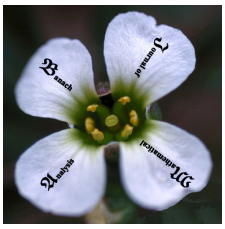

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HOMOGENIZATION IN ALGEBRAS WITH MEAN VALUE

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Abstract. In several works, the theory of strongly continuous groups is used to build a framework for solving random homogenization problems. Following this idea, we present a detailed and comprehensive framework enabling one to solve homogenization problems in algebras with mean value, regardless of whether they are ergodic or not. We also state and prove a compactness result for Young measures in these algebras. As an important achievement we study the homogenization problem associated with a stochastic Ladyzhenskaya model for incompressible viscous flow, and we present and solve a few examples of homogenization problems related to nonergodic algebras.

1. Introduction

The theory of strongly continuous N-parameter groups of operators is a very important tool in solving partial differential equations (PDEs). In [45] (see also [10]), it is used to solve PDEs in spaces of almost periodic functions. In the random homogenization theory, one constructs through a dynamical system, a strongly continuous N-parameters group. One then uses its infinitesimal generators to build up a framework for solving random homogenization problems. We refer, e.g., to the works [13, 25, 50] (see also [36]) for an exposition of this idea.

Given an algebra with mean value, the uniform continuity property of its elements allows the construction [on the generalized Besicovitch spaces associated to this algebra] of a strongly continuous N-parameters group. We therefore rely

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on the properties of this group to construct as in the random case, a comprehensive and detailed framework for solving deterministic homogenization problems as well as homogenization problems related to stochastic partial differential equations (SPDEs). The results obtained generalize the already existing ones, and provide more clarity and conciseness to the latter. It is very important to precise that some of these results have already been presented in some of our earlier works without clear justification. One very important achievement will be to work out the homogenization problems related to a stochastic Ladyzhenskaya model for incompressible non-Newtonian fluid, without help of any ergodicity assumption.

For the sake of clarity, let us state two of our main results. The first one deals with Young measures in algebras with mean value. It reads as.

Theorem 1.1. *Let Q be an open bounded subset of $\mathbb{R}^N$. Let $1 \leq p < \infty$, and let A be an algebra with mean value on $\mathbb{R}^N_y$. Finally let $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $L^p(Q;\mathbb{R}^m)$. There exist a subsequence E' from E and a family $\nu = (\nu_{x,s})_{x\in Q, s\in\Delta(A)} \in L^\infty(Q \times \Delta(A); \mathcal{P}(\mathbb{R}^m))$ such that, as $E' \ni \varepsilon \to 0$,*

$$\int_Q \Phi\left(x, \frac{x}{\varepsilon_1}, u_\varepsilon(x)\right) dx \to \int_Q \int_{\Delta(A)} \int_{\mathbb{R}^m} \widehat{\Phi}(x,s,\lambda) d\nu_{x,s}(\lambda) d\beta(s) dx$$

for all $\Phi \in E_p$, where E_p stands for the space of continuous functions $\Phi : \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ belonging to $\mathcal{C}(\overline{Q} \times \mathbb{R}^m; A)$ and for which the limit $\lim_{|\lambda|\to\infty} \frac{\Phi(x,y,\lambda)}{1+|\lambda|^p}$ exists uniformly in $(x,y) \in \overline{Q} \times \mathbb{R}^N$.

The approach used to prove the previous result is based on a result by Valadier [43] regarding the disintegration of measures.

The next one is an application to homogenization of the so-called sigma-convergence for stochastic processes' concept.

Theorem 1.2. *Assume $p \geq 3$. For each $\varepsilon > 0$ let $\mathbf{u}_\varepsilon$ be the unique solution of the following stochastic PDE:*

$$\begin{cases} d\mathbf{u}_\varepsilon + (P^\varepsilon \mathbf{u}_\varepsilon + \mathcal{A}^\varepsilon \mathbf{u}_\varepsilon + B(\mathbf{u}_\varepsilon))dt = \mathbf{f}dt + g^\varepsilon(\mathbf{u}_\varepsilon)dW, \ 0 < t < T \\ \mathbf{u}_\varepsilon(0) = \mathbf{u}^0. \end{cases}$$

Under assumption (6.26) (see Section 6), the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ converges in probability to $\mathbf{u}_0$ in $L^2(Q_T)$ where $\mathbf{u}_0$ is the unique strong probabilistic solution of the following problem:

$$\begin{cases} d\mathbf{u}_0 + [-\operatorname{div}(\mathsf{m}\nabla\mathbf{u}_0 - M(\nabla\mathbf{u}_0)) + B(\mathbf{u}_0)]dt = \mathbf{f}dt + \widetilde{g}(\mathbf{u}_0)dW \\ \mathbf{u}_0(0) = \mathbf{u}^0. \end{cases}$$

In view of the above result, one might be tempted to believe that the homogenization process for SPDEs is summarized in the homogenization of its deterministic part, added to the average of its stochastic part. This is far to be true in general. Indeed, one can obtain after passing to the limit, a homogenized equation of a type completely different from that of the initial problem; see e.g., [47].

The paper is organized as follows. In Section 2 we give the key tools which will be used in the following sections, namely we apply the semigroup theory

to the generalized Besicovitch spaces. Section 3 is devoted to the concept of Σ-convergence. In Section 4 we prove Theorem 1.1 and we give some of its important corollaries. Finally, Sections 5 and 6 are devoted to the applications of the results of the earlier sections to homogenization theory.

In the sequel, unless otherwise specified, the field of scalars acting on vector spaces is the set of complex numbers and scalar functions are complex-valued.

Finally, let us precise that the present work is a *clean* version of a more detailed version which has been posted in ArXiv under the reference "arXiv: 1207.5397v1".

2. The semigroup theory applied to the generalized Besicovitch spaces

2.1. **Preliminaries.** Let A be an algebra with mean value (algebra wmv, in short) on $\mathbb{R}^N$ [25, 14, 35, 52], that is, A is a closed subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ which contains the constants, is closed under complex conjugation ($\overline{u} \in A$ whenever $u \in A$), is translation invariant ($u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$) and is such that each element possesses a mean value in the following sense:

(MV) For each $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ (where $u^\varepsilon(x) = u(x/\varepsilon_1)$, $x \in \mathbb{R}^N$) weakly $*$-converges in $L^\infty(\mathbb{R}^N)$ to some constant function $M(u) \in \mathbb{C}$ (the complex field) as $\varepsilon \to 0$, $\varepsilon_1 = \varepsilon_1(\varepsilon)$ being a positive function of ε tending to zero with ε.

It is known that A (endowed with the sup norm topology) is a commutative $\mathcal{C}^*$-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle , \rangle$ denotes the duality pairing between A' and A. We endow $\Delta(A)$ with the relative weak$*$ topology on A'. Then using the well-known theorem of Stone (see e.g., either [26] or more precisely [19, Theorem IV.6.18, p. 274]) one can easily show that the spectrum $\Delta(A)$ is a compact topological space, and the Gelfand transformation $\mathcal{G}$ is an isometric $*$-isomorphism identifying A with $\mathcal{C}(\Delta(A))$ (the continuous functions on $\Delta(A)$) as $\mathcal{C}^*$-algebras. Next, since each element of A possesses a mean value, this yields an application $u \mapsto M(u)$ (denoted by M and called the mean value) which is a nonnegative continuous linear functional on A with $M(1) = 1$, and so provides us with a linear nonnegative functional $\psi \mapsto M_1(\psi) = M(\mathcal{G}^{-1}(\psi))$ defined on $\mathcal{C}(\Delta(A)) = \mathcal{G}(A)$, which is clearly bounded. Therefore, by the Riesz–Markov theorem, $M_1(\psi)$ is representable by integration with respect to some Radon measure β (of total mass 1) in $\Delta(A)$, called the *M-measure* for A [28]. It is evident that we have

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta \text{ for } u \in A.$$

Next, to any algebra with mean value A are associated the following subspaces: $A^m = \{\psi \in \mathcal{C}^m(\mathbb{R}^N) : D_y^\alpha \psi \in A \text{ for every } \alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \leq$

$m\}$ (where $D_y^\alpha\psi = \partial^{|\alpha|}\psi/\partial y_1^{\alpha_1}\cdots\partial y_N^{\alpha_N}$). Endowed with the norm $|||u|||_m = \sup_{|\alpha|\le m}\left\|D_y^\alpha\psi\right\|_\infty$, A^m is a Banach space. We also define the space A^∞ as the space of $\psi\in\mathcal{C}^\infty(\mathbb{R}^N_y)$ such that $D_y^\alpha\psi\in A$ for every $\alpha=(\alpha_1,\cdots,\alpha_N)\in\mathbb{N}^N$. Endowed with a suitable locally convex topology defined by the family of norms $|||\cdot|||_m$, A^∞ is a Fréchet space.

