assumption of (3.2), the Cauchy-Schwarz-inequality and $\mu^*(K_0 \cap [\psi \neq 0]) = 0$ imply that

$$\mu^*(S) = \int_T \chi_S \, \mathrm{d}\mu^* = \int_{G_0} \chi_S \, \mathrm{d}\mu^* = \int_{G_0 \setminus K_0} \chi_S \, \mathrm{d}\mu^* + \int_{K_0} \chi_S \, \mathrm{d}\mu^* \leq \varepsilon + \int_{K_0} \varphi \, \mathrm{d}\mu^*$$

$$\leq \varepsilon + \int_T \varphi \, \mathrm{d}\mu^* = \varepsilon + \mu(\varphi) = \varepsilon + \lim_{n\to+\infty} |\mu(\psi_n^*\varphi)| = \varepsilon + \lim_{n\to+\infty} |\int_T \varphi\overline{\psi_n} \, \mathrm{d}\mu^*|$$

$$= \varepsilon + \lim_{n\to+\infty} |\int_{G_0} \varphi\overline{\psi_n} \, \mathrm{d}\mu^*| \leq \varepsilon + \limsup_{n\to+\infty} \int_{G_0} |\varphi\overline{\psi_n}| \, \mathrm{d}\mu^*$$

$$= \varepsilon + \limsup_{n\to+\infty} (\int_{G_0\setminus K_0} |\varphi\overline{\psi_n}| \, \mathrm{d}\mu^* + \int_{K_0} |\varphi\overline{\psi_n}| \, \mathrm{d}\mu^*)$$

$$= \varepsilon + \limsup_{n\to+\infty} (\int_T |\varphi\overline{\psi_n}\chi_{G_0\setminus K_0}| \, \mathrm{d}\mu^* + \int_{K_0} |\varphi\overline{\psi_n}| \, \mathrm{d}\mu^*)$$

$$\leq \varepsilon + \limsup_{n\to+\infty} \left(\sqrt{\int_T |\varphi\overline{\psi_n}|^2 \, \mathrm{d}\mu^*} \sqrt{\int_T |\chi_{G_0\setminus K_0}|^2 \, \mathrm{d}\mu^*} + \sqrt{\int_{K_0} |\varphi|^2 \, \mathrm{d}\mu^*} \sqrt{\int_{K_0} |\psi_n|^2 \, \mathrm{d}\mu^*} \right)$$

$$\leq \varepsilon + \sqrt{\mu^*(G_0 \setminus K_0)} \sqrt{\int_T |\psi|^2 \, \mathrm{d}\mu^*} + \sqrt{\mu^*(K_0)} \sqrt{\int_{K_0} |\psi|^2 \, \mathrm{d}\mu^*}$$

$$= \varepsilon + \sqrt{\varepsilon} \sqrt{\int_T |\psi|^2 \, \mathrm{d}\mu^*} + 0,$$

so from $\varepsilon \to 0$ we conclude $\mu^*(S) = 0$.

$\Leftarrow$: Suppose that μ^* and ν^* are singular, i.e., there exists an $S \subseteq T$ Baire set with the properties $\mu^*(S) = 0$, $\nu^*(T \setminus S) = 0$ and $T = S \cup^* (T \setminus S)$. By (3.1) there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathscr{K}(T;\mathbb{C})$ such that

$$(0 \leq \varphi_n \leq 1) \ \wedge \ (\lim_{n\to+\infty} \varphi_n = \chi_{T\setminus S})$$

holds true in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$ and in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \nu^*)$, as well. We prove that this sequence shows the singularity of μ and ν (Theorem 1.15 *(iii)*):

$$\nu(\varphi_n^*\varphi_n) = \int_T |\varphi_n|^2 \, \mathrm{d}\nu^* \to \int_T |\chi_{T\setminus S}|^2 \, \mathrm{d}\nu^* = \nu^*(T \setminus S) = 0,$$

$$\mu((\varphi_n - \varphi_m)^*(\varphi_n - \varphi_m)) = \int_T |\varphi_n - \varphi_m|^2 \mathrm{d}\mu^* \to 0.$$

What we still have to show is that for any $\varphi \in \mathscr{K}(T;\mathbb{C})$

$$\lim_{n\to+\infty} \mu(\varphi_n^*\varphi) = \mu(\varphi)$$

is true. Indeed, by the boundedness of μ^* we have $1_T \in \mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$, thus

$$\begin{aligned}
\limsup_{n\to+\infty} |\mu(\varphi) - \mu(\varphi_n^*\varphi)|^2 &= \limsup_{n\to+\infty} |\int_T \overline{(1_T - \varphi_n)}\varphi \, \mathrm{d}\mu^*|^2 \\
&\leq \limsup_{n\to+\infty}((\int_T |1_T - \varphi_n|^2 \mathrm{d}\mu^*)(\int_T |\varphi|^2 \, \mathrm{d}\mu^*)) \\
&\leq \mu^*(T)|||\varphi|||_T^2 \limsup_{n\to+\infty} \int_T |1_T - \varphi_n|^2 \mathrm{d}\mu^* \\
&= \mu^*(T)|||\varphi|||_T^2 \lim_{n\to+\infty} \|1_T - \varphi_n\|^2_{\mathscr{L}^2_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)} \\
&= \mu^*(T)|||\varphi|||_T^2 \|1_T - \chi_{T\setminus S}\|^2_{\mathscr{L}^2_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)} \\
&= \mu^*(T)|||\varphi|||_T^2 \|\chi_S\|^2_{\mathscr{L}^2_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)} \\
&= \mu^*(T)|||\varphi|||_T^2 \mu^*(S) = 0.
\end{aligned}$$

(c): $\Rightarrow$: Let $S \in \mathscr{B}_0(T)$ be an arbitrary set. By (3.1) there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathscr{K}(T;\mathbb{C})$ such that for every $n \in \mathbb{N}$

$$(0 \leq \varphi_n \leq 1) \ \wedge \ (\lim_{n\to+\infty} \varphi_n = \chi_S)$$

is true in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*)$ and in $\mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \nu^*)$, as well. Then

$$\begin{aligned}
\mu^*(S) = \int_T |\chi_S|^2 \, \mathrm{d}\mu^* &= \lim_{n\to+\infty} \int_T |\varphi_n|^2 \, \mathrm{d}\mu^* = \lim_{n\to+\infty} \mu(\varphi_n^*\varphi_n) \\
&\leq \alpha \lim_{n\to+\infty} \nu(\varphi_n^*\varphi_n) = \alpha \lim_{n\to+\infty} \int_T |\varphi_n|^2 \, \mathrm{d}\nu^* = \alpha \int_T |\chi_S|^2 \, \mathrm{d}\nu^* = \alpha\nu^*(S),
\end{aligned}$$

so $\mu^* \leq \alpha\nu^*$.

$\Leftarrow$: It is well-known that for any $\varphi \in \mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \mu^*) \cap \mathscr{L}^2_{\mathbb{C}}(T, \mathscr{B}_0(T), \nu^*)$

$$\int_T |\varphi|^2 \, \mathrm{d}\mu^* \leq \alpha \int_T |\varphi|^2 \, \mathrm{d}\nu^*$$

holds. In particular, for every $\varphi \in \mathscr{K}(T;\mathbb{C})$, hence

$$\mu(\varphi^*\varphi) = \int_T |\varphi|^2 \, \mathrm{d}\mu^* \leq \alpha \int_T |\varphi|^2 \, \mathrm{d}\nu^* = \alpha\nu(\varphi^*\varphi),$$

that is $\mu \leq \alpha\nu$. The proof is complete. □

To begin investigations of the state space of a unital C^*-algebra we need the following general theorem for Hausdorff locally convex spaces, which is the base point of the Choquet theory of compact convex sets ([2, Chapter 4]).

Theorem 3.4. *Let E be a Hausdorff locally convex space. Then for every $K \subseteq E$ nonempty, compact convex set and for every probability Radon measure μ on K there exists a unique $x \in K$ such that for the real part $u_{\mathbb{R}}$ of an arbitrary* *continuous linear functional* $u \in E'$ *the equality* $u_{\mathbb{R}}(x) = \mu((u_{\mathbb{R}})|_K)$ *holds. We say that* x *is the* barycenter *of* μ*, and we use the notation* $\mathbf{b}(\mu)$ *for* x*.*

Remark 3.5. For every element $x \in K$ there exists a probability Radon measure μ on K such that $\mathbf{b}(\mu) = x$. For example, let ε_x be the probability Radon measure such that for any $\varphi \in \mathscr{C}(K;\mathbb{C})$ the equation $\varepsilon_x(\varphi) := \varphi(x)$ holds.

Let A be a C^*-algebra with unit. It is obvious that $\widehat{a}|_{\mathrm{E}(A)} \in \mathscr{C}(\mathrm{E}(A);\mathbb{C})$ for any $a \in A$. Furthermore, it is well known that every continuous linear functional on the locally convex space A' (equipped with the $\sigma(A', A)$ topology) is of the form $\widehat{a}$ for some $a \in A$ (see [2]). Hence a functional $f \in \mathrm{E}(A)$ is the barycenter of a probability Radon measure μ on $\mathrm{E}(A)$ iff for every $a \in A$ we have

$$\mu((\widehat{a}_{\mathbb{R}})|_{\mathrm{E}(A)}) = (\widehat{a}_{\mathbb{R}})|_{\mathrm{E}(A)}(f).$$

On the other hand, $(\widehat{a}_{\mathbb{R}})|_{\mathrm{E}(A)}$ is just $(\widehat{\frac{a+a^*}{2}})|_{\mathrm{E}(A)}$. This means that f is the barycenter if and only if for any $a \in A_{sa}$ the equality $\mu((\widehat{a})|_{\mathrm{E}(A)}) = f(a)$ holds true. In particular, if $\nu \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})^*$ is an $\mathbb{R}$-linear functional such that for any positive function $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ the inequality $\nu(\varphi) \geq 0$ is true, moreover for every $a \in A_{sa}$ the equation $\nu((\widehat{a})|_{\mathrm{E}(A)}) = f(a)$ holds, then the unique positive $\mathbb{C}$-linear extension of ν to $\mathscr{C}(\mathrm{E}(A);\mathbb{C})$ is a probability Radon measure on $\mathrm{E}(A)$ with barycenter f.

We note that the constant $1_{\mathrm{E}(A)}$ function is $(\widehat{\mathbf{1}})|_{\mathrm{E}(A)}$.

Henceforth for any probability Radon measure μ on $\mathrm{E}(A)$ denote by μ^* the associated Baire measure by Theorem 3.2. To prove our characterizations of domination and absolute continuity we recall the following extension theorem due to Ky Fan ([5, Lemma on page 1]).

Theorem 3.6. *Let* E *be a real Hausdorff topological vector space, let* L *be a linear subspace of* E *and let* C *be a convex subset of* E *such that* $L \cap \operatorname{int} C \neq \emptyset$*. Let* $\omega \in L^*$ *and suppose that for any* $x \in L \cap C$ *the inequality* $\omega(x) \geq 0$ *holds. Then there exists an extension* $\widetilde{\omega} \in E'$ *of* ω *such that* $\widetilde{\omega}(x) \geq 0$ *is true for any* $x \in C$*.*

Similar to our result on singularity of states (Theorem 7 in [25]), the following two theorems give equivalent conditions to domination and absolute continuity by means of the Choquet theory of the state space. The relevance of these facts is that the fundamental concepts of the Lebesgue decomposition theory of positive functionals on unital C^*-algebras can be characterized via positive functionals on unital *commutative* C^*-algebras.

Theorem 3.7 (Domination of states)**.** *Let* A *be* C^**-algebra with unit and let* $f, g \in \mathrm{E}(A)$*,* $\alpha \in \mathbb{R}^+$*. The following statements are equivalent.*

(i) $f \leq \alpha g$.