Let B_A^p $(1\le p<\infty)$ denote the Besicovitch space associated to A, that is the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p = \left(\limsup_{r\to+\infty}\frac{1}{|B_r|}\int_{B_r}|u(y)|^p\,dy\right)^{1/p}.$$

It is known that B_A^p is a complete seminormed vector space verifying $B_A^q\subset B_A^p$ for $1\le p\le q<\infty$. From this last property one may naturally define the space B_A^∞ as follows:

$$B_A^\infty = \{f\in\cap_{1\le p<\infty}B_A^p : \sup_{1\le p<\infty}\|f\|_p<\infty\}.$$

We endow B_A^∞ with the seminorm $[f]_\infty = \sup_{1\le p<\infty}\|f\|_p$, which makes it a complete seminormed space. We recall that the spaces B_A^p $(1\le p\le\infty)$ are not in general Fréchet spaces since they are not separated in general. The following properties are worth noticing (see e.g. [31, Section 2] or [35, Section 2]):

- **(1)** The Gelfand transformation $\mathcal{G}: A\to\mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping, still denoted by $\mathcal{G}$, of B_A^p into $L^p(\Delta(A))$, which in turn induces an isometric isomorphism $\mathcal{G}_1$, of $B_A^p/\mathcal{N} = \mathcal{B}_A^p$ onto $L^p(\Delta(A))$ (where $\mathcal{N}=\{u\in B_A^p : \mathcal{G}(u)=0\}$). Furthermore if $u\in B_A^p\cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u)\in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))}\le\|u\|_{L^\infty(\mathbb{R}^N)}$.
- **(2)** The mean value M viewed as defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u)=\int_{\Delta(A)}\mathcal{G}(u)d\beta$ $(u\in B_A^p)$. Furthermore, $M(\tau_a u)=M(u)$ for each $u\in B_A^p$ and all $a\in\mathbb{R}^N$, where $\tau_a u(z)=u(z+a)$ for almost all $z\in\mathbb{R}^N$. Moreover for $u\in B_A^p$ we have $\|u\|_p=[M(|u|^p)]^{1/p}$.

2.2. **The semigroup theory.** Let $1\le p\le\infty$. We consider the N-parameter group of isometries $\{T(y): y\in\mathbb{R}^N\}$ defined by

$$T(y):\mathcal{B}_A^p\to\mathcal{B}_A^p,\ T(y)(u+\mathcal{N})=\tau_y u+\mathcal{N}\ \text{for } u\in B_A^p.$$

Since the elements of A are uniformly continuous, $\{T(y): y\in\mathbb{R}^N\}$ is a strongly continuous group of operators in $\mathcal{L}(\mathcal{B}_A^p,\mathcal{B}_A^p)$ (the Banach space of continuous linear functionals of $\mathcal{B}_A^p$ into $\mathcal{B}_A^p$) in the sense of semigroups: $T(y)(u+\mathcal{N})\to u+\mathcal{N}$ in $\mathcal{B}_A^p$ as $|y|\to 0$. To $\{T(y): y\in\mathbb{R}^N\}$ is associated the following N-parameter group $\{\overline{T}(y): y\in\mathbb{R}^N\}$ defined as

$$\overline{T}(y): L^p(\Delta(A))\to L^p(\Delta(A))$$
$$\overline{T}(y)\mathcal{G}_1(u+\mathcal{N})=\mathcal{G}_1(T(y)(u+\mathcal{N}))=\mathcal{G}_1(\tau_y u+\mathcal{N})\ \text{for } u\in B_A^p.$$

The group $\{\overline{T}(y): y\in\mathbb{R}^N\}$ is also strongly continuous. The infinitesimal generator of $T(y)$ (resp. $\overline{T}(y)$) along the ith coordinate direction, denoted by $D_{i,p}$ (resp. $\partial_{i,p}$), is defined as $D_{i,p}u=\lim_{t\to 0}t^{-1}(T(te_i)u-u)$ in $\mathcal{B}_A^p$ (resp. $\partial_{i,p}v=$

$\lim_{t\to 0} t^{-1}\left(\overline{T}(te_i)v - v\right)$ in $L^p(\Delta(A)))$, where here we have used the same letter u to denote the equivalence class of an element $u \in B_A^p$ in $\mathcal{B}_A^p$, $e_i = (\delta_{ij})_{1\le j\le N}$ (δ_{ij} being the Kronecker δ). The domain of $D_{i,p}$ (resp. $\partial_{i,p}$) in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$) is denoted by $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$). By using the general theory of semigroups [19, Chap. VIII, Section 1], we easily see that the set $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is a vector subspace of $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$). Moreover $D_{i,p} : \mathcal{D}_{i,p} \to \mathcal{B}_A^p$ (resp. $\partial_{i,p} : \mathcal{W}_{i,p} \to L^p(\Delta(A))$) is a linear operator, $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is dense in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$), and the graph of $D_{i,p}$ (resp. $\partial_{i,p}$) is closed in $\mathcal{B}_A^p \times \mathcal{B}_A^p$ (resp. $L^p(\Delta(A)) \times L^p(\Delta(A))$).

In the sequel we denote by ϱ the canonical mapping of B_A^p onto $\mathcal{B}_A^p$, that is, $\varrho(u) = u + \mathcal{N}$ for $u \in B_A^p$. This being so, it is a fact that, for any $1 \le i \le N$, if $u \in A^1$ then $\varrho(u) \in \mathcal{D}_{i,p}$ and

$$D_{i,p}\varrho(u) = \varrho\left(\frac{\partial u}{\partial y_i}\right). \tag{2.1}$$

From (2.1) we deduce that $D_{i,p} \circ \varrho = \varrho \circ \partial/\partial y_i$, which means that $D_{i,p}$ is a generalization of the usual partial derivative.

One can naturally define higher order derivatives by setting $D_p^\alpha = D_{1,p}^{\alpha_1} \circ \cdots \circ D_{N,p}^{\alpha_N}$ (resp. $\partial_p^\alpha = \partial_{1,p}^{\alpha_1} \circ \cdots \circ \partial_{N,p}^{\alpha_N}$) for $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$ with $D_{i,p}^{\alpha_i} = D_{i,p} \circ \cdots \circ D_{i,p}$, α_i-times. Now, let

$$\mathcal{B}_A^{1,p} = \cap_{i=1}^N \mathcal{D}_{i,p} = \{u \in \mathcal{B}_A^p : D_{i,p}u \in \mathcal{B}_A^p \ \forall 1 \le i \le N\}$$

and

$$\mathcal{D}_A(\mathbb{R}^N) = \{u \in \mathcal{B}_A^\infty : D_\infty^\alpha u \in \mathcal{B}_A^\infty \ \forall \alpha \in \mathbb{N}^N\}.$$

It can be shown that $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^p$, $1 \le p < \infty$. We also have that $\mathcal{B}_A^{1,p}$ is a Banach space under the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left(\|u\|_p^p + \sum_{i=1}^N \|D_{i,p}u\|_p^p\right)^{1/p} \qquad (u \in \mathcal{B}_A^{1,p});$$

this comes from the fact that the graph of $D_{i,p}$ is closed.

The counter-part of the above properties also holds with

$$W^{1,p}(\Delta(A)) = \cap_{i=1}^N \mathcal{W}_{i,p} \text{ in place of } \mathcal{B}_A^{1,p}$$

and

$$\mathcal{D}(\Delta(A)) = \{u \in L^\infty(\Delta(A)) : \partial_\infty^\alpha u \in L^\infty(\Delta(A)) \ \forall \alpha \in \mathbb{N}^N\} \text{ in that of } \mathcal{D}_A(\mathbb{R}^N).$$

We have the following relation between $D_{i,p}$ and $\partial_{i,p}$.

Lemma 2.1. *Let $u \in \mathcal{D}_{i,p}$. Then $\mathcal{G}_1(u) \in \mathcal{W}_{i,p}$ and $\mathcal{G}_1(D_{i,p}u) = \partial_{i,p}\mathcal{G}_1(u)$.*

Proof. We have

$$\begin{aligned}
\left\|t^{-1}(T(te_i)u - u) - D_{i,p}u\right\|_p &= \left\|\mathcal{G}_1(t^{-1}(T(te_i)u - u)) - \mathcal{G}_1(D_{i,p}u)\right\|_{L^p(\Delta(A))} \\
&= \left\|t^{-1}(\mathcal{G}_1(T(te_i)u) - \mathcal{G}_1(u)) - \mathcal{G}_1(D_{i,p}u)\right\|_{L^p(\Delta(A))} \\
&= \left\|t^{-1}(\overline{T}(te_i)\mathcal{G}_1(u) - \mathcal{G}_1(u)) - \mathcal{G}_1(D_{i,p}u)\right\|_{L^p(\Delta(A))}.
\end{aligned}$$

Since $u \in \mathcal{D}_{i,p}$ we have $\|t^{-1}(T(te_i)u - u) - D_{i,p}u\|_p \to 0$ as $t \to 0$. Therefore

$$\left\|t^{-1}(\overline{T}(te_i)\mathcal{G}_1(u) - \mathcal{G}_1(u)) - \mathcal{G}_1(D_{i,p}u)\right\|_{L^p(\Delta(A))} \to 0 \text{ as } t \to 0,$$

so that $\mathcal{G}_1(u) \in \mathcal{W}_{i,p}$ with $\partial_{i,p}\mathcal{G}_1(u) = \mathcal{G}_1(D_{i,p}u)$. □

Let $u \in \mathcal{D}_{i,p}$ ($p \geq 1$, $1 \leq i \leq N$). Then we easily see that $D_{i,1}u = D_{i,p}u$, so that $D_{i,p}$ is the restriction to $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ of $D_{i,1}$. Therefore, for all $u \in \mathcal{D}_{i,\infty}$ we have $u \in \mathcal{D}_{i,p}$ ($p \geq 1$) and $D_{i,\infty}u = D_{i,p}u$ $\forall 1 \leq i \leq N$. We also have that $\mathcal{D}_{AP}(\mathbb{R}^N) = \varrho(AP^\infty(\mathbb{R}^N))$ (see [32, Lemma 2]). The following properties also hold true:

(i) $\int_{\Delta(A)} \partial^\alpha_\infty \widehat{u} d\beta = 0$ for all $u \in \mathcal{D}_A(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$;
(ii) $\int_{\Delta(A)} \partial_{i,p}\widehat{u} d\beta = 0$ for all $u \in \mathcal{D}_{i,p}$ and $1 \leq i \leq N$;
(iii) $D_{i,p}(u\phi) = uD_{i,\infty}\phi + \phi D_{i,p}u$ for all $(\phi, u) \in \mathcal{D}_A(\mathbb{R}^N) \times \mathcal{D}_{i,p}$ and $1 \leq i \leq N$.

It emerges from (iii) above that

$$\int_{\Delta(A)} \widehat{\phi}\partial_{i,p}\widehat{u} d\beta = -\int_{\Delta(A)} \widehat{u}\partial_{i,\infty}\widehat{\phi} d\beta \quad \forall (u, \phi) \in \mathcal{D}_{i,p} \times \mathcal{D}_A(\mathbb{R}^N).$$

This suggests us to define the concept weak derivatives of continuous linear functionals defined on A. Before we can do that, let us endow $\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$ with its natural topology defined by the family of norms

$$N_n(u) = \sup_{|\alpha| \leq n} \sup_{y \in \mathbb{R}^N} |D^\alpha_\infty u(y)|, \quad n \in \mathbb{N}.$$

In this topology, $\mathcal{D}_A(\mathbb{R}^N)$ is a Fréchet space. We denote by $\mathcal{D}'_A(\mathbb{R}^N)$ the topological dual of $\mathcal{D}_A(\mathbb{R}^N)$. We endow it with the strong dual topology. One can now define the weak derivative of $f \in \mathcal{D}'_A(\mathbb{R}^N)$ as follows: for any $\alpha \in \mathbb{N}^N$, $D^\alpha f$ stands for the generalized derivative of order α of f defined by the formula

$$\langle D^\alpha f, \phi\rangle = (-1)^{|\alpha|} \langle f, D^\alpha_\infty \phi\rangle \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

Since $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}^p_A$ ($1 \leq p < \infty$), it is immediate that $\mathcal{B}^p_A \subset \mathcal{D}'_A(\mathbb{R}^N)$ with continuous embedding, so that one may define the weak derivative of any $f \in \mathcal{B}^p_A$, and it verifies the following functional equation:

$$\langle D^\alpha f, \phi\rangle = (-1)^{|\alpha|} \int_{\Delta(A)} \widehat{f}\partial^\alpha_\infty \widehat{\phi} d\beta \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

In particular, for $f \in \mathcal{D}_{i,p}$ we have

$$-\int_{\Delta(A)} \widehat{f}\partial_{i,p}\widehat{\phi} d\beta = \int_{\Delta(A)} \widehat{\phi}\partial_{i,p}\widehat{f} d\beta \quad \forall \phi \in \mathcal{D}_A(\mathbb{R}^N),$$

so that we may identify $D_{i,p}f$ with $D^{\alpha_i}f$, $\alpha_i = (\delta_{ij})_{1 \leq j \leq N}$, δ_{ij} the Kronecker delta. Conversely, if $f \in \mathcal{B}^p_A$ is such that there exists $f_i \in \mathcal{B}^p_A$ with $\langle D^{\alpha_i}f, \phi\rangle = -\int_{\Delta(A)} \widehat{f_i}\widehat{\phi} d\beta$ for all $\phi \in \mathcal{D}_A(\mathbb{R}^N)$, then $f \in \mathcal{D}_{i,p}$ and $D_{i,p}f = f_i$.

Now, in order to deal with the homogenization theory, we need to define the space of correctors. Before we can do this, however, we need some further notions.

A function $f \in \mathcal{B}^1_A$ is said to be *invariant* if for any $y \in \mathbb{R}^N$, $T(y)f = f$. It is immediate that the above notion of invariance is the well-known one relative to

dynamical systems. An algebra with mean value will therefore said to be *ergodic* if every invariant function f is constant in $\mathcal{B}^1_A$. As in [13, Lemma 2.3 (a)] one can show that $f \in \mathcal{B}^1_A$ is invariant if and only if $D_{i,1}f = 0$ for all $1 \leq i \leq N$. We denote by I^p_A the set of $f \in \mathcal{B}^1_A$ that are invariant. The set I^p_A is a closed vector subspace of $\mathcal{B}^p_A$ satisfying the following property:

$$f \in I^p_A \text{ if and only if } D_{i,p}f = 0 \text{ for all } 1 \leq i \leq N. \tag{2.2}$$

The above property is due to the fact that $D_{i,p}$ is the restriction to $\mathcal{B}^p_A$ of $D_{i,1}$. So the mapping

$$u \mapsto \|u\|_{\#,p} := \left(\sum_{i=1}^{N} \|D_{i,p}u\|^p_p\right)^{1/p}$$

considered as defined on $\mathcal{D}_A(\mathbb{R}^N)$, is a norm on the subspace $\mathcal{D}_A(\mathbb{R}^N)/I^p_A$ of $\mathcal{D}_A(\mathbb{R}^N)$ consisting of functions $u \in \mathcal{D}_A(\mathbb{R}^N)$ that agree on I^p_A. Unfortunately, under this norm, $\mathcal{D}_A(\mathbb{R}^N)/I^p_A$ is a normed vector space which is in general not complete. We denote by $\mathcal{B}^{1,p}_{\#A}$ its completion with respect to $\|\cdot\|_{\#,p}$. Moreover, as $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}^{1,p}_A$ and further $\|u\|_{\#,p} = 0$ if and only if $u \in I^p_A$, we have that $\mathcal{B}^{1,p}_{\#A}$ is also the completion of $\mathcal{B}^{1,p}_A/I^p_A$ with respect to $\|\cdot\|_{\#,p}$. We denote by J_p the canonical embedding of $\mathcal{B}^{1,p}_A/I^p_A$ into its completion $\mathcal{B}^{1,p}_{\#A}$ (which allows us to viewed $\mathcal{B}^{1,p}_A/I^p_A$ as a subspace of $\mathcal{B}^{1,p}_{\#A}$). The following properties are due to the theory of completion of uniform spaces (see [12, Chap. II]):