(ii) *There exist probability Radon measures* μ_f, μ_g *on* $\mathrm{E}(A)$*, such that the barycenters* $\mathbf{b}(\mu_f) = f, \mathbf{b}(\mu_g) = g$ *and* $\mu_f \leq \alpha\mu_g$*.*

(iii) *There exist probability Radon measures* μ_f, μ_g *on* $\mathrm{E}(A)$*, such that the barycenters* $\mathbf{b}(\mu_f) = f, \mathbf{b}(\mu_g) = g$ *and* $\mu_f^* \leq \alpha\mu_g^*$*.*

Proof. $(i) \Rightarrow (ii)$: By the arguments previous to Theorem 3.6 it is enough to prove the existence of $\mathbb{R}$-linear functionals $\nu_f, \nu_g \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})^*$ such that

(a) for every $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ the inequalities $\nu_f(\varphi) \geq 0$ and $\nu_g(\varphi) \geq 0$ hold;
(b) for every $a \in A_{sa}$ the equalities $\nu_f(\widehat{a}|_{\mathrm{E}(A)}) = f(a)$, $\nu_g(\widehat{a}|_{\mathrm{E}(A)}) = g(a)$ hold;
(c) for every $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ the inequality $\nu_f(\varphi) \leq \alpha\nu_g(\varphi)$ holds.

Let $E := \mathscr{C}(\mathrm{E}(A);\mathbb{R})$, $L := \{\widehat{a}|_{\mathrm{E}(A)} | a \in A_{sa}\}$ and $C := \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$. The real vector space E equipped with the supremum-norm is Hausdorff and locally convex, L is a linear subspace of E, C is a convex subset of E such that $\widehat{\mathbf{1}}|_{\mathrm{E}(A)} \in (L \cap \operatorname{int} C)$. Let

$$\omega_f : L \to \mathbb{R};\ \widehat{a}|_{\mathrm{E}(A)} \mapsto f(a).$$

This mapping is linear, moreover for $x \in L \cap C$ we have $x = \widehat{a}|_{\mathrm{E}(A)}$ for some $a \in A_+$, hence the positivity of f implies $\omega_f(x) \geq 0$. By Theorem 3.6 there exists an $\widetilde{\omega_f} : E \to \mathbb{R}$ continuous linear functional which extends ω_f, and for every $x \in C$ the inequality $\widetilde{\omega_f}(x) \geq 0$ holds. Let

$$\nu_f := \widetilde{\omega_f}.$$

From the assumption $f \leq \alpha g$ we conclude that the mapping

$$\omega_{\alpha g - f} : L \to \mathbb{R};\ \widehat{a}|_{\mathrm{E}(A)} \mapsto (\alpha g - f)(a)$$

is a linear functional such that for any $x \in L \cap C$ we have $\omega_{\alpha g - f}(x) \geq 0$. Theorem 3.6 implies the existence of a continuous linear functional $\widetilde{\omega_{\alpha g - f}} : E \to \mathbb{R}$, which is an extension of $\omega_{\alpha g - f}$, furthermore for $x \in C$ we get $\widetilde{\omega_{\alpha g - f}}(x) \geq 0$. Let

$$\nu_g := \frac{1}{\alpha}(\widetilde{\omega_{\alpha g - f}} + \nu_f).$$

It is clear that ν_f and ν_g fulfill (a). The property (b) is clear to ν_f, and for ν_g it is a simple calculation, since for $a \in A_{sa}$

$$\nu_g(\widehat{a}|_{\mathrm{E}(A)}) = \frac{1}{\alpha}(\widetilde{\omega_{\alpha g - f}} + \nu_f)(\widehat{a}|_{\mathrm{E}(A)}) = \frac{1}{\alpha}((\alpha g - f)(a) + f(a)) = g(a).$$

Point (c) is also an immediately consequence, since the positivity implies for any $\varphi \in C$ that

$$\alpha\nu_g(\varphi) = (\widetilde{\omega_{\alpha g - f}} + \nu_f)(\varphi) = \widetilde{\omega_{\alpha g - f}}(\varphi) + \nu_f(\varphi) \geq \nu_f(\varphi).$$

$(ii) \Leftrightarrow (iii)$: By (c) of Lemma 3.3 the statement is obvious, since for any $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{C})$

$$\mu_f(\varphi) = \int\limits_{\mathrm{E}(A)} \varphi \,\mathrm{d}\mu_f^*;\ \mu_g(\varphi) = \int\limits_{\mathrm{E}(A)} \varphi \,\mathrm{d}\mu_g^*.$$

$(iii) \Rightarrow (i)$: If μ_f and μ_g are probability Radon measures on $\mathrm{E}(A)$ such that the barycenters are f and g, respectively, moreover $\mu_f^* \leq \alpha\mu_g^*$ holds true, then for any nonnegative function $\varphi \in \mathscr{L}^1_{\mathbb{C}}(\mathrm{E}(A), \mathscr{B}_0(\mathrm{E}(A)), \mu_f^*) \cap \mathscr{L}^1_{\mathbb{C}}(\mathrm{E}(A), \mathscr{B}_0(\mathrm{E}(A)), \mu_g^*)$ the inequality

$$\int\limits_{\mathrm{E}(A)} \varphi \,\mathrm{d}\mu_f^* \leq \alpha \int\limits_{\mathrm{E}(A)} \varphi \,\mathrm{d}\mu_g^*$$

is true, in particular for the functions $\widehat{a^*a}|_{\mathrm{E}(A)}$ $(a \in A)$, that is:

$$f(a^*a) = \mu_f(\widehat{a^*a}|_{\mathrm{E}(A)}) = \int\limits_{\mathrm{E}(A)} \widehat{a^*a}|_{\mathrm{E}(A)}\ \mathrm{d}\mu_f^*$$

$$\leq \alpha \int\limits_{\mathrm{E}(A)} \widehat{a^*a}|_{\mathrm{E}(A)}\ \mathrm{d}\mu_g^* = \alpha\mu_g(\widehat{a^*a}|_{\mathrm{E}(A)}) = \alpha g(a^*a),$$

thus $f \leq \alpha g$. □

Theorem 3.8 (Absolute continuity of states). *Let A be C^*-algebra with unit, and let $f, g \in \mathrm{E}(A)$. The following statements are equivalent.*

(i) *f is absolutely continuous with respect to g.*

(ii) *There exist probability Radon measures μ_f, μ_g on $\mathrm{E}(A)$ such that the barycenters $\mathbf{b}(\mu_f) = f, \mathbf{b}(\mu_g) = g$ and μ_f is absolutely continuous with respect to μ_g.*

(iii) *There exist probability Radon measures μ_f, μ_g on $\mathrm{E}(A)$ such that the barycenters $\mathbf{b}(\mu_f) = f, \mathbf{b}(\mu_g) = g$ and μ_f^* is absolutely continuous with respect to μ_g^*.*

Proof. $(i) \Rightarrow (ii)$: By Theorem 2.15 (ii) there exists an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of positive functionals on A and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that for any $n \in \mathbb{N}$ the inequality $f_n \leq \alpha_n g$ and the equality $f = \sup_{n\in\mathbb{N}} f_n$ hold. We may assume that $\alpha_n \neq 0$ for every $n \in \mathbb{N}$.

By the arguments previous to Theorem 3.6 and by Theorem 2.15 it is enough to prove the existence of a sequence $(\nu_{f_n})_{n\in\mathbb{N}}$ in $\mathscr{C}(\mathrm{E}(A);\mathbb{R})^*$ and an $\mathbb{R}$-linear functional $\nu_g \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})^*$ satisfying

(a) for every $n \in \mathbb{N}$, $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ and $a \in A_{sa}$ the properties $\nu_{f_n}(\varphi) \geq 0$, $\nu_g(\varphi) \geq 0$, $\nu_{f_n}(\widehat{a}|_{\mathrm{E}(A)}) = f_n(a)$ and $\nu_g(\widehat{a}|_{\mathrm{E}(A)}) = g(a)$ hold;

(b) for every $n \in \mathbb{N}$ $\nu_{f_n} \leq \nu_{f_{n+1}}$;

(c) $\nu_f := \sup_{n\in\mathbb{N}} \nu_{f_n} \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})^*$, moreover for every $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ $\nu_f(\varphi) \geq 0$ and for any $a \in A_{sa}$ $\nu_f(\widehat{a}|_{\mathrm{E}(A)}) = f(a)$ is true.

(d) There exists a sequence $(\widetilde{\alpha_n})_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that for any $n \in \mathbb{N}$ and for arbitrary $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ the inequality $\nu_{f_n}(\varphi) \leq \widetilde{\alpha_n}\nu_g(\varphi)$ is true.

We show the existence of $(\nu_{f_n})_{n\in\mathbb{N}}$ by induction. Let $E := \mathscr{C}(\mathrm{E}(A);\mathbb{R})$, furthermore $L := \{\widehat{a}|_{\mathrm{E}(A)} | a \in A_{sa}\}$ and $C := \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$. The real vector space E equipped with the supremum-norm is Hausdorff and locally convex, L is a linear subspace of E, C is a convex subset of E such that $\widehat{\mathbf{1}}|_{\mathrm{E}(A)} \in (L \cap \operatorname{int} C)$. Let

$$\omega_{f_0} : L \to \mathbb{R};\ \widehat{a}|_{\mathrm{E}(A)} \mapsto f_0(a).$$

This mapping is linear, moreover for $x \in L\cap C$ $x = \widehat{a}|_{\mathrm{E}(A)}$ for some $a \in A_+$, hence the positivity of f_0 implies $\omega_{f_0}(x) \geq 0$. By Theorem 3.6 there exists a continuous linear functional $\widetilde{\omega_{f_0}} : E \to \mathbb{R}$ which extends ω_{f_0}, furthermore for any $x \in C$ we have $\widetilde{\omega_{f_0}}(x) \geq 0$. Let

$$\nu_{f_0} := \widetilde{\omega_{f_0}};$$

this function obviously fulfills the properties of (a).

Let $n \in \mathbb{N}$ be arbitrary, and suppose that the system $(\nu_{f_k})_{0\le k\le n}$ satisfies for $0 \le k \le n$ that $\nu_{f_k} \in \mathscr{C}(\mathrm{E}(A));\mathbb{R})^*$, for $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ that $\nu_{f_k}(\varphi) \ge 0$, for any $a \in A_{sa}$ that $\nu_{f_k}(\widehat{a}|_{\mathrm{E}(A)}) = f_k(a)$ and for any $k < n$ that $\nu_{f_k} \le \nu_{f_{k+1}}$. Let

$$\omega_{n+1} : L \to \mathbb{R};\ \widehat{a}|_{\mathrm{E}(A)} \mapsto (f_{n+1} - f_n)(a).$$

From the property for $(f_n)_{n\in\mathbb{N}}$ it follows that for $x \in L \cap C$ $\omega_{n+1}(x) \ge 0$. Thus applying again Theorem 3.6, we obtain the existence of a continuous linear functional $\widetilde{\omega_{n+1}} : E \to \mathbb{R}$ which extends ω_{n+1}, moreover for arbitrary $x \in C$ the inequality $\widetilde{\omega_{n+1}}(x) \ge 0$ holds. Let

$$\nu_{f_{n+1}} := \widetilde{\omega_{n+1}} + \nu_{f_n}.$$

For this functional we have the inequality $\nu_{f_{n+1}}(\varphi) \ge 0$ for any $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$, furthermore $\nu_{f_n} \le \nu_{f_n} + \widetilde{\omega_{n+1}} = \nu_{f_{n+1}}$. If $a \in A_{sa}$ arbitrary, then

$$\nu_{f_{n+1}}(\widehat{a}|_{\mathrm{E}(A)}) = \widetilde{\omega_{n+1}}(\widehat{a}|_{\mathrm{E}(A)}) + \nu_{f_n}(\widehat{a}|_{\mathrm{E}(A)}) = (f_{n+1} - f_n)(a) + f_n(a) = f_{n+1}(a),$$

hence the properties of (a) and (b) are fulfilled. We proved the existence of the sequence $(\nu_{f_n})_{n\in\mathbb{N}}$.