- (P_1) The gradient operator $D_p = (D_{1,p}, \cdots, D_{N,p}) : \mathcal{D}_A(\mathbb{R}^N)/I^p_A \to (\mathcal{B}^p_A)^N$ extends by continuity to a unique mapping $\overline{D}_p : \mathcal{B}^{1,p}_{\#A} \to (\mathcal{B}^p_A)^N$ with the properties
$$D_{i,p} = \overline{D}_{i,p} \circ J_p$$
and
$$\|u\|_{\#,p} = \left(\sum_{i=1}^{N} \left\|\overline{D}_{i,p}u\right\|^p_p\right)^{1/p} \quad \text{for } u \in \mathcal{B}^{1,p}_{\#A}.$$
- (P_2) The space $\mathcal{D}_A(\mathbb{R}^N)/I^p_A$ (and hence $\mathcal{B}^{1,p}_A/I^p_A$) is dense in $\mathcal{B}^{1,p}_{\#A}$: in fact by the embedding J_p, $\mathcal{D}_A(\mathbb{R}^N)/I^p_A$ is viewed as a subspace of $\mathcal{B}^{1,p}_{\#A}$ (as said above), and by the theory of completion, $J_p(\mathcal{D}_A(\mathbb{R}^N)/I^p_A) \equiv \mathcal{D}_A(\mathbb{R}^N)/I^p_A$ is dense in $\mathcal{B}^{1,p}_{\#A}$.

Moreover the mapping $\overline{D}_p$ is an isometric embedding of $\mathcal{B}^{1,p}_{\#A}$ onto a closed subspace of $(\mathcal{B}^p_A)^N$, so that $\mathcal{B}^{1,p}_{\#A}$ is a reflexive Banach space. By duality we define the divergence operator $\mathrm{div}_{p'} : (\mathcal{B}^{p'}_A)^N \to (\mathcal{B}^{1,p}_{\#A})'$ ($p' = p/(p-1)$) by

$$\langle \mathrm{div}_{p'}\, u, v\rangle = -\left\langle u, \overline{D}_p v\right\rangle \text{ for } v \in \mathcal{B}^{1,p}_{\#A} \text{ and } u = (u_i) \in (\mathcal{B}^{p'}_A)^N, \tag{2.3}$$

where $\left\langle u, \overline{D}_p v\right\rangle = \sum_{i=1}^N \int_{\Delta(A)} \widehat{u}_i \partial_{i,p}\widehat{v} d\beta$. The operator $\mathrm{div}_{p'}$ just defined extends the natural divergence operator defined in $\mathcal{D}_A(\mathbb{R}^N)$ since $D_{i,p}f = \overline{D}_{i,p}(J_p f)$ for all $f \in \mathcal{D}_A(\mathbb{R}^N)$ where here, we write f under the form $f = (f - \overline{f}) + \overline{f}$ with $\overline{f} \in I^p_A$ and we know that in that case, $D_{i,p}f = D_{i,p}(f - \overline{f})$, so that $\overline{D}_{i,p}(J_p f) = \overline{D}_{i,p}(J_p(f - \overline{f}))$.

Now if in (2.3) we take $u = D_{p'}w$ with $w \in \mathcal{B}^{p'}_A$ being such that $D_{p'}w \in (\mathcal{B}^{p'}_A)^N$ then this allows us to define the Laplacian operator on $\mathcal{B}^{p'}_A$, denoted here by $\Delta_{p'}$, as follows:

$$\langle \Delta_{p'}w, v\rangle = \langle \mathrm{div}_{p'}(D_{p'}w), v\rangle = -\left\langle D_{p'}w, \overline{D}_p v\right\rangle \text{ for all } v \in \mathcal{B}^{1,p}_{\#A}.$$

If in addition $v = J_p(\phi)$ with $\phi \in \mathcal{D}_A(\mathbb{R}^N)/I^p_A$ then $\langle \Delta_{p'}w, J_p(\phi)\rangle = -\langle D_{p'}w, D_p\phi\rangle$, so that, for $p = 2$, we get

$$\langle \Delta_2 w, J_2(\phi)\rangle = \langle w, \Delta_2\phi\rangle \text{ for all } w \in \mathcal{B}^2_A \text{ and } \phi \in \mathcal{D}_A(\mathbb{R}^N)/I^2_A. \tag{2.4}$$

Remark 2.2. If the algebra A is ergodic, then the space $\mathcal{B}^{1,p}_{\#A}$ is just the one defined in [31, 35] as the completion of $\mathcal{B}^{1,p}_A/\mathbb{C} = \{u \in \mathcal{B}^{1,p}_A : M(u) = 0\}$ with respect to $\|\cdot\|_{\#,p}$. Indeed in that case the elements of I^p_A are constant functions.

We end this subsection with some notations. Let $f \in \mathcal{B}^p_A$. We know that $D^{\alpha_i}f$ exists (in the weak sense) and that $D^{\alpha_i}f = D_{i,p}f$ if $f \in \mathcal{D}_{i,p}$. So we can drop the subscript p and therefore denote $D_{i,p}$ (resp. $\partial_{i,p}$) by $\overline{\partial}/\partial y_i$ (resp. ∂_i). Thus, $\overline{D}_y$ will stand for the gradient operator $(\overline{\partial}/\partial y_i)_{1\le i\le N}$ and $\overline{\mathrm{div}}_y$ for the divergence operator div_p. We will also denote the operator $\overline{D}_{i,p}$ by $\overline{\partial}/\partial y_i$, and the canonical mapping J_p will be merely denote by J, so that in property (P_1), we will have

$$\frac{\overline{\partial}}{\partial y_i} = \frac{\overline{\partial}}{\partial y_i} \circ J \quad \text{(in the above notations)}. \tag{2.5}$$

This will lead to the notation $\overline{D}_p = (\overline{\partial}/\partial y_i)_{1\le i\le N}$. Finally, we will denote the Laplacian operator on $\mathcal{B}^p_A$ by $\overline{\Delta}_y$.

3. The Σ-convergence

This section deals with two concepts of Σ-convergence: the usual one [28] which is revisited, and its generalization to stochastic processes.

3.1. The Σ-convergence revisited.

In all that follows Q is an open subset of $\mathbb{R}^N$ (integer $N \ge 1$) and A is an algebra wmv on $\mathbb{R}^N_y$. The notations are the one of the preceding section.

Definition 3.1. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ $(1 \le p < \infty)$ is said to:

(1) *weakly* Σ*-converge* in $L^p(Q)$ to some $u_0 \in L^p(Q; \mathcal{B}^p_A)$ if as $\varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon_1}\right) dx \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x,s)\widehat{f}(x,s)dxd\beta \tag{3.1}$$

for every $f \in L^{p'}(Q; A)$ $(1/p' = 1 - 1/p)$, where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G}_1 \circ (\varrho \circ f) = \mathcal{G} \circ f$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ;

(2) *strongly* Σ*-converge* in $L^p(Q)$ to some $u_0 \in L^p(Q; \mathcal{B}^p_A)$ if it is weakly Σ-convergent towards u_0 and further satisfies the following condition:

$$\|u_\varepsilon\|_{L^p(Q)} \to \|\widehat{u}_0\|_{L^p(Q\times\Delta(A))}. \tag{3.2}$$

We denote this by $u_\varepsilon \to u_0$ in $L^p(Q)$-strong Σ.

Throughout the paper the letter E will denote any ordinary sequence $E = (\varepsilon_n)$ (integers $n \geq 0$) with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. Such a sequence will be termed a *fundamental sequence*. The following result is proved exactly as its homologue in [31, Theorem 3.1].

Theorem 3.2. *Any bounded sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^p(Q)$ (where E is a fundamental sequence and $1 < p < \infty$) admits a subsequence which is weakly Σ-convergent in $L^p(Q)$.*

The next result can be proven as in [14, Theorem 4.10].

Theorem 3.3. *Any uniformly integrable sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^1(Q)$ admits a subsequence which is weakly Σ-convergent in $L^1(Q)$.*

We recall that a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(Q)$ is said to be uniformly integrable if $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^1(Q)$ and further $\sup_{\varepsilon>0}\int_X |u_\varepsilon|\,dx \to 0$ as $|X| \to 0$ (X being an integrable set in Q with $|X|$ denoting the Lebesgue measure of X).

Remark 3.4. (1) By the above definition, the uniqueness of the limit of such a sequence is ensured. (2) By [28] it is immediate that for any $u \in L^p(Q; A)$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ is strongly Σ-convergent to $\varrho(u)$.

The next result will be of capital interest in the homogenization process.

Theorem 3.5 ([37, Theorem 6])**.** *Let $1 < p, q < \infty$ and $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Assume $(u_\varepsilon)_{\varepsilon\in E} \subset L^q(Q)$ is weakly Σ-convergent in $L^q(Q)$ to some $u_0 \in L^q(Q; \mathcal{B}^q_A)$, and $(v_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ is strongly Σ-convergent in $L^p(Q)$ to some $v_0 \in L^p(Q; \mathcal{B}^p_A)$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is weakly Σ-convergent in $L^r(Q)$ to $u_0 v_0$.*

The following result will be of great interest in practise. It is a mere consequence of the preceding theorem.

Corollary 3.6. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^{p'}(Q) \cap L^\infty(Q)$ ($1 < p < \infty$ and $p' = p/(p-1)$) be two sequences such that:*

(i) *$u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ;*
(ii) *$v_\varepsilon \to v_0$ in $L^{p'}(Q)$-strong Σ;*
(iii) *$(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^\infty(Q)$.*

Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(Q)$-weak Σ.

The next result gives the characterization of the Σ-limit of sequences involving gradients.

Theorem 3.7 ([18, Theorem 3.3])**.** *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $W^{1,p}(\Omega)$. Then there exist a subsequence E' of E, and a couple $(u_0, u_1) \in W^{1,p}(\Omega; I^p_A) \times L^p(\Omega; \mathcal{B}^{1,p}_{\#A})$ such that, as $E' \ni \varepsilon \to 0$,*

(i) *$u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ;*
(ii) *$\partial u_\varepsilon/\partial x_i \to \partial u_0/\partial x_i + \overline{\partial} u_1/\partial y_i$ in $L^p(Q)$-weak Σ, $1 \leq i \leq N$.*

3.2. **The Σ-convergence for stochastic processes.** In order to deal with homogenization problems related to *stochastic* PDEs we need to give a suitable notion of Σ-convergence adapted to stochastic processes. In all that follows, Q and T are as above.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ will throughout be denoted by $\mathbb{E}$. Let us first recall the definition of the Banach space of bounded $\mathcal{F}$-measurable functions. Denoting by $F(\Omega)$ the Banach space of all bounded functions $f : \Omega \to \mathbb{R}$ (with the sup norm), we define $B(\Omega)$ as the closure in $F(\Omega)$ of the vector space $H(\Omega)$ consisting of all finite linear combinations of the characteristic functions 1_X of sets $X \in \mathcal{F}$. Since $\mathcal{F}$ is an σ-algebra, $B(\Omega)$ is the Banach space of all bounded $\mathcal{F}$-measurable functions. Likewise we define the space $B(\Omega; Z)$ of all bounded $(\mathcal{F}, B_Z)$-measurable functions $f : \Omega \to Z$, where Z is a Banach space endowed with the σ-algebra of Borelians B_Z. The tensor product $B(\Omega) \otimes Z$ is a dense subspace of $B(\Omega; Z)$: this follows from the obvious fact that $B(\Omega)$ can be viewed as a space of continuous functions over the *gamma-compactification* [51] of the measurable space $(\Omega, \mathcal{F})$, which is a compact topological space. Next, for X a Banach space, we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; X)$ the space of X-valued random variables u such that $\|u\|_X$ is $L^p(\Omega, \mathcal{F}, \mathbb{P})$-integrable.

This being so, we still recall some preliminary as in the preceding subsection. Let A_y and A_τ be two algebras wmv on $\mathbb{R}^N_y$ and $\mathbb{R}_\tau$ respectively, and let $A = A_y \odot A_\tau$ be their product [28, 31, 49]. We know that A is the closure in $BUC(\mathbb{R}^{N+1}_{y,\tau})$ of the tensor product $A_y \otimes A_\tau$. We denote by $\Delta(A_y)$ (resp. $\Delta(A_\tau)$, $\Delta(A)$) the spectrum of A_y (resp. A_τ, A). The same letter $\mathcal{G}$ will denote the Gelfand transformation on A_y, A_τ and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) are denoted by s (resp. s_0). The M-measure on the compact space $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) is denoted by β_y (resp. β_τ). We know that $\Delta(A) = \Delta(A_y) \times \Delta(A_\tau)$ and the M-measure on $\Delta(A)$ is the product measure $\beta = \beta_y \otimes \beta_\tau$. Points in Ω are as usual denoted by ω. Unless otherwise stated, random variables will always be considered on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We keep using the previous notations.

Definition 3.8. A sequence of random variables $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(Q_T))$ $(1 \le p < \infty)$ is said to *weakly* Σ*-converge* in $L^p(Q_T \times \Omega)$ to some random variable $u_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(Q_T; \mathcal{B}^p_A))$ if as $\varepsilon \to 0$, we have

$$\begin{aligned} &\int_{Q_T \times \Omega} u_\varepsilon(x,t,\omega) f\left(x, t, \tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, \omega\right) dxdtd\mathbb{P} \\ &\qquad \to \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{u}_0(x,t,s,s_0,\omega) \widehat{f}(x,t,s,s_0,\omega) dxdtd\mathbb{P}d\beta \end{aligned} \tag{3.3}$$

for every $f \in L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(Q_T; A))$ $(1/p' = 1 - 1/p)$, where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G}_1 \circ (\varrho \circ f) = \mathcal{G} \circ f$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q_T \times \Omega)$-weak Σ.

In order to simplify the notation, we will henceforth denote $L^p(\Omega, \mathcal{F}, \mathbb{P}; X)$ merely by $L^p(\Omega; X)$ if it is understood from the context and there is no fear of confusion.

The proofs of the following results are copied on that of their counterparts in [33] (see especially Theorems 2 and 3 therein).

Theorem 3.9. *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega; L^p(Q_T))$ be a sequence of random variables verifying the following boundedness condition:*

$$\sup_{\varepsilon\in E} \mathbb{E}\, \|u_\varepsilon\|^p_{L^p(Q_T)} < \infty.$$

Then there exists a subsequence E' from E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ is weakly Σ-convergent in $L^p(Q_T \times \Omega)$.

Theorem 3.10. *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega; L^p(0,T; W^{1,p}(Q)))$ be a sequence of random variables which satisfies the following estimate:*

$$\sup_{\varepsilon\in E} \mathbb{E}\, \|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}(Q))} < \infty.$$

Then there exist a subsequence E' of E and a couple of random variables (u_0, u_1) with $u_0 \in L^p(\Omega; L^p(0,T; W^{1,p}(Q; I^p_A)))$ and $u_1 \in L^p(\Omega; L^p(Q_T; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#A_y})))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p(Q_T \times \Omega)\text{-weak } \Sigma$$

and

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \text{ in } L^p(Q_T \times \Omega)\text{-weak } \Sigma,\ 1 \le i \le N.$$

Theorem 3.10 will be very useful in the last section of this work.

4. Young measures generated by an algebra with mean value

In this section we assume that the algebra A is separable. This assumption is not fundamental, but it is made just to simplify the presentation of the foregoing section.

Let E_p $(1 \le p < \infty)$ denote the space of continuous functions $\Phi : \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ satisfying the following conditions:

(C1) $\Phi \in \mathcal{C}(\overline{Q} \times \mathbb{R}^m; A)$

(C2) $\lim_{|\lambda|\to\infty} \frac{\Phi(x,y,\lambda)}{1+|\lambda|^p}$ exists uniformly in $(x,y) \in \overline{Q} \times \mathbb{R}^N$.