Since for any $n \in \mathbb{N}$ the equation

$$\|\nu_{f_n}\| = \nu_{f_n}(\widehat{\mathbf{1}}|_{\mathrm{E}(A)}) = f_n(\mathbf{1}) \le f(\mathbf{1}) = 1$$

is true, thus the linear functional

$$\nu_f : E \to \mathbb{R};\ \nu_f := \sup_{n\in\mathbb{N}} \nu_{f_n}$$

exists. If $a \in A_{sa}$ arbitrary, then the previous arguments and $f = \lim_{n\to+\infty} f_n$, $f(\mathbf{1}) = 1$ imply

$$\nu_f(\widehat{a}|_{\mathrm{E}(A)}) := \lim_{n\to+\infty} \nu_{f_n}(\widehat{a}|_{\mathrm{E}(A)}) = \lim_{n\to+\infty} f_n(a) = f(a).$$

Moreover, by definition for any $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{R})_+$ the inequality $\nu_f(\varphi) \ge 0$ obviously occurs. This proves (c).

For the definition of ν_g let $n \in \mathbb{N}$ be any number. Since $\alpha_n g - f_n$ is a positive linear functional on A, thus let us denote by ν_{g_n} an arbitrary extension to E of the mapping

$$L \to \mathbb{R};\ \widehat{a}|_{\mathrm{E}(A)} \mapsto \alpha_n g(a) - f_n(a)$$

in virtue of Theorem 3.6. Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\sum_{n=0}^{+\infty} \delta_n = 1$. Let

$$\nu_g := \sum_{n=0}^{+\infty} \delta_n\Big(\frac{1}{\alpha_n}(\nu_{g_n} + \nu_{f_n})\Big),$$

where the sum is meant in the functional norm. We state that ν_g is a linear functional on $\mathscr{C}(\mathrm{E}(A),\mathbb{R})$ which fulfills the desired properties of (a) and (d).

The series determining ν_g is absolute convergent in the functional norm, hence convergent. Indeed, it is a sum of positive functionals, thus $g(\mathbf{1}) = 1$ implies

$$\sum_{n=0}^{+\infty} \|\delta_n(\frac{1}{\alpha_n}(\nu_{g_n} + \nu_{f_n}))\| = \sum_{n=0}^{+\infty} \delta_n(\frac{1}{\alpha_n}(\nu_{g_n} + \nu_{f_n})(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})) = \sum_{n=0}^{+\infty} \delta_n(\frac{1}{\alpha_n}((\alpha_n g - f_n)(\mathbf{1}) + f_n(\mathbf{1}))) = 1.$$

It is clear from the definition that ν_g is a positive linear functional on $\mathscr{C}(\mathrm{E}(A), \mathbb{R})$, moreover for $a \in A$

$$\sum_{n=0}^{+\infty} \delta_n(\frac{1}{\alpha_n}(\nu_{g_n} + \nu_{f_n})(\widehat{a}|_{\mathrm{E}(A)})) = \sum_{n=0}^{+\infty} \delta_n(\frac{1}{\alpha_n}((\alpha_n g - f_n)(a) + f_n(a))) = \sum_{n=0}^{+\infty} \delta_n g(a) = g(a),$$

i. e., (a) holds.

For $n \in \mathbb{N}$ let $\widetilde{\alpha_n} := \frac{\alpha_n}{\delta_n}$. We state that for any $n \in \mathbb{N}$ the inequality $\nu_{f_n} \leq \widetilde{\alpha_n}\nu_g$ is true. Indeed, this is an easy calculation:

$$\nu_{f_n} = \widetilde{\alpha_n}(\frac{\delta_n}{\alpha_n}\nu_{f_n}) \leq \widetilde{\alpha_n}(\delta_n\frac{1}{\alpha_n}(\nu_{g_n} + \nu_{f_n})) \leq \widetilde{\alpha_n}\nu_g.$$

Hence we get (d).

$(ii) \Rightarrow (i)$: If μ_f and μ_g are probability Radon measures on $\mathrm{E}(A)$ (with barycenters f and g, respectively), such that μ_f is absolutely continuous with respect to μ_g, then Theorem 2.15 imply the existence of an increasing sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive functionals in $\mathscr{C}(\mathrm{E}(A); \mathbb{C})'$ and of a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that for every $n \in \mathbb{N}$ $\mu_n \leq \alpha_n\mu_g$ holds, moreover $\mu_f = \sup_{n\in\mathbb{N}} \mu_n$. We may suppose for $n \in \mathbb{N}$ that $\mu_n \neq 0$ and $\alpha_n \neq 0$, since $\mu_f \neq 0$. Hence for $n \in \mathbb{N}$ we have $0 \neq \|\mu_n\| = \mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})$. Since for any $n \in \mathbb{N}$ $\mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})^{-1}\mu_n$ is a probability Radon measure on $\mathrm{E}(A)$, thus by Theorem 3.4 we conclude the existence of its barycenter $f_n \in \mathrm{E}(A)$, i. e., for $a \in A$ we obtain

$$f_n(a) = (\widehat{a}|_{\mathrm{E}(A)})(f_n) = \mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})^{-1}\mu_n(\widehat{a}|_{\mathrm{E}(A)}).$$

We prove that the sequence $(\mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})f_n)_{n\in\mathbb{N}}$ is of positive functionals in A' such that

(a) it is increasing;
(b) for every $n \in \mathbb{N}$ the inequality $\mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})f_n \leq \alpha_n g$ holds;
(c) $\sup_{n\in\mathbb{N}} \mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})f_n = f$.

Thus by Theorem 2.15 the functional f is absolutely continuous with respect to g.

We note that $1 = \mu_f(\widehat{\mathbf{1}}|_{\mathrm{E}(A)}) = \sup_{n\in\mathbb{N}} \mu_n(\widehat{\mathbf{1}}|_{\mathrm{E}(A)})$. Let $n \in \mathbb{N}$ and $a \in A$ be fixed elements. Then

$$\begin{aligned}(a):\ (\mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})f_n)(a^*a) &= \mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})((\widehat{a^*a}|_{\mathrm{E}(A)})(f_n)) = \mu_n(\widehat{a^*a}|_{\mathrm{E}(A)})\\ &\leq \mu_{n+1}(\widehat{a^*a}|_{\mathrm{E}(A)}) = \mu_{n+1}(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})((\widehat{a^*a}|_{\mathrm{E}(A)})(f_{n+1}))\\ &= (\mu_{n+1}(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})f_{n+1})(a^*a).\end{aligned}$$

$$\begin{aligned}(b):\ (\mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})f_n)(a^*a) &= \mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})((\widehat{a^*a}|_{\mathrm{E}(A)})(f_n)) = \mu_n(\widehat{a^*a}|_{\mathrm{E}(A)})\\ &\leq \alpha_n\mu_g(\widehat{a^*a}|_{\mathrm{E}(A)}) = \alpha_n((\widehat{a^*a}|_{\mathrm{E}(A)})(g)) = \alpha_n g(a^*a).\end{aligned}$$

$$\begin{aligned}(c):\ \sup_{n\in\mathbb{N}}(\mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})f_n)(a^*a) &= \sup_{n\in\mathbb{N}}\mu_n(\widehat{\mathbb{1}}|_{\mathrm{E}(A)})((\widehat{a^*a}|_{\mathrm{E}(A)})(f_n))\\ &= \sup_{n\in\mathbb{N}}\mu_n(\widehat{a^*a}|_{\mathrm{E}(A)}) = \mu_f(\widehat{a^*a}|_{\mathrm{E}(A)}) = (\widehat{a^*a}|_{\mathrm{E}(A)})(f) = f(a^*a).\end{aligned}$$

$(ii) \Leftrightarrow (iii)$: By Lemma 3.3 (a) it is obvious, since for any $\varphi \in \mathscr{C}(\mathrm{E}(A);\mathbb{C})$ we have

$$\mu_f(\varphi) = \int_{\mathrm{E}(A)} \varphi\ \mathrm{d}\mu_f^*;\ \ \mu_g(\varphi) = \int_{\mathrm{E}(A)} \varphi\ \mathrm{d}\mu_g^*.$$

The proof is complete. □

Remark 3.9. We note that the equivalence $(i) \Leftrightarrow (ii)$ in the latter theorem can be found in van Handel's work [9, Proposition 4.1], but the proof is very different from ours. Namely, the arguments in [9] were based on a result of Gudder [8] and a representation theorem in [20], while we used our characterization Theorem 2.15, our statement Lemma 3.3 and Ky Fan's result Theorem 3.6.

4. Characterizations of absolute continuity, measure algebras of compact groups

By the aid of singularity, we characterize the absolute continuity of positive functionals defined on the measure algebra of a compact (always Hausdorff) group. The significance of this is that the measure algebra is a kind of Banach *-algebra which is neither a C^*-algebra, nor commutative in general. In the last two sections of the paper we will apply our statements to achieve results on the Lebesgue decomposition's uniqueness and on faithful positive functionals (Theorems 5.15, 6.1 and Corollary 6.2).

Now we recall some fundamental concepts and results in the context of the measure and the Hilbert algebra of a compact group. The reader is referred to the works [3], [6], [7], [12], [13], [14], [16], [17].

Let G be a compact group. Denote by β the normalized Haar measure (*functional*, or *integral*, cf. Theorem 3.2) on G, and let $\mathscr{C}_{\mathbb{C}}(G;\beta)$ be the convolution *-algebra of the $G \to \mathbb{C}$ continuous functions (we set "$*_\beta$" for the convolution and "*" for the involution).

- We use the notation $(L^1_{\mathbb{C}}(G;\beta), \|\cdot\|_{\beta,1})$ for the *measure algebra* of G, i. e., the completion of the *-algebra $\mathscr{C}_{\mathbb{C}}(G;\beta)$ equipped with the norm

$$\mathscr{C}_{\mathbb{C}}(G;\beta) \ni \varphi \mapsto \beta(|\varphi|) =: \int_G |\varphi|\ \mathrm{d}\beta.$$

The involution of the Banach *-algebra $L^1_{\mathbb{C}}(G;\beta)$ is isometric and proper (that is, $\varphi^* *_\beta \varphi \neq 0$ if $\varphi \in L^1_{\mathbb{C}}(G;\beta)\setminus\{0\}$). Furthermore, $L^1_{\mathbb{C}}(G;\beta)$ admits an approximate identity $(\varphi_i)_{i\in I}$ of continuous functions in the closed unit ball such that for any $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ the nets $(\varphi_i *_\beta \varphi)_{i\in I}$ and $(\varphi *_\beta \varphi_i)_{i\in I}$ converge uniformly to φ on G.

We also note that *every* positive functional on $L^1_{\mathbb{C}}(G;\beta)$ is continuous and representable ([17, Corollary 11.3.8]).

- We use the notation $(L^2_{\mathbb{C}}(G;\beta),(\cdot|\cdot)_{\beta,2})$ for the *Hilbert algebra* of G, i. e., the completion of the pre-Hilbert space $\mathscr{C}_{\mathbb{C}}(G;\beta)$ equipped with the inner product

$$\mathscr{C}_{\mathbb{C}}(G;\beta)\times\mathscr{C}_{\mathbb{C}}(G;\beta) \ni (\varphi,\psi) \mapsto \beta(\varphi\overline{\psi}) =: \int_G \varphi\overline{\psi}\,\mathrm{d}\beta =: (\varphi|\psi)_{\beta,2},$$

which is also a Banach *-algebra under the extensions of the convolution and involution (isometric and proper, as well), moreover it can be embedded into $L^1_{\mathbb{C}}(G;\beta)$.

Furthermore, the above mentioned approximate identity for $L^1_{\mathbb{C}}(G;\beta)$ is an approximate identity in $L^2_{\mathbb{C}}(G;\beta)$, as well.

We note that the following property holds for any $\varphi_1,\varphi_2,\varphi_3 \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ (see [17, page 1484]):

$$(\varphi_1^* *_\beta \varphi_2|\varphi_3)_{\beta,2} = (\varphi_2|\varphi_1 *_\beta \varphi_3)_{\beta,2} = (\varphi_2 *_\beta \varphi_3^*|\varphi_1)_{\beta,2}.$$

Let us fix a nonzero positive functional p on $L^1_{\mathbb{C}}(G;\beta)$ until Lemma 4.1. The representation π_p associated to p is nondegenerate, hence by well-known results we infer that there exists a unique continuous unitary representation V_p of G on the Hilbert space $\mathcal{H}_p$ such that for every $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ and $\xi,\eta \in \mathcal{H}_p$ the equation

$$(\pi_p(\varphi)\xi|\eta)_p = \int_G \varphi(s)(V_p(s)\xi|\eta)_p\,\mathrm{d}\beta(s) = \beta(\varphi(\cdot)(V_p(\cdot)\xi|\eta)_p)$$

holds (cf. [17, page 1388]). In particular, we have that

$$\begin{aligned} p(\varphi) = (\pi_p(\varphi)\xi_p|\xi_p)_p &= \int_G \varphi(s)(V_p(s)\xi_p|\xi_p)_p\,\mathrm{d}\beta(s) \\ &= \int_G \varphi(s)\overline{\overline{(V_p(s)\xi_p|\xi_p)_p}}\,\mathrm{d}\beta(s). \end{aligned} \tag{4.1}$$

Define

$$\varphi_p : G \to \mathbb{C};\ s \mapsto \overline{(V_p(s)\xi_p|\xi_p)_p}. \tag{4.2}$$

It is easy to check that this function is continuous, self-adjoint (and what is more, *positive-definite*), moreover for any $\lambda \in \mathbb{R}^+$ the equation $\varphi_{\lambda p} = \lambda\varphi_p$ holds (see [17, page 1389]).

It is also well-known that π_p is topologically irreducible if and only if V_p is a topologically irreducible representation of G ([17, page 1391, 1392]). In this case, from Theorem 12.4.3 in [17] we conclude that $\mathcal{H}_p$ is finite-dimensional (hence we can omit the word "topologically"), moreover there is a positive constant

$\lambda_p \in \mathbb{R}^+$ such that $\lambda_p\varphi_p$ is projection in the convolution *-algebra $\mathscr{C}_{\mathbb{C}}(G;\beta)$ (in fact $\lambda_p = \frac{\dim(\mathcal{H}_p)}{\|\xi_p\|_p^2}$, see chapter 4, page 943 in [7]).

Furthermore, for any nonzero positive functional f on $L^1_{\mathbb{C}}(G;\beta)$ there exists a nonzero positive functional p on $L^1_{\mathbb{C}}(G;\beta)$ such that $p \leq f$ and π_p is an irreducible representation.

Now let R_{φ_p} be the right multiplication operator on $L^2_{\mathbb{C}}(G;\beta)$ associated to φ_p, i.e., for $\psi \in L^2_{\mathbb{C}}(G;\beta)$, $R_{\varphi_p}(\psi) := \psi *_\beta \varphi_p$. This operator is continuous ([13]), moreover it is positive, since (4.1) and the self-adjointness of φ_p imply for any $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ that

$$\begin{aligned}0 \leq p(\psi^* *_\beta \psi) &= \int_G (\psi^* *_\beta \psi)\overline{\varphi_p}\,\mathrm{d}\beta \\ &= (\psi^* *_\beta \psi|\varphi_p)_{\beta,2} = (\psi|\psi *_\beta \varphi_p)_{\beta,2} = (\psi *_\beta \varphi_p|\psi)_{\beta,2} = (R_{\varphi_p}(\psi)|\psi)_{\beta,2}.\end{aligned}$$

Furthermore, if φ_p is a projection, then R_{φ_p} is also a projection. Indeed, it is positive, moreover for any $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$

$$R_{\varphi_p}(R_{\varphi_p}(\psi)) = (\psi *_\beta \varphi_p) *_\beta \varphi_p = \psi *_\beta \varphi_p = R_{\varphi_p}(\psi),$$

hence it is idempotent on a dense subspace.

Denote by L the *left regular representation* of $\mathscr{C}_{\mathbb{C}}(G;\beta)$ on the Hilbert algebra $L^2_{\mathbb{C}}(G;\beta)$, i.e., for any $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ and $\varphi \in L^2_{\mathbb{C}}(G;\beta)$ let $L_\psi(\varphi) := \psi *_\beta \varphi$. Let

$$\begin{aligned}\mathcal{H}_{\varphi_p} :&= \overline{\{L_\psi(\varphi_p)|\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)\}} = \overline{\{\psi *_\beta \varphi_p|\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)\}} = \\ &= \overline{\{R_{\varphi_p}(\psi)|\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)\}} = \overline{\operatorname{ran}} R_{\varphi_p},\end{aligned}$$

where the closures stand for the closure in $L^2_{\mathbb{C}}(G;\beta)$. This closed linear subspace $\mathcal{H}_{\varphi_p}$ of $L^2_{\mathbb{C}}(G;\beta)$ is invariant under the continuous operators L_ψ, so denote by $L|\mathcal{H}_{\varphi_p}$ the subrepresentation of L on $\mathcal{H}_{\varphi_p}$.

Since $\mathscr{C}_{\mathbb{C}}(G;\beta)$ is dense in $L^1_{\mathbb{C}}(G;\beta)$, the operator

$$\begin{aligned}&U_0 : \mathcal{H}_p \to \mathcal{H}_{\varphi_p}; \\ &\psi + \ker\|\cdot\|_p^\bullet \mapsto \psi *_\beta \varphi_p = L_\psi(\varphi_p) = R_{\varphi_p}(\psi) \qquad (\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta))\end{aligned} \tag{4.3}$$

is densely and well-defined, moreover continuous. Indeed, the bounded operator R_{φ_p} is positive, thus for any $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ we conclude

$$\begin{aligned}\|\sqrt{R_{\varphi_p}}\|^2\|\psi + \ker\|\cdot\|_p^\bullet\|_p^2 &= \|\sqrt{R_{\varphi_p}}\|^2 p(\psi^* *_\beta \psi) = \|\sqrt{R_{\varphi_p}}\|^2(\pi_p(\psi^* *_\beta \psi)\xi_p|\xi_p)_p \\ &= \|\sqrt{R_{\varphi_p}}\|^2 \int_G (\psi^* *_\beta \psi)\overline{\varphi_p}\,\mathrm{d}\beta \\ &= \|\sqrt{R_{\varphi_p}}\|^2(\psi^* *_\beta \psi|\varphi_p)_{\beta,2} \\ &= \|\sqrt{R_{\varphi_p}}\|^2(\psi|\psi *_\beta \varphi_p)_{\beta,2} = \|\sqrt{R_{\varphi_p}}\|^2(\psi *_\beta \varphi_p|\psi)_{\beta,2} \\ &= \|\sqrt{R_{\varphi_p}}\|^2(R_{\varphi_p}(\psi)|\psi)_{\beta,2} = \|\sqrt{R_{\varphi_p}}\|^2\|\sqrt{R_{\varphi_p}}(\psi)\|_{\beta,2}^2 \\ &\geq \|R_{\varphi_p}(\psi)\|_{\beta,2}^2 = \|\psi *_\beta \varphi_p\|_{\beta,2}^2,\end{aligned}$$

Hence there exists a unique $U_p : \mathcal{H}_p \to \mathcal{H}_{\varphi_p}$ continuous linear extension of U_0.

We state that for every $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$

$$U_p \circ \pi_p(\psi) = L_\psi|_{\mathcal{H}_{\varphi_p}} \circ U_p \tag{4.4}$$

holds (that is, U_p intertwines the representations $\pi_p|_{\mathscr{C}_{\mathbb{C}}(G;\beta)}$ and $L|\mathcal{H}_{\varphi_p}$), moreover $U_p(\xi_p) = \varphi_p$ is true. Indeed, first of all (4.4) is obvious from definition (4.3). As we mentioned before, $L^1_{\mathbb{C}}(G;\beta)$ admits an approximate identity $(\varphi_i)_{i\in I}$ of continuous functions in the closed ball, which is also an approximate identity for $L^2_{\mathbb{C}}(G;\beta)$. Hence from Lemma 2.5 it follows that

$$\varphi_i + \ker\|\cdot\|_p^\bullet = \pi_p(\varphi_i)\xi_p \to \xi_p,$$

thus the continuity of U_p implies that

$$U_p(\xi_p) = U_p(\lim_{i,I}(\varphi_i + \ker\|\cdot\|_p^\bullet)) = \lim_{i,I}(\varphi_i *_\beta \varphi_p) = \varphi_p. \tag{4.5}$$

We also note here that the arguments above immediately imply that $\pi_p|_{\mathscr{C}_{\mathbb{C}}(G;\beta)}$ and $L|\mathcal{H}_{\varphi_p}$ are cyclic *-representations of $\mathscr{C}_{\mathbb{C}}(G;\beta)$ with cyclic vectors ξ_p and φ_p, respectively.

If φ_p is a projection, then the operator U_p is unitary onto $\mathcal{H}_{\varphi_p}$. Indeed, let $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ be arbitrary. Then

$$\begin{aligned}(\pi_p(\psi)\xi_p|\xi_p)_p &= \int_G \psi\overline{\varphi_p}\,\mathrm{d}\beta = (\psi|\varphi_p)_{\beta,2} = \\ &= (\psi|\varphi_p *_\beta \varphi_p)_{\beta,2} = (\psi *_\beta \varphi_p|\varphi_p)_{\beta,2} = (L_\psi(\varphi_p)|\varphi_p)_{\beta,2},\end{aligned} \tag{4.6}$$

hence from [3, Proposition 2.4.1 (ii)] we conclude by means of (4.3), (4.4) and (4.5) that U_p is unitary onto $\mathcal{H}_{\varphi_p}$.

Now suppose that π_p is an irreducible representation of $L^1_{\mathbb{C}}(G;\beta)$. We show that R_{φ_p} is a finite-rank operator (projection). We may assume that φ_p is a projection, since there is a $\lambda_p \in \mathbb{R}^+$ such that $\lambda_p\varphi_p$ is a projection, moreover $R_{\lambda_p\varphi_p} = \lambda_p R_{\varphi_p}$ (see the paragraph after (4.2)). But in this case the operator U_p is unitary from $\mathcal{H}_p$ onto $\mathcal{H}_{\varphi_p} = \overline{\operatorname{ran}} R_{\varphi_p}$, and by the irreducibility the space $\mathcal{H}_p$ is finite dimensional (paragraph after (4.2)).

Now we are in position the investigate the absolute continuity of positive functionals on $L^1_{\mathbb{C}}(G;\beta)$. Our following key lemma deals with the singularity.

Lemma 4.1. *Let G be a compact group and let β be the normed Haar measure on G. Assume that p and g are nonzero singular positive functionals on $L^1_{\mathbb{C}}(G;\beta)$. Let φ_p and φ_g be the functions associated to p and g according to (4.2). Then the following statements hold.*

(a) *There exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{C}_{\mathbb{C}}(G;\beta)$ such that*

$$(\lim_{n\to+\infty} R_{\varphi_p}(\psi_n) = \varphi_p) \ \wedge \ (\lim_{n\to+\infty} R_{\varphi_g}(\psi_n) = 0),$$

that is,

$$(R_{\varphi_p}(\psi_n) = \psi_n *_\beta \varphi_p \to \varphi_p) \ \wedge \ (R_{\varphi_g}(\psi_n) = \psi_n *_\beta \varphi_g \to 0)$$

holds in the Hilbert space $L^2_{\mathbb{C}}(G;\beta)$.