Let $K = \widehat{\mathbb{R}^m}$ be the Alexandroff one point compactification of $\mathbb{R}^m$: $\widehat{\mathbb{R}^m} = \mathbb{R}^m \cup \{\infty\}$. Each element Φ of E_p extends to a unique element Ψ of $\mathcal{C}(\overline{Q} \times K; A)$ as follows:

$$\Psi(x,y,\lambda) = \begin{cases} \frac{\Phi(x,y,\lambda)}{1+|\lambda|^p} \text{ if } (x,y,\lambda) \in \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^m \\ \lim_{|\lambda|\to\infty} \frac{\Phi(x,y,\lambda)}{1+|\lambda|^p} \text{ if } \lambda = \infty. \end{cases}$$

Besides Ψ verifies the property that there exists $c > 0$ (depending on Φ) such that

$$|\Psi(x,y,\lambda)| \le c(1 + |\lambda|^p) \quad \forall (x,y,\lambda) \in \overline{Q} \times \mathbb{R}^N \times K.$$

On the other hand, the Gelfand transformation $\mathcal{G}$ being an isometric isomorphism of A onto $\mathcal{C}(\Delta(A))$, we construct an isometric isomorphism of $\mathcal{C}(\overline{Q} \times K; A)$ onto $\mathcal{C}(\overline{Q} \times K; \mathcal{C}(\Delta(A))) = \mathcal{C}(\overline{Q} \times \Delta(A) \times K)$ (which is separable), so that E_p is separable.

Equipped with the norm

$$\|\Phi\| = \sup_{x\in\overline{Q},y\in\mathbb{R}^N,\lambda\in\mathbb{R}^m} \frac{|\Phi(x,y,\lambda)|}{1+|\lambda|^p},$$

E_p is a Banach space. We denote by $\mathcal{P}(\mathbb{R}^m)$ the space of probability measures on $\mathbb{R}^m$. This being so, we have the following result.

Theorem 4.1. *Let Q be an open bounded subset of $\mathbb{R}^N$. Let $1 \le p < \infty$, and let A be an algebra wmv on $\mathbb{R}^N_y$. Finally let $(u_\varepsilon)_{\varepsilon\in E}$ (E being a fundamental sequence) be a bounded sequence in $L^p(Q;\mathbb{R}^m)$. There exist a subsequence E' from E and a family $\nu = (\nu_{x,s})_{x\in Q,s\in\Delta(A)} \in L^\infty(Q\times\Delta(A);\mathcal{P}(\mathbb{R}^m))$ such that, as $E' \ni \varepsilon \to 0$,*

$$\int_Q \Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx \to \int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m} \widehat{\Phi}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx \tag{4.1}$$

for all $\Phi \in E_p$.

Proof. For fixed $\Psi_0 \in \mathcal{C}(\overline{Q}\times K;A)$, let us define μ_ε ($\varepsilon\in E$) as follows:

$$\left\langle \mu_\varepsilon, \widehat{\Psi}_0\right\rangle = \int_Q \Psi_0\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx.$$

We have

$$\left|\left\langle \mu_\varepsilon, \widehat{\Psi}_0\right\rangle\right| \le \int_Q \sup_{y,\lambda}|\Psi_0(x,y,\lambda)|\,dx.$$

$\mathcal{G}$ being an isometric isomorphism of A onto $\mathcal{C}(\Delta(A))$ and as $\Psi_0(x,\cdot,\lambda)\in A$, we have $\sup_{y\in\mathbb{R}^N}|\Psi_0(x,y,\lambda)| = \sup_{s\in\Delta(A)}\left|\widehat{\Psi}_0(x,s,\lambda)\right|$, hence

$$\left|\left\langle \mu_\varepsilon, \widehat{\Psi}_0\right\rangle\right| \le \int_Q \sup_{s,\lambda}\left|\widehat{\Psi}_0(x,s,\lambda)\right|dx = \left\|\widehat{\Psi}_0\right\|_{L^1(Q;\mathcal{C}(\Delta(A)\times K))}.$$

Thus μ_ε (continuous functional on the subspace $\{t\widehat{\Psi}_0 : t\in\mathbb{R}\}$ of $L^1(Q;\mathcal{C}(\Delta(A)\times K))$) extends (in a non unique way) to a continuous linear form on $L^1(Q;\mathcal{C}(\Delta(A)\times K))$, denoted by $\widetilde{\mu}_\varepsilon$, and satisfying

$$\|\widetilde{\mu}_\varepsilon\|_{L^\infty(Q;\mathcal{M}(\Delta(A)\times K))} \le 1 \quad (\varepsilon\in E)$$

where $\mathcal{M}(\Delta(A)\times K)$ (the dual space of $\mathcal{C}(\Delta(A)\times K)$) is the space of Radon measures defined on the compact space $\Delta(A)\times K$. Because of the Banach-Alaoglu theorem, there exist a subsequence $E'(\Psi_0)$ of E and some $\mu\in L^\infty(Q;\mathcal{M}(\Delta(A)\times K))$ such that, as $E'(\Psi_0)\ni\varepsilon\to 0$,

$$\widetilde{\mu}_\varepsilon \to \mu \text{ in } L^\infty(Q;\mathcal{M}(\Delta(A)\times K))\text{-weak}*.$$

In particular we have, as $E'(\Psi_0)\ni\varepsilon\to 0$,

$$\int_Q \Psi_0\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx \to \int_Q\int_{\Delta(A)\times K}\widehat{\Psi}_0(x,s,\lambda)d\mu_x(s,\lambda)dx. \tag{4.2}$$

E_p being separable, let $\{\Psi_k : k\in\mathbb{N}\}$ be a countable dense subset of E_p. For simplification we put $E = (\varepsilon_n)_{n\in\mathbb{N}}$. We have in hand a family $\{E'(\Psi_k) : k\in\mathbb{N}\}$ of subsequences of E obtained by repeating the argument used to get (4.2) and satisfying the following relation: $E'(\Psi_{k+1})\subset E'(\Psi_k)$ for each $k\in\mathbb{N}$. By

the well-known diagonal process we construct a subsequence E' from the family $\{E'(\Psi_k) : k \in \mathbb{N}\}$ satisfying, as $E' \ni \varepsilon \to 0$,

$$\int_Q \Psi_k\left(x, \frac{x}{\varepsilon_1}, u_\varepsilon(x)\right) dx \to \int_Q \int_{\Delta(A)\times K} \widehat{\Psi}_k(x,s,\lambda) d\mu_x(s,\lambda) dx \ \forall k \in \mathbb{N}. \tag{4.3}$$

By a mere routine we get (4.3) by replacing Ψ_k with any Ψ in E_p. It is evident that, for a.e. $x \in Q$, μ_x is a probability measure: in fact, taking in (4.3) $\Psi \equiv 1$, we are led (by the uniqueness of the limit) to $\int_{\Delta(A)\times K} d\mu_x(s,\lambda) = 1$.

Next, using the same argument as the one used in the proof of [42, Theorem 7] we see that the boundedness of $(u_\varepsilon)_{\varepsilon\in E}$ in $L^p(Q;\mathbb{R}^m)$ implies that μ (thus constructed) is supported by $Q \times \Delta(A) \times \mathbb{R}^m$, so that

$$\int_Q \int_{\Delta(A)\times K} \widehat{\Psi}(x,s,\lambda) d\mu_x(s,\lambda) dx = \int_Q \int_{\Delta(A)\times\mathbb{R}^m} \widehat{\Psi}(x,s,\lambda) d\mu_x(s,\lambda) dx$$

for all $\Psi \in \mathcal{C}(\overline{Q};\mathcal{C}(K;A))$. Thus, by [42, Theorem 3], μ is the weak $*$-limit of $(\widetilde{\mu}_\varepsilon)_{\varepsilon\in E'}$ in $L^\infty(Q;\mathcal{M}(\Delta(A)\times\mathbb{R}^m))$ and thereby, defines a family of probability measures $(\mu_x)_{x\in Q}$ with $\mu_x \in \mathcal{M}(\Delta(A)\times\mathbb{R}^m)$. Let ν_x denote the projection of μ_x onto $\Delta(A)$. Let us show that $\nu_x = \beta$. For that, let p_1 denote the projection of $\Delta(A)\times\mathbb{R}^m$ onto $\Delta(A)$: $p_1(s,\lambda) = s$, $(s,\lambda) \in \Delta(A)\times\mathbb{R}^m$. We have the obvious equality

$$\int_{\Delta(A)\times\mathbb{R}^m} (g\circ p_1)(s,\lambda) d\mu_x(s,\lambda) = \int_{\Delta(A)} g(s) d\nu_x(s) \quad (g \in \mathcal{C}(\Delta(A))). \tag{4.4}$$