(b) *If π_p is an irreducible representation of $L^1_{\mathbb{C}}(G;\beta)$, then there exists a continuous function $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ such that*

$$(g(\varphi^* *_\beta \varphi) = 0) \ \wedge \ (p(\varphi^* *_\beta \varphi) \neq 0).$$

Proof. (a): Since p and g are singular, moreover $\mathscr{C}_{\mathbb{C}}(G;\beta)$ is dense in $L^1_{\mathbb{C}}(G;\beta)$, Lemma 2.10 ($b$) implies that $p|_{\mathscr{C}_{\mathbb{C}}(G;\beta)}$ and $g|_{\mathscr{C}_{\mathbb{C}}(G;\beta)}$ are singular, as well. Hence by Theorem 1.15 (ii) there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{C}_{\mathbb{C}}(G;\beta)$ such that

$$(\lim_{n\to+\infty} \pi_p(\psi_n)\xi_p = \xi_p) \ \wedge \ (\lim_{n\to+\infty} \pi_g(\psi_n)\xi_g = 0). \tag{4.7}$$

From (4.3), (4.4) and (4.5) follow the existence of continuous operators $U_p : \mathcal{H}_p \to \mathcal{H}_{\varphi_p}$ and $U_g : \mathcal{H}_g \to \mathcal{H}_{\varphi_g}$ such that for any $\psi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ the equations

$$(U_p(\pi_p(\psi)\xi_p) = \psi *_\beta \varphi_p) \ \wedge \ (U_p(\xi_p) = \varphi_p); \qquad U_g(\pi_g(\psi)\xi_g) = \psi *_\beta \varphi_g$$

hold. Hence by (4.7) and the continuity we have

$$(\psi_n *_\beta \varphi_p \to \varphi_p) \ \wedge \ (\psi_n *_\beta \varphi_g \to 0).$$

(b): Let π_p be an irreducible representation. By the arguments after (4.2) we may assume that φ_p is a projection, since it is enough to prove the statement for a constant multiple of p (Remark 1.10).

By (a) there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{C}_{\mathbb{C}}(G;\beta)$ such that

$$(\lim_{n\to+\infty} R_{\varphi_p}(\psi_n) = \varphi_p) \ \wedge \ (\lim_{n\to+\infty} R_{\varphi_g}(\psi_n) = 0),$$

that is,

$$(R_{\varphi_p}(\psi_n) = \psi_n *_\beta \varphi_p \to \varphi_p) \ \wedge \ (R_{\varphi_g}(\psi_n) = \psi_n *_\beta \varphi_g \to 0) \tag{4.8}$$

holds true in $L^2_{\mathbb{C}}(G;\beta)$.

The set $H := \{\varphi_g^m *_\beta \varphi_p | m \in \mathbb{N}\setminus\{0\}\}$ is a subset of the finite dimensional space $\operatorname{ran} R_{\varphi_p}$ (see the last argument before the Lemma's statement), hence there exist an $m \in \mathbb{N}\setminus\{0\}$ and a non-identically zero system $(\lambda_j)_{1\leq j\leq m}$ in $\mathbb{C}$ such that

$$\sum_{j=1}^{m} \lambda_j(\varphi_g^j *_\beta \varphi_p) = 0.$$

Indeed, if for some $m \in \mathbb{N}\setminus\{0\}$ the equation $\varphi_g^m *_\beta \varphi_p = 0$ holds, then we are done. If there exist different $l, m \in \mathbb{N}\setminus\{0\}$ numbers such that $\varphi_g^m *_\beta \varphi_p = \varphi_g^l *_\beta \varphi_p$, then $\varphi_g^m *_\beta \varphi_p - \varphi_g^l *_\beta \varphi_p = 0$ is suitable. Thus the remained case is that H contains infinite nonzero vectors, but since $\operatorname{ran} R_{\varphi_p}$ is finite dimensional, these vectors are linearly dependent. So what we stated is true.

Let m_0 be the minimal index between 1 and m such that $\lambda_{m_0} \neq 0$. So we have the following identity:

$$0 = \sum_{j=m_0}^{m} \lambda_j(\varphi_g^j *_\beta \varphi_p) = \varphi_g^{m_0} *_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)),$$

where the sum on the right side is possibly void.

- We show that the function in the brackets on the right are nonzero. If we assume the contrary, that is,

$$\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p) = 0,$$

then we get

$$\lambda_{m_0}\varphi_p = -\sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p).$$

For any $n \in \mathbb{N}$ multiplying from the left by ψ_n we have

$$\lambda_{m_0}(\psi_n *_\beta \varphi_p) = -\sum_{j=m_0+1}^{m} \lambda_j(\psi_n *_\beta (\varphi_g^{j-m_0} *_\beta \varphi_p)).$$

Since $\lambda_{m_0} \neq 0$, the property (4.8) implies that the right side of the equation tends to zero, but the left side is not, which is a contradiction.
- Let M_0 be the minimal positive integer such that

$$\varphi_g^{M_0} *_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)) = 0.$$

We show that $M_0 = 1$. Indeed, if $M_0 \geq 2$, then the minimality of M_0 forces that

$$\psi := \varphi_g^{M_0-1} *_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)) \neq 0.$$

On the other hand, the selfadjointness of φ_g and $2M_0 - 2 \geq M_0$ imply

$$\begin{aligned}\psi^* *_\beta \psi = &\Big(\big(\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)\big)^* *_\beta \varphi_g^{M_0-1}\Big) *_\beta \\ &*_\beta (\varphi_g^{M_0-1} *_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p))) \\ = &\big(\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)\big)^* *_\beta \varphi_g^{2M_0-2} *_\beta \\ &*_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)) = 0,\end{aligned}$$

which is impossible, since the involution of $L^1_{\mathbb{C}}(G;\beta)$ is proper.

Thus we have the following:

($*$) $\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p) \neq 0$,

($**$) $\varphi_g *_\beta (\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)) = 0$.

We state that the element

$$\varphi := \Big(\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)\Big)^* = \overline{\lambda_{m_0}}\varphi_p + \sum_{j=m_0+1}^{m} \overline{\lambda_j}(\varphi_p *_\beta \varphi_g^{j-m_0})$$

of the convolution *-algebra $\mathscr{C}_{\mathbb{C}}(G;\beta)$ fulfills the properties

(I) : $g(\varphi^* *_\beta \varphi) = 0$,
(II) : $p(\varphi^* *_\beta \varphi) > 0$.

(I): Since φ^* is the function in $(*)$ $(\varphi^* \neq 0)$ and $(**)$ $(\varphi_g *_\beta \varphi^* = 0)$, hence the self-adjointness of φ_g and $(**)$ imply

$$g(\varphi^* *_\beta \varphi) = \int_G (\varphi^* *_\beta \varphi)\overline{\varphi_g}\,\mathrm{d}\beta = (\varphi^* *_\beta \varphi|\varphi_g)_{\beta,2} = (\varphi^*|\varphi_g *_\beta \varphi^*)_{\beta,2} = 0.$$

(II): Since φ_p is a projection, from (4.6) we obtain that

$$p(\varphi^* *_\beta \varphi) = (\pi_p(\varphi^* *_\beta \varphi)\xi_p|\xi_p)_p = (L_{(\varphi^* *_\beta \varphi)}(\varphi_p)|\varphi_p)_{\beta,2} = \|\varphi *_\beta \varphi_p\|^2_{\beta,2}.$$

Since $\|\cdot\|_{\beta,2}$ is a norm, we only have to show that $\varphi *_\beta \varphi_p$ is nonzero. But from $(*)$, $(**)$, $\lambda_{m_0} \neq 0$ and the argument below this is an immediately consequence, since the involution of $L^2(G;\beta)$ is proper:

$$\begin{aligned}
0 \neq \varphi *_\beta \varphi^* &= \Big(\lambda_{m_0}\varphi_p + \sum_{j=m_0+1}^{m} \lambda_j(\varphi_g^{j-m_0} *_\beta \varphi_p)\Big)^* *_\beta \varphi^* \\
&= \Big(\overline{\lambda_{m_0}}\varphi_p + \sum_{j=m_0+1}^{m} \overline{\lambda_j}(\varphi_p *_\beta \varphi_g^{j-m_0})\Big) *_\beta \varphi^* \\
&= \overline{\lambda_{m_0}}(\varphi_p *_\beta \varphi^*) + (\sum_{j=m_0+1}^{m} \overline{\lambda_j}(\varphi_p *_\beta \varphi_g^{j-m_0})) *_\beta \varphi^* \\
&= \overline{\lambda_{m_0}}(\varphi_p *_\beta \varphi^*) + (\sum_{j=m_0+1}^{m} \overline{\lambda_j}(\varphi_p *_\beta \varphi_g^{j-m_0} *_\beta \varphi^*)) \\
&= \overline{\lambda_{m_0}}(\varphi_p *_\beta \varphi^*) + 0 = \overline{\lambda_{m_0}}(\varphi *_\beta \varphi_p)^*.
\end{aligned}$$

The proof is complete. □

Corollary 4.2. *Let G be a compact group and β the normed Haar measure on G. Let f and g be positive functionals on $L^1_{\mathbb{C}}(G,\beta)$. If f and g are singular, then there exists $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ such that*

$$(g(\varphi^* *_\beta \varphi) = 0) \ \wedge \ (f(\varphi^* *_\beta \varphi) \neq 0).$$

Proof. As we mentioned in the second paragraph after (4.2), there exists a nonzero positive functional p on $L^1_{\mathbb{C}}(G,\beta)$ such that π_p is an irreducible *-representation and $p \leq f$. Since f and g are singular, then p and g are singular, as well (Remark 1.10). Thus by (b) of Lemma 4.1 there exists $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ such that

$$(g(\varphi^* *_\beta \varphi) = 0) \ \wedge \ (p(\varphi^* *_\beta \varphi) \neq 0).$$

From this and from $p \leq f$ it follows that $f(\varphi^* *_\beta \varphi) \neq 0$. □

We present our result on characterization of the absolute continuity of positive functionals on measure algebras of compact groups. This will be also useful during the investigations on the uniqueness of the Lebesgue decomposition (Theorem 5.15), as well as in the proof of results on faithful positive functionals (Section 6).

Theorem 4.3 (Characterization of absolute continuity)**.** *Let G be a compact group and β the normed Haar measure on G. Let f and g be positive functionals on $L^1_{\mathbb{C}}(G,\beta)$. The following statements are equivalent.*

(i) *f is absolutely continuous with respect to g.*

(ii) $\ker\|\cdot\|^\bullet_g \subseteq \ker\|\cdot\|^\bullet_f$.

Proof. $(i) \Rightarrow (ii)$: It is obvious by Remark 1.10.

$(ii) \Rightarrow (i)$: Assume that $\ker\|\cdot\|^\bullet_g \subseteq \ker\|\cdot\|^\bullet_f$ holds. Let $f = f_{reg} + f_{sing}$ be the Lebesgue decomposition of f with respect to g by Corollary 2.17. If f is not absolutely continuous with respect to g, then $f_{sing} \neq 0$ is true by Proposition 1.13. Since g and f_{sing} are singular, Corollary 4.2 implies the existence of a continuous function $\varphi \in \mathscr{C}_{\mathbb{C}}(G;\beta)$ such that

$$(g(\varphi^* *_\beta \varphi) = 0) \;\wedge\; (f_{sing}(\varphi^* *_\beta \varphi) \neq 0).$$

From $f_{sing} \leq f$ we have that $f(\varphi^* *_\beta \varphi) \neq 0$, but this contradicts (ii). The proof is complete. □

One of the definitions of absolute continuity exactly means that the densely defined operator

$$J_{g,f} : \mathcal{H}_g \to \mathcal{H}_f;\; a + \ker\|\cdot\|^\bullet_g \mapsto a + \ker\|\cdot\|^\bullet_f$$

is well-defined and closable (Remark 1.4). The previous theorem shows the strong fact that the well-definedness automatically implies the closability. This will be the key in the last two sections.

5. Uniqueness of the Lebesgue decomposition on commutative *-algebras and measure algebras of compact groups

Hassi, Sebestyén and de Snoo in [10] answered the question of uniqueness about their Lebesgue decomposition related to forms (Theorem 1.11 in this paper). Namely, if $\mathfrak{t}$ and $\mathfrak{w}$ are forms on the complex vector space E and $\mathfrak{t} = \mathfrak{t}_{reg} + \mathfrak{t}_{sing}$ is the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$ by Theorem 1.11, then are there a different form decomposition $\mathfrak{t} = \mathfrak{t}_r + \mathfrak{t}_s$ where $\mathfrak{t}_r$ is absolutely continuous with respect to $\mathfrak{w}$ and $\mathfrak{t}_s$ is singular to $\mathfrak{w}$? The answer is the next remarkable characterization ([10, Section 4]).

Proposition 5.1 (Uniqueness of the Lebesgue decomposition for forms)**.** *The Lebesgue decomposition of the form $\mathfrak{t}$ with respect to $\mathfrak{w}$ is unique if and only if $\mathfrak{w}$ dominates $\mathfrak{t}_{reg}$.*

From this result it follows that on finite dimensional complex vector spaces the Lebesgue decomposition of forms is unique. Indeed, let $\dim(E) < +\infty$. The absolute continuity of $\mathfrak{t}_{reg}$ with respect to $\mathfrak{w}$ is equivalent to that the densely defined mapping

$$J_{\mathfrak{w},\mathfrak{t}_{reg}} : \mathcal{H}_{\mathfrak{w}} \to \mathcal{H}_{\mathfrak{t}_{reg}};\ x + \ker \|\cdot\|_{\mathfrak{w}}^{\bullet} \mapsto x + \ker \|\cdot\|_{\mathfrak{t}_{reg}}^{\bullet}$$

is well-defined and closable (Remark 1.4). But the spaces are finite dimensional, hence the linear operator above is everywhere defined, thus it is continuous. So for any $x \in E$ we get

$$\mathfrak{t}_{reg}[x] = \|x + \ker \|\cdot\|_{\mathfrak{t}_{reg}}^{\bullet}\|_{\mathfrak{t}_{reg}}^2 = \|J_{\mathfrak{w},\mathfrak{t}_{reg}}(x + \ker \|\cdot\|_{\mathfrak{w}}^{\bullet})\|_{\mathfrak{t}_{reg}}^2 \leq$$
$$\leq \|J_{\mathfrak{w},\mathfrak{t}_{reg}}\|^2 \|x + \ker \|\cdot\|_{\mathfrak{w}}^{\bullet}\|_{\mathfrak{w}}^2 = \|J_{\mathfrak{w},\mathfrak{t}_{reg}}\|^2 \mathfrak{w}[x],$$

that is, $\mathfrak{w}$ dominates $\mathfrak{t}_{reg}$.

On the other hand, if $\dim(E) = +\infty$, then there exist forms $\mathfrak{t}$ and $\mathfrak{w}$ on E such that $\mathfrak{t}$ is absolutely continuous with respect to $\mathfrak{w}$ (hence $\mathfrak{t} = \mathfrak{t}_{reg}$ in the Lebesgue decomposition with respect to $\mathfrak{w}$), but $\mathfrak{w}$ not dominates $\mathfrak{t}$, thus this Lebesgue decomposition is not unique (Proposition 5.1). Indeed, let $(e_i)_{i\in I}$ be an algebraic basis in E. Choose a countably infinite subsystem $(e_j)_{j\in J}$, and let $\sigma : \mathbb{N} \to J$ be a bijection. Let $\mathfrak{t}$ and $\mathfrak{w}$ be the forms associated to the $E \times E \to \mathbb{C}$ semi-inner products determined by the following properties:

$$(e_i|e_k)_{\mathfrak{t}} := \begin{cases} 1, & \text{if } \mathrm{i} = \mathrm{k} \text{ and } \mathrm{i} \in \operatorname{ran}\sigma. \\ 0 & \text{otherwise.} \end{cases}$$

$$(e_i|e_k)_{\mathfrak{w}} := \begin{cases} \frac{1}{2^n}, & \text{if } \mathrm{i} = \mathrm{k} \text{ and } \mathrm{i} \in \operatorname{ran}\sigma,\ \sigma^{-1}(\mathrm{i}) = \mathrm{n}. \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for a fixed $n \in \mathbb{N}$ let $\mathfrak{p}_n$ be the form associated to the semi-inner product with the following property:

$$(e_i|e_k)_{\mathfrak{p}_n} := \begin{cases} 1, & \text{if } \mathrm{i} = \mathrm{k} \text{ and } \mathrm{i} = \sigma(\mathrm{n}). \\ 0 & \text{otherwise.} \end{cases}$$

Define the form $\mathfrak{t}_n$ with the formula $\mathfrak{t}_n := \sum_{m=0}^{n} \mathfrak{p}_m$. It is easy to check that $\mathfrak{w}$ not dominates $\mathfrak{t}$, moreover $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ is increasing, $\mathfrak{t}_n \leq 2^n\mathfrak{w}$ $(n \in \mathbb{N})$ and $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$, hence $\mathfrak{t}$ is absolutely continuous with respect to $\mathfrak{w}$ (Theorem 1.2). Thus on an infinite dimensional complex vector space the Lebesgue decomposition is not unique for arbitrary forms.

The case of positive functionals is more complicated. As we mentioned after Theorem 1.16, all of the Lebesgue decompositions of (representable) positive functionals (Corollary 3 in [8]; Theorem 3.5 in [15]; Theorem 1 in [11]; Theorem 1.16 (and Corollary 2.17) in this paper) are coincide with the same settings. Hence the question of the uniqueness is the following: if f and g are arbitrary representable positive functionals on the *-algebra A and $f = f_{reg} + f_{sing}$ is the Lebesgue decomposition of f with respect to g by Corollary 2.17, then are there a different representable positive functional decomposition $f = f_r + f_s$ where f_r is absolutely continuous with respect to g and f_s is singular to g? What we can definitely observe is that the inequalities $f_r \leq f_{reg}$ and $f_s \geq f_{sing}$ are true by the extremal property of the regular part (Corollary 2.17).

Since Corollary 2.17 is based on Theorem 1.11, the finite dimensional case is clear, because the argument after Proposition 5.1 implies that the decomposition is unique. On the other hand, on infinite dimensional *-algebras we cannot use Proposition 5.1, since in its proof ([10, Section 4]) the form which shows the non-uniqueness is not a positive functional induced form in general. In particular, Example 5.14 points out that the functional decomposition can be unique, while the form decomposition can not.

From the papers above we can conclude the following. Gudder in [8, paragraph after Corollary 3] noted that he could not prove the uniqueness. Henle in [11, Theorem 2] stated that his decomposition is unique. However, this statement is false in general. Namely, since on σ-finite von Neumann algebras for normal states Henle's decomposition is just the same as Kosaki's decomposition, and Kosaki's example ([15, 10.6]) shows the non-uniqueness of his decomposition, Henle's result on the uniqueness cannot be true in general.

Hence from the facts above we conclude that the Lebesgue decomposition of representable positive functionals is not unique in full generality. But then the question arises: what are the classes of *-algebras, whereon the Lebesgue decomposition of representable positive functionals is unique? Finite dimensional *-algebras are of this kind (argument after Proposition 5.1). Our purpose in this part of the paper is to show that commutative *-algebras and measure algebras of compact groups also have this property. We will use our results from Sections 3 and 4.

Our next lemma characterizes the uniqueness on general *-algebras.

Lemma 5.2 (Uniqueness of the Lebesgue decomposition for positive functionals)**.** *Let A be a *-algebra. The following statements are equivalent.*

(i) *For every f and g representable positive functionals on A the Lebesgue decomposition of f with respect to g is unique.*
(ii) *For every f, p and g representable positive functionals on A, if f is absolutely continuous with respect to g and g is singular to p, then f and p are also singular.*
(iii) *For every f, p and g representable positive functionals on A, if f is absolutely continuous with respect to g, $p \leq f$ and g is singular to p, then $p = 0$.*
(iv) *For every f, t and g representable positive functionals on A, if f is absolutely continuous with respect to g and $t \leq f$, then t is absolutely continuous with respect to g.*
(v) *For every f, t and g representable positive functionals on A, if f is absolutely continuous with respect to g and t is absolutely continuous with respect to f, then t is absolutely continuous with respect to g.*

Proof. $(i) \Rightarrow (ii)$: Let f, p and g be representable positive functionals on A, and suppose that f is absolutely continuous with respect to g and g is singular to p. Let $r := f + p$; the functional r is positive and representable, since it is a sum of this kind of functionals (Theorem 2.3 (a)). The assumption implies that the Lebesgue decomposition of r with respect to g is unique, hence the regular part of r is just $f = r_{reg}$, meanwhile the singular part is $p = r_{sing}$. On the other hand, by Proposition 1.13 these functionals are singular to each other, that is f and p are singular.

$(ii) \Rightarrow (iii)$: If f, p and g are representable positive functionals on A, $p \leq f$, f is absolutely continuous with respect to g and p is singular to g, then by (ii) it follows that p is singular to f. By Remark 1.10 $p = 0$ follows.

$(iii) \Rightarrow (iv)$: Assume that t, f and g are representable positive functionals on A such that $t \leq f$ and f is absolutely continuous with respect to g. From Corollary 2.17 we have that t is a sum of representable positive functionals: $t = t_{reg} + t_{sing}$, where t_{reg} is absolutely continuous with respect to g, t_{sing} and g are singular. From $t_{sing} \leq t \leq f$ it follows that $t_{sing} \leq f$, thus by (iii) $t_{sing} = 0$, that is $t = t_{reg}$, hence t is absolutely continuous with respect to g by Proposition 1.13.

$(iv) \Rightarrow (i)$: Let f and g be representable positive functionals on A. Assume that f can be written in a sum $f = f_{reg} + f_{sing} = f_r + f_s$, where $f_{reg} + f_{sing}$ is the decomposition with respect to g by Corollary 2.17, meanwhile f_r and f_s are representable positive functionals on A such that f_r is absolutely continuous, f_s is singular with respect to g. By Corollary 2.17 $f_r \leq f_{reg}$ holds, so for the representable positive functional $f_{reg} - f_r$ (Theorem 2.3) the inequality $f_{reg} - f_r \leq f_{reg}$ is true, hence (iv) implies that $f_{reg} - f_r$ is absolutely continuous with respect to g. On the other hand, $f_{reg} - f_r = f_s - f_{sing}$ holds, and by $f_s - f_{sing} \leq f_s$ the right side of the equality is singular to g (Remark 1.10). But also from Remark 1.10 this occurs only when $f_{reg} - f_r = f_s - f_{sing} = 0$.

$(v) \Rightarrow (iv)$: It is obvious, since $t \leq f$ implies that t is absolutely continuous with respect to f (Remark 1.10).

$(ii) + (iv) \Rightarrow (v)$: Assume that t, f and g are representable positive functionals on A such that t is absolutely continuous with respect to f and f is absolutely continuous with respect to g. Let $t = t_{reg} + t_{sing}$ be the Lebesgue decomposition of t with respect to g by Corollary 2.17. Since t_{sing} and g are singular, hence by (ii) t_{sing} and f are also singular. On the other hand, due to $t_{sing} \leq t$, (iv) implies that t_{sing} is absolutely continuous with respect to f. From Remark 1.10 this comes true if $t_{sing} = 0$, i.e., $t = t_{reg}$, hence by Proposition 1.13 the proof is complete. ☐

Remark 5.3. In [11, Lemma (a), section 4] Henle stated that (ii) is always occurs for positive functionals on unital C^*-algebras. However, Kosaki's Lemma 10.5 in [15] shows that (iv) fails in generality on σ-finite von Neumann algebras. Since (ii) and (iv) are equivalent, thus the argument on the uniqueness in [11] is false.

5.1. ***G**-algebras.** Before we turn to the uniqueness on commutative *-algebras, we examine the Lebesgue decomposition on a special class of *-algebras, namely on G^*-algebras. Every Banach *-algebra (complete normed algebra with an involution) is in this class, hence it is worth to study this case. Following Palmer ([17, Section 10.1]), we introduce some definitions and theorems.