This being so, let $h \in A$, and let $\varphi \in \mathcal{K}(Q)$ (the space continuous functions on $\mathbb{R}^N$ with compact support contained in Q). Set

$$\Phi(x,y,\lambda) = \varphi(x)h(y) \quad (x \in Q, y \in \mathbb{R}^N, \lambda \in \mathbb{R}^m).$$

Then $\Phi \in \mathcal{C}(\overline{Q};\mathcal{C}(K;A))$ and so, as $E' \ni \varepsilon \to 0$,

$$\int_Q \varphi(x) h\left(\frac{x}{\varepsilon_1}\right) dx \to \int_Q \int_{\Delta(A)\times\mathbb{R}^m} \varphi(x)\widehat{h}(s) d\mu_x(s,\lambda).$$

On the other hand, we have, as $E' \ni \varepsilon \to 0$,

$$\int_Q \varphi(x) h\left(\frac{x}{\varepsilon_1}\right) dx \to \int_Q \int_{\Delta(A)} \varphi(x)\widehat{h}(s) d\beta(s) dx.$$

We deduce that

$$\int_{\Delta(A)\times\mathbb{R}^m} \widehat{h}(s) d\mu_x(s,\lambda) = \int_{\Delta(A)} \widehat{h}(s) d\beta(s) \quad \text{a.e. } x \in Q,$$

or, taking into account (4.4),

$$\int_{\Delta(A)} \widehat{h}(s) d\nu_x(s) = \int_{\Delta(A)} \widehat{h}(s) d\beta(s).$$

Since the above equality holds for every $h \in A$, we deduce that $\nu_x = \beta$ a.e. $x \in Q$ (hence ν_x is homogeneous, i.e. is independent of x). Thus, using the Valadier's

result on disintegration of measures [43, Theorem 2], there exists a probability measure $\nu_{x,s}$ ($s \in \Delta(A)$) on $\mathbb{R}^m$ such that

$$\mu_x = \nu_{x,s} \otimes \beta.$$

We are therefore led to (4.1) for all $\Phi \in E_p$. This completes the proof of the theorem. □

Theorem 4.1 yields the following

Definition 4.2. The family of probability measures $\{\nu_{x,s}\}_{x\in Q, s\in\Delta(A)}$ is called the Young measure associated with $(u_\varepsilon)_{\varepsilon\in E}$ at length scale ε_1.

The concept of Young measures is weaker than the one of weak Σ-limit as shown by the following result.

Corollary 4.3. *Let $1 < p < \infty$. The function $u_0 \in L^p(Q; (\mathcal{B}^p_A)^m)$ defined by*

$$\mathcal{G}_1(u_0)(x,s) = \int_{\mathbb{R}^m} \lambda d\nu_{x,s}(\lambda) \quad ((x,s) \in Q \times \Delta(A))$$

is the weak Σ-limit of $(u_\varepsilon)_{\varepsilon\in E'}$.

Proof. Let $g \in \mathcal{K}(Q; A)$. Set

$$\Phi^i(x,y,\lambda) = g(x,y)\lambda_i \quad ((x,y,\lambda) \in Q \times \mathbb{R}^N \times \mathbb{R}^m)\ (1 \le i \le m),$$

where $\lambda = (\lambda_i)_{1\le i\le m}$. Then Φ^i is continuous on $Q\times\mathbb{R}^N\times\mathbb{R}^m$ (so is of Carathéodory's type on $Q \times \mathbb{R}^N \times \mathbb{R}^m$). Besides, as $|\lambda_i| \le 1 + |\lambda|^p$ for all $\lambda \in \mathbb{R}^m$, we have

$$\left|\Phi^i\left(x, \frac{x}{\varepsilon_1}, u_\varepsilon(x)\right)\right| \le c(1 + |u_\varepsilon(x)|^p)$$

where $c = \sup_{x\in Q, y\in\mathbb{R}^N} |g(x,y)| < \infty$. The sequence $(\Phi^i(\cdot, \cdot/\varepsilon_1, u_\varepsilon))_{\varepsilon\in E'}$ is therefore uniformly integrable since $p > 1$. We deduce from [44, Theorem 17] that, as $E' \ni \varepsilon \to 0$,

$$\int_Q g\left(x, \frac{x}{\varepsilon_1}\right) u^i_\varepsilon(x)dx \to \int_Q \int_{\Delta(A)} \int_{\mathbb{R}^m} \widehat{g}(x,s)\lambda_i d\nu_{x,s}(\lambda)d\beta(s)dx$$

where u^i_ε is the ith component of u_ε, $1 \le i \le m$. But by the definition of the weak Σ-limit, if $u^i_0 \in L^p(Q; \mathcal{B}^p_A)$ is the weak Σ-limit of $(u^i_\varepsilon)_{\varepsilon\in E'}$, then, as $E' \ni \varepsilon \to 0$,

$$\int_Q g\left(x, \frac{x}{\varepsilon_1}\right) u^i_\varepsilon(x)dx \to \int_Q \int_{\Delta(A)} \widehat{g}(x,s)\widehat{u}^i_0(x,s)d\beta(s)dx,$$

hence, comparing the above convergence results, we deduce that

$$\widehat{u}^i_0(x,s) = \int_{\mathbb{R}^m} \lambda_i d\nu_{x,s}(\lambda),$$

which completes the proof. □

Remark 4.4. Due to the preceding corollary, it is now clear that Theorem 3.9 generalizes Theorem 3.2 of the preceding section.

The next result is very useful in practice.

Theorem 4.5. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^p(Q;\mathbb{R}^m)$ $(1\le p<\infty)$ with associated Young measure $(\nu_{x,s})_{x\in Q,s\in\Delta(A)}$. The following properties hold:*

(i) *Let $\Phi : Q\times\mathbb{R}^N\times\mathbb{R}^m\to[0,+\infty)$ be a Carathéodory integrand, i.e., $\Phi(\cdot,y,\cdot)$ is continuous for all $y\in\mathbb{R}^N$ and $\Phi(x,\cdot,\lambda)$ is measurable for all $(x,\lambda)\in Q\times\mathbb{R}^m$. Assume further that*

$$\Phi(x,\cdot,\lambda)\in B^1_A\quad\forall(x,\lambda)\in Q\times\mathbb{R}^m.$$

Then

$$\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m}\widehat{\Phi}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx\le\liminf_{E\ni\varepsilon\to0}\int_Q\Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx.$$

(ii) *If in addition to* (i)*, the sequence $(\Phi(\cdot,\cdot/\varepsilon_1,u_\varepsilon))_{\varepsilon\in E}$ is uniformly integrable, then $\widehat{\Phi}(x,s,\cdot)$ is $\nu_{x,s}$-integrable for a.e. $(x,s)\in Q\times\Delta(A)$. Besides there exists $\chi\in L^1(Q;\mathcal{B}^1_A)$ such that*

$$\mathcal{G}_1(\chi)(x,s)=\int_{\mathbb{R}^m}\widehat{\Phi}(x,s,\lambda)d\nu_{x,s}(\lambda)\quad a.e.\ (x,s)\in Q\times\Delta(A)\tag{4.5}$$

and

$$\Phi(\cdot,\cdot/\varepsilon_1,u_\varepsilon)\to\chi\ \text{ in } L^1(Q)\text{-weak }\Sigma.\tag{4.6}$$

(iii) *The barycenter $(x,s)\mapsto\int_{\mathbb{R}^m}\lambda d\nu_{x,s}(\lambda)$ belongs to $L^p(Q\times\Delta(A);\mathbb{R}^m)$.*