Definition 5.4. We say that the pair (B, j) is an *enveloping C^*-algebra* of the *-algebra A, if B is a C^*-algebra, $j : A \to B$ is a *-algebra morphism, and for every C^*-algebra C and for every $\pi : A \to C$ *-algebra morphism there exists a $\widetilde{\pi} : B \to C$ *-algebra morphism such that $\pi = \widetilde{\pi} \circ j$.

Proposition 5.5. *If A is a *-algebra and (B_1, j_1), (B_2, j_2) are enveloping C^*-algebras of A, then there exists a unique *-algebra morphism $\pi : B_1 \to B_2$ such that $\pi \circ j_1 = j_2$.*

From this proposition it follows that if (B, j) is an enveloping C^*-algebra of the *-algebra A, then $\operatorname{ran} j$ is a dense *-subalgebra of B.

We introduce the concept of G^*-algebras by the aid of the next theorem.

Theorem 5.6. *Let A be a *-algebra. The following are equivalent.*

(i) *There exists enveloping C^*-algebra of A.*
(ii) *There exists greatest C^*-seminorm on A.*
(iii) *There exists $r : \{a^*a | a \in A\} \to \mathbb{R}_+$ such that*
  (iii)' *for any $a \in A$ $\liminf_{n\to+\infty} r((a^*a)^{2^n})^{2^{-n}}$ is finite;*
  (iii)" *for any representable positive functional f on A and for any $b \in A$ there exists $B \in \mathbb{R}_+$ such that for every $a \in A$ $f(b^*a^*ab) \leq Br(a^*a)$ holds.*

Definition 5.7. The *-algebra A is a *G^*-algebra* if one (hence all) of the properties is fulfilled by A.

Corollary 5.8. *Every Banach *-algebra is a G^*-algebra.*

Proof. It is well known that if r stands for the spectral radius of a Banach *-algebra A, then A has the properties in *(iii)* of Theorem 5.6. □

For commutative Banach *-algebras the following theorem is also known ([17, Proposition 10.1.14]).

Theorem 5.9. *If A is commutative Banach *-algebra and $\mathrm{X}_{sa}(A)$ stands for the locally compact space of the nonzero selfadjoint characters (equipped with the Gelfand topology), then the pair $(\overline{\mathscr{K}}(\mathrm{X}_{sa}(A); \mathbb{C}), \widehat{\ })$ is an enveloping C^*-algebra of A, where $\widehat{\ } : A \to \overline{\mathscr{K}}(\mathrm{X}_{sa}(A); \mathbb{C})$ is the selfadjoint Gelfand homomorphism of A.*

The following theorem shows the connection between representable positive functionals on a G^*-algebra and positive functionals on one of its enveloping C^*-algebra ([17, Theorem 10.1.12]).

Theorem 5.10. *Let A be a G^*-algebra and let (B, j) be an enveloping C^*-algebra of A. Denote by $(B^*)_+$ the set of positive functionals on B, and let $J : (B^*)_+ \to A^*$; $F \mapsto F \circ j$. Then:*

(a) *The mapping J is a bijection between $(B^*)_+$ and the representable positive functionals on A (we say that J is the* ***canonical bijection****).*
(b) *For the Hilbert bound (see Remark 2.2) of $J(F)$ we have $\|J(F)\|_H = \|F\|$.*

The following result characterizes the concepts of the Lebesgue decomposition on G^*-algebras via enveloping C^*-algebras.

Theorem 5.11 (Lebesgue decomposition of positive functionals on G*-algebras)**.** *Let A be a G^*-algebra and let (B, j) be an enveloping C^*-algebra of A. Let $J : (B^*)_+ \to A^*$; $F \mapsto F \circ j$ be the canonical bijection (Theorem 5.10). Let f and g be representable positive functionals on A. Then the following statements hold.*

(a) *f is absolutely continuous with respect to g $\Leftrightarrow$ $J^{-1}(f)$ is absolutely continuous with respect to $J^{-1}(g)$.*
(b) *f is singular to g $\Leftrightarrow$ $J^{-1}(f)$ is singular to $J^{-1}(g)$.*
(c) *If the Lebesgue decomposition (Corollary 2.17) of f (resp. $J^{-1}(f)$) is $f = f_{reg} + f_{sing}$ (resp. $J^{-1}(f) = F_{reg} + F_{sing}$) with respect to g (resp. $J^{-1}(g)$), then:*
 (c') *$J^{-1}(f_{reg}) = F_{reg}, J(F_{reg}) = f_{reg}$;*
 (c") *$J^{-1}(f_{sing}) = F_{sing}, J(F_{sing}) = f_{sing}$.*

Proof. By Lemma 2.10 it is enough to investigate the absolute continuity and the singularity of $J^{-1}(f)$ and $J^{-1}(g)$ on the dense *-subalgebra $\operatorname{ran} j$.

(a) and (b): Since for $a \in A$

$$(J^{-1}(f))(j(a)) = ((J^{-1}(f)) \circ j)(a) = (J(J^{-1}(f)))(a) = f(a),$$
$$(J^{-1}(g))(j(a)) = ((J^{-1}(g)) \circ j)(a) = (J(J^{-1}(g)))(a) = g(a),$$

hold, thus by the definitions the statement is clear.

(c): The mapping J is order-preserving, since for positive functionals F and G on B such that $F \leq G$ we infer $0 \leq J(G - F) = (G - F) \circ j = G \circ j - F \circ j = J(G) - J(F)$.

It is enough to show that the equation $J^{-1}(f_{reg}) = F_{reg}$ is true, the other equation $J^{-1}(f_{sing}) = F_{sing}$ follows from the definition of singularity and the order-preserving.

The positive functional f_{reg} is the greatest among the representable positive functionals p on A such that $p \leq f$ hold and p is absolutely continuous with respect to g (Corollary 2.17). Since J is an order-preserving bijection, the positive functional $J^{-1}(f_{reg})$ is the greatest among the positive functionals on B what are lower or equal than $J^{-1}(f)$ and absolutely continuous with respect to $J^{-1}(g)$. Hence from the decomposition $J^{-1}(f_{reg}) = F_{reg}$ follows. □

This means that the Lebesgue decomposition theory of representable positive functionals on G^*-algebras "coincides" with the Lebesgue decomposition theory of positive functionals on C^*-algebras.

From the following theorem we are able to show the existence of a noncommutative, infinite dimensional C^*-algebra whereon the Lebesgue decomposition is unique (Corollary 5.16).

Theorem 5.12. *Let A be a G^*-algebra and let (B, j) be an enveloping C^*-algebra of A. The following statements are equivalent.*
(i) *The Lebesgue decomposition on A is unique.*
(ii) *The Lebesgue decomposition on B is unique.*

Proof. Let $J : (B^*)_+ \to A^*$; $F \mapsto F \circ j$ be the canonical bijection. We will use (v) of Lemma 5.2.

If f, g and t are arbitrary representable positive functionals on A such that t is absolutely continuous with respect to f and f is absolutely continuous with respect to g, then by Theorem 5.11 these properties hold iff $J^{-1}(t)$ is absolutely continuous with respect to $J^{-1}(f)$ and $J^{-1}(f)$ is absolutely continuous with respect $J^{-1}(g)$, as well. The statement "t is absolutely continuous with respect to

g" (which guarantees the uniqueness on A) is true if and only if $J^{-1}(t)$ is absolutely continuous with respect to $J^{-1}(g)$ (which guarantees the uniqueness on B). □

5.2. **Commutative *-algebras.** We have seen for a locally compact space T that the absolute continuity and the singularity of positive functionals defined on $\overline{\mathscr{K}}(T;\mathbb{C})$ are in strong connection with the absolute continuity and the singularity of the bounded Baire measures on T (Riesz representation theorem 3.2 and Lemma 3.3). Since the Lebesgue decomposition for set measures is unique, one may expect that the Lebesgue decomposition on $\overline{\mathscr{K}}(T;\mathbb{C})$ is also unique. Our result shows this in a more general setting.

Theorem 5.13. *Let A be a commutative *-algebra. Then for every representable positive functionals f and g on A the Lebesgue decomposition of f with respect to g is unique.*

Proof. First assume that A is a Banach *-algebra. Since every Banach *-algebra is a G^*-algebra, by Theorem 5.12 the Lebesgue decomposition on A is unique if and only if the Lebesgue decomposition on any enveloping C^*-algebra of A is unique. By Theorem 5.9 the pair $(\overline{\mathscr{K}}(\mathrm{X}_{sa}(A);\mathbb{C}),\widehat{\ })$ is an enveloping C^*-algebra of A, hence we have to prove that if T is a locally compact space, then the Lebesgue decomposition on $\overline{\mathscr{K}}(T,\mathbb{C})$ is unique. We will use Lemma 5.2 (v).

Let $\widetilde{\mu},\widetilde{\vartheta}$ and $\widetilde{\nu}$ be positive functionals on $\overline{\mathscr{K}}(T,\mathbb{C})$ such that $\widetilde{\vartheta}$ is absolutely continuous with respect to $\widetilde{\mu}$, and the latter is absolutely continuous with respect to $\widetilde{\nu}$. Then by Theorem 3.2 there exist $\mu^*,\vartheta^*,\nu^* : \mathscr{B}_0(T)\to\mathbb{R}_+$ bounded Baire measures such that the inclusion

$$\overline{\mathscr{K}}(T;\mathbb{C})\subseteq\mathscr{L}^1_{\mathbb{C}}(T,\mathscr{B}_0(T),\mu^*)\cap\mathscr{L}^1_{\mathbb{C}}(T,\mathscr{B}_0(T),\vartheta^*)\cap\mathscr{L}^1_{\mathbb{C}}(T,\mathscr{B}_0(T),\nu^*)$$

holds, and for any $\varphi\in\overline{\mathscr{K}}(T;\mathbb{C})$ the equations

$$(\widetilde{\mu}(\varphi)=\int\limits_T\varphi\ \mathrm{d}\mu^*)\ \wedge\ (\widetilde{\vartheta}(\varphi)=\int\limits_T\varphi\ \mathrm{d}\vartheta^*)\ \wedge\ (\widetilde{\nu}(\varphi)=\int\limits_T\varphi\ \mathrm{d}\nu^*)$$

are true. From the assumptions for the positive functionals and Lemma 3.3 we conclude that ϑ^* is absolutely continuous with respect to μ^* and μ^* is absolutely continuous with respect to ν^*. Since absolute continuity between set measures is a transitive property, thus we infer that ϑ^* is absolutely continuous with respect to ν^*. Thus, applying once again Lemma 3.3 it follows that $\widetilde{\vartheta}$ is absolutely continuous with respect to $\widetilde{\nu}$, that is the Lebesgue decomposition on $\overline{\mathscr{K}}(T;\mathbb{C})$ is unique, hence on any commutative Banach *-algebra, as well.

Now let A be an arbitrary commutative *-algebra, and suppose that f,t and g are representable positive functionals on A such that t is absolutely continuous with respect to f and f is absolutely continuous with respect to g. By Lemma 5.2 (v) we have to show that t is absolutely continuous with respect to g. Theorem 2.1 implies the following: *there exist C^*-seminorms $p_f,p_t,p_g : A\to\mathbb{C}$ and nonnegative numbers $M_f,M_t,M_g\in\mathbb{R}_+$ such that for every $a\in A$ the inequalities $|f(a)|\le M_fp_f(a)$, $|t(a)|\le M_tp_t(a)$, $|g(a)|\le M_gp_g(a)$ hold.* Thus for the C^*-seminorm $p := \sup\{p_f, p_g, p_t\}$ and the nonnegative number $M := \sup\{M_f, M_t, M_g\}$ we conclude for any $a \in A$

$$(|f(a)| \leq Mp(a)) \ \wedge \ (|t(a)| \leq Mp(a)) \ \wedge \ (|g(a)| \leq Mp(a)).$$

Let us denote by $(B, \|\cdot\|_p)$ the completion of the pre-C*-algebra $A/\ker p$ (equipped with the factor-norm derived from p). According to Remark 2.12, let f', t' and g' be the corresponding $B \to \mathbb{C}$ positive functionals to f, t and g, respectively, that is for $a \in A$

$$(f'(a + \ker p) := f(a)) \ \wedge \ (t'(a + \ker p) := t(a)) \ \wedge \ (g'(a + \ker p) := g(a)).$$

Theorem 2.15 shows that t is absolutely continuous with respect to g iff t' is absolutely continuous with respect to g'. On the one hand B is a commutative C^*-algebra, hence the first part of the proof implies that the Lebesgue decompostion on B is unique. Thus Lemma 5.2 shows that t' is absolutely continuous with respect to g', hence (also from Lemma 5.2) we get the uniqueness on A. $\square$

The next example shows that the Lebesgue decomposition can be unique in the context of positive functionals, while it is not unique in the context of forms.

Example 5.14. If $A := \mathscr{C}([0,1];\mathbb{C})$ is the unital C^*-algebra of the $[0,1] \to \mathbb{C}$ continuous functions and $\mu_{[0,1]}$ is the positive functional on A generated by the Lebesgue measure $\mu^*_{[0,1]}$ (on the Baire sets), then for the function

$$\varphi_{\frac{1}{2}} : [0,1] \to \mathbb{C}; \ \varphi_{\frac{1}{2}}(x) := \begin{cases} 0, & \text{if x} = 0. \\ \frac{1}{\sqrt{x}}, & \text{if x} \in]0,1]. \end{cases}$$

we have $\varphi_{\frac{1}{2}} \in \mathscr{L}^1_{\mathbb{C}}([0,1], \mathscr{B}_0([0,1]), \mu^*_{[0,1]})$, hence the mapping

$$\nu : A \to \mathbb{C}; \ a \mapsto \int\limits_{[0,1]} a\varphi_{\frac{1}{2}} \, \mathrm{d}\mu^*_{[0,1]}$$

is a positive functional on A. The Baire measure $\nu^* : \mathscr{B}_0([0,1]) \to \mathbb{R}$ (Theorem 3.2) is absolutely continuous with respect to $\mu^*_{[0,1]}$, since it is given by the Radon–Nikodym derivative. Thus Lemma 3.3 implies that ν is absolutely continuous with respect to $\mu_{[0,1]}$, that is, the Lebesgue decomposition of ν with respect to $\mu_{[0,1]}$ is $\nu = \nu_{reg}$ by Proposition 1.13. From the previous theorem it follows that this decomposition is unique in the context of positive functionals. On the other hand, it is obvious that $\mu_{[0,1]}$ not dominates ν, hence by Proposition 5.1 we obtain that the decomposition is not unique in the context of forms.

5.3. **Measure algebras of compact groups.** The uniqueness on measure algebras of compact groups is an immediately consequence of our result, Theorem 4.3.

Theorem 5.15. *The Lebesgue decomposition of positive functionals defined on the measure algebra of a compact group is unique.*

Proof. Let G be a compact group and let β be the normed Haar measure on G. By Lemma 5.2 (v) we have to show that if f, t, g are positive functionals on $L^1_{\mathbb{C}}(G, \beta)$ such that f is absolutely continuous with respect to g, t is absolutely continuous with respect to f, then t is absolutely continuous with respect to g. From Theorem 4.3 it follows that the assumptions is equivalent to the inclusions $\ker \|\cdot\|_g^\bullet \subseteq \ker \|\cdot\|_f^\bullet$ and $\ker \|\cdot\|_f^\bullet \subseteq \ker \|\cdot\|_t^\bullet$. Thus $\ker \|\cdot\|_g^\bullet \subseteq \ker \|\cdot\|_t^\bullet$ is true, but then Theorem 4.3 yields that t is absolutely continuous with respect to g. □

Since every Banach *-algebra is a G^*-algebra (Corollary 5.8), thus we have the following

Corollary 5.16. *If A stands for the measure algebra of a compact group, then the Lebesgue decomposition of positive functionals on any enveloping C^*-algebra of A is unique. In particular, if the group is noncommutative and infinite, then any enveloping C^*-algebra of A is a noncommutative, infinite dimensional C^*-algebra such that the Lebesgue decomposition of the positive functionals on it is unique.*

Proof. By the previous Theorem and Theorem 5.12 it is clear. □

We note that a common point in the proved cases of the uniqueness is that the topologically irreducible *-representations of the above-mentioned *-algebras are *finite dimensional*.

6. An application to faithful positive functionals on measure algebras of compact groups

In the end of the paper we introduce a characterization of faithful positive functionals defined on measure algebras of compact groups, by the aid of our result Theorem 4.3. As a consequence, we immediately gain a classical result on irreducible subrepresentations of measure algebras of compact groups with countable basis.

Theorem 6.1. *Let G be a compact group and β the normed Haar measure on G. Let g be a positive functional on $L^1_{\mathbb{C}}(G,\beta)$. The following statements are equivalent.*

(i) *g is faithful, that is, $g(a^* *_\beta a) = 0$ implies $a = 0$ for any $a \in L^1_{\mathbb{C}}(G,\beta)$.*
(ii) *Every positive functional f on $L^1_{\mathbb{C}}(G,\beta)$ is absolutely continuous with respect to g.*
(iii) *If p is nonzero positive functional on $L^1_{\mathbb{C}}(G,\beta)$ such that π_p is irreducible, then there exists a number $\lambda_p \in \mathbb{R}^+$ such that $\lambda_p p \leq g$.*

Proof. $(i) \Rightarrow (ii)$: If g is faithful, that is, $\ker \|\cdot\|_g^\bullet = \{0\}$, then for any positive functional f on $L^1_{\mathbb{C}}(G,\beta)$ the inclusion $\ker \|\cdot\|_g^\bullet \subseteq \ker \|\cdot\|_f^\bullet$ is true, hence Theorem 4.3 implies that f is absolutely continuous with respect to g.

$(ii) \Rightarrow (iii)$: Let p be nonzero positive functional on $L^1_{\mathbb{C}}(G,\beta)$ such that π_p is irreducible. Theorem 9.6.4 in [17] shows that this property is equivalent to that p is pure (see also the Introduction in our paper [25]). Then our result, Theorem 4 in [25] implies that g can be written in a form $g = \lambda_p p + q$ with a number $\lambda_p \in \mathbb{R}_+$ and with a positive functional q where p and q are singular. If the equation $\lambda_p = 0$ holds, then we have $g = q$, hence g and p are singular. But by Remark 1.10 this is absurd, since p is nonzero and it is absolutely continuous with respect to g from the assumptions.

$(iii) \Rightarrow (i)$: Assume indirectly that g is not faithful. Hence there is a nonzero element $a \in L^1_{\mathbb{C}}(G,\beta)$ such that $g(a^* *_\beta a) = 0$. Since the involution of $L^1_{\mathbb{C}}(G,\beta)$ is proper, then by the well known Gelfand–Raikov theorem there is a positive functional p on $L^1_{\mathbb{C}}(G,\beta)$ such that π_p is irreducible and $p(a^* *_\beta a) \neq 0$. But this contradicts (iii). □

It is well known that if a compact group has countable basis, then $L^1_{\mathbb{C}}(G,\beta)$ is separable. Hence similar arguments of 3.7 in [18] show that there exists a faithful positive functional on $L^1_{\mathbb{C}}(G,\beta)$ in this case. Our last result in the paper is a classical statement with a new proof.

Corollary 6.2. *Let G be a compact group with countable basis and β the normed Haar measure on G. Let g be a faithful positive functional on $L^1_{\mathbb{C}}(G,\beta)$. Then for any irreducible representation π of $L^1_{\mathbb{C}}(G,\beta)$ on a Hilbert space $\mathcal{H}$ there exists a subrepresentation of π_g which is unitarily equivalent with π. That is, there is a π_g-invariant closed linear subspace $\mathcal{H}_\pi$ of $\mathcal{H}_g$ and an unitary operator V of $\mathcal{H}$ onto $\mathcal{H}_\pi$ such that for any $a \in L^1_{\mathbb{C}}(G,\beta)$ the equation*

$$V \circ \pi(a) = \pi_g(a)|_{\mathcal{H}_\pi} \circ V$$

is true.

Proof. Let $\xi \in \mathcal{H}$ be any nonzero (hence cylic) vector. Let $p : L^1_{\mathbb{C}}(G,\beta) \to \mathbb{C}$ be the positive functional

$$a \mapsto (\pi(a)\xi|\xi).$$

Hence the GNS construction implies that for any $a \in L^1_{\mathbb{C}}(G,\beta)$

$$(\pi(a)\xi|\xi) = (\pi_p(a)\xi_p|\xi_p)_p. \tag{6.1}$$

Then by Proposition 2.4.1 in [3] there exists a unique unitary operator U of $\mathcal{H}$ onto $\mathcal{H}_p$ such that $U(\xi) = \xi_p$ and for any $a \in L^1_{\mathbb{C}}(G,\beta)$

$$U \circ \pi(a) = \pi_p(a) \circ U$$

holds, that is, π and π_p are unitarily equivalent. Thus π_p is irreducible (hence cyclic), as well. Our Theorem 6.1 shows the existence of a positive number $\lambda_p \in \mathbb{R}^+$ such that $\lambda_p p \leq g$. Since every positive functional on $L^1_{\mathbb{C}}(G,\beta)$ is representable, thus from Remark 2.2 it follows that $\|\lambda_p p\|_H < +\infty$. Thus by Theorem 9.4.20 in [17] there exists an operator $Q_p \in \mathscr{B}(\mathcal{H}_g)$ such that $0 \leq Q_p \leq \mathrm{id}_{\mathcal{H}_g}$, and for any $a \in L^1_{\mathbb{C}}(G,\beta)$

$$\lambda_p p(a) = ((\pi_g(a) \circ Q_p)\xi_g|Q_p(\xi_g)); \qquad \pi_g(a) \circ Q_p = Q_p \circ \pi_g(a). \tag{6.2}$$

This and (6.1) imply for the vector $\xi_\pi := \frac{1}{\sqrt{\lambda_p}} Q_p(\xi_g)$ that for every $a \in L^1_{\mathbb{C}}(G,\beta)$

$$(\pi(a)\xi|\xi) = (\pi_g(a)\xi_\pi|\xi_\pi)_p. \tag{6.3}$$

Denote by $\mathcal{H}_\pi$ the closed linear subspace $\overline{\{\pi_g(a)\xi_\pi|a \in A\}}$ in $\mathcal{H}_g$. It is a π_g-invariant subspace, hence the mapping

$$\pi_g|\mathcal{H}_\pi : L^1_{\mathbb{C}}(G,\beta) \to \mathscr{B}(\mathcal{H}_\pi) \;\; a \mapsto \pi_g(a)|_{\mathcal{H}_\pi}$$

is a subrepresentation of π_g. We state that it is unitarily equivalent with π. To see this, we note first that the cyclicity of ξ_g implies the existence of a sequence

$(a_n)_{n\in\mathbb{N}}$ in $L^1_{\mathbb{C}}(G,\beta)$ such that $\pi(a_n)\xi_g \to \xi_g$. Hence the continuity of the operator Q_p and (6.2) conclude that

$$\pi_g(a_n)\xi_\pi = \pi_g(a_n)(\frac{1}{\sqrt{\lambda_p}}Q_p(\xi_g)) = \frac{1}{\sqrt{\lambda_p}}Q_p(\pi_g(a_n)\xi_g) \to \frac{1}{\sqrt{\lambda_p}}Q_p(\xi_g) = \xi_\pi,$$

which means that $\xi_\pi \in \mathcal{H}_\pi$, so it is a cyclic vector for the subrepresentation $\pi_g|\mathcal{H}_\pi$. Thus using (6.3) and Proposition 2.4.1 in [3] there exists a unique unitary operator V of $\mathcal{H}$ onto $\mathcal{H}_\pi$ such that $V(\xi) = \xi_\pi$ and for any $a \in L^1_{\mathbb{C}}(G,\beta)$

$$V \circ \pi(a) = ((\pi_g|\mathcal{H}_\pi)(a)) \circ V$$

holds, that is, π and π_p are unitarily equivalent. □

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