Proof. Thanks to the density of A in B^1_A, (i) is a direct consequence of [44, Theorem 16]. As for (ii), the integrability of $\widehat{\Phi}(x,s,\cdot)$ is a consequence of [44, Theorem 17] (see also [42, 4, 3]). Let us check (4.5) and (4.6). Let $g\in\mathcal{K}(Q;A)$; define $\Psi : Q\times\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$ by $\Psi(x,y,\lambda)=g(x,y)\Phi(x,y,\lambda)$. Then Ψ is of Carathéodory's type on $Q\times\mathbb{R}^N\times\mathbb{R}^m$ and $\Psi(\cdot,\cdot/\varepsilon_1,u_\varepsilon)$ is uniformly integrable. In view of [44, Theorem 17] we have, as $E\ni\varepsilon\to0$,

$$\begin{aligned}\int_Q\Psi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx&\to\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m}\widehat{\Psi}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx\\&=\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m}\widehat{g}(x,s)\widehat{\Phi}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx.\end{aligned}$$

But, $(\Phi(\cdot,\cdot/\varepsilon_1,u_\varepsilon))_{\varepsilon\in E}$ is uniformly integrable, so because of the Theorem 3.3, there exist a subsequence E' from E and a function $\chi\in L^1(Q;\mathcal{B}^1_A)$ such that

$$\Phi\left(\cdot,\frac{\cdot}{\varepsilon_1},u_\varepsilon\right)\to\chi\text{ in }L^1(Q)\text{- weak }\Sigma\text{ as }E'\ni\varepsilon\to0.$$

Thus, for g, we have, when $E'\ni\varepsilon\to0$,

$$\int_Q g\left(x,\frac{x}{\varepsilon_1}\right)\Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx\to\int_Q\int_{\Delta(A)}\widehat{g}(x,s)\widehat{\chi}(x,s)d\beta(s)dx,$$

hence, comparing the above convergence results yields

$$\widehat{\chi}(x,s)=\int_{\mathbb{R}^m}\widehat{\Phi}(x,s,\lambda)d\nu_{x,s}(\lambda),$$

form which, (4.5) and (4.6). Finally, let us check (iii). If in (ii) we choose $\Phi(x,y,\lambda)=|\lambda|^p$ then by using Jensen's inequality we are led to

$$\begin{aligned}\int_Q\int_{\Delta(A)}\left|\int_{\mathbb{R}^m}\lambda d\nu_{x,s}(\lambda)\right|^p d\beta dx &\leq \int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m}|\lambda|^p\,d\nu_{x,s}(\lambda)d\beta(s)dx\\ &\leq \liminf_{E\ni\varepsilon\to 0}\int_Q|u_\varepsilon(x)|^p\,dx<+\infty,\end{aligned}$$

hence (iii). □

The following result characterizes the strong convergence.

Proposition 4.6. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^p(Q;\mathbb{R}^m)$ $(1\leq p<\infty)$, and let $(\nu_{x,s})_{x\in Q,s\in\Delta(A)}$ be the associated Young measure. Assume that either $v\in L^p(Q;(A)^m)$ or $v\in\mathcal{C}(\overline{Q};(B_A^p)^m)$. Then*

$$\nu_{x,s}=\delta_{\widehat{v}(x,s)}\ \textit{if and only if}\ \lim_{E\ni\varepsilon\to 0}\|u_\varepsilon-v^\varepsilon\|_{L^1(Q)^m}=0,$$

where $v^\varepsilon(x)=v(x,x/\varepsilon_1)$, $x\in Q$.

Proof. Assume that $\nu_{x,s}=\delta_{\widehat{v}(x,s)}$ where v is either in $L^p(Q;(A)^m)$ or in $\mathcal{C}(\overline{Q};(B_A^p)^m)$. Set $\Phi(x,y,\lambda)=|\lambda-v(x,y)|$; then Φ is a Carathéodory function and further, $\Phi\geq 0$. Moreover the sequence $(\Phi(\cdot,\cdot/\varepsilon_1,u_\varepsilon))_{\varepsilon\in E}$ is uniformly integrable since by setting $\varphi(t)=t^p$ $(t\geq 0)$, φ is inf-compact and

$$\begin{aligned}\int_Q\varphi\left(\left|\Phi\left(\cdot,\frac{\cdot}{\varepsilon_1},u_\varepsilon\right)\right|\right)dx &= \int_Q\left|u_\varepsilon(x)-v\left(x,\frac{x}{\varepsilon_1}\right)\right|^p dx\\ &\leq 2^p\left(\int_Q|u_\varepsilon(x)|^p\,dx+\int_Q\left|v\left(x,\frac{x}{\varepsilon_1}\right)\right|^p dx\right)\\ &\leq M\end{aligned}$$

where M is a positive constant independent of ε. Applying [part (ii) of] Theorem 4.5, we get, as $E\ni\varepsilon\to 0$,

$$\int_Q\Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx\to\int_Q\int_{\Delta(A)}\left\langle\nu_{x,s},\widehat{\Phi}(x,s,\cdot)\right\rangle d\beta(s)dx.$$

But

$$\begin{aligned}\left\langle\nu_{x,s},\widehat{\Phi}(x,s,\cdot)\right\rangle &= \left\langle\delta_{\widehat{v}(x,s)},\widehat{\Phi}(x,s,\cdot)\right\rangle=\widehat{\Phi}(x,s,\widehat{v}(x,s))\\ &= |\widehat{v}(x,s)-\widehat{v}(x,s)|=0\ \text{a.e. }(x,s)\in Q\times\Delta(A).\end{aligned}$$

On the other hand

$$\int_Q\Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx=\int_Q\left|u_\varepsilon(x)-v\left(x,\frac{x}{\varepsilon_1}\right)\right|dx=\|u_\varepsilon-v^\varepsilon\|_{L^1(Q)}.$$

Now assume that $\|u_\varepsilon-v^\varepsilon\|_{L^1(Q)}\to 0$ as $E\ni\varepsilon\to 0$. Let $g\in\mathcal{K}(Q;A)$; Applying once more [part (ii) of] Theorem 4.5 with $\Phi(x,y,\lambda)=g(x,y)\,|\lambda-v(x,y)|$ we get, when $E\ni\varepsilon\to 0$

$$\int_Q\Phi\left(x,\frac{x}{\varepsilon_1},u_\varepsilon(x)\right)dx\to\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^m}\widehat{g}(x,s)\,|\lambda-\widehat{v}(x,s)|\,d\nu_{x,s}(\lambda)d\beta(s)dx.$$

But

$$\left|\int_Q \Phi\left(x, \frac{x}{\varepsilon_1}, u_\varepsilon(x)\right) dx\right| \leq \|g\|_\infty \|u_\varepsilon - v^\varepsilon\|_{L^1(Q)} \to 0 \text{ when } E \ni \varepsilon \to 0.$$

Thus

$$\int_Q \int_{\Delta(A)} \widehat{g}(x,s) \langle \nu_{x,s}, |\lambda - \widehat{v}(x,s)| \rangle \, d\beta dx = 0 \ \forall g \in \mathcal{K}(Q;A),$$

hence $\langle \nu_{x,s}, |\lambda - \widehat{v}(x,s)| \rangle = 0$ a.e. $(x,s) \in Q \times \Delta(A)$, which leads to $\nu_{x,s} = \delta_{\widehat{v}(x,s)}$. □

5. Homogenization of a convex integral functional revisited

All function spaces and scalar functions are real-valued in this section.

5.1. **Setting of the problem.** Our main concern here is the study of the asymptotic behavior (as $\varepsilon \to 0$) of the sequence of solutions of the problems

$$\min\left\{F_\varepsilon(v) : v \in W_0^{1,p}(Q;\mathbb{R}^n)\right\}$$

where the functional F_ε is defined on $L^p(Q;\mathbb{R}^n)$ by

$$F_\varepsilon(v) = \begin{cases} \int_Q f\left(x, \frac{x}{\varepsilon}, Dv(x)\right) dx, \ v \in W_0^{1,p}(Q;\mathbb{R}^n) \\ +\infty \ \text{ elsewhere,} \end{cases} \tag{5.1}$$

Q being a bounded open set in $\mathbb{R}^N$ and $f : \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0, +\infty)$ a Carathéodory function (i.e., $f(x,\cdot,\lambda)$ is measurable and $f(\cdot,y,\cdot)$ is continuous) satisfying the following conditions:

(H_1) $f(x,y,\cdot)$ is strictly convex for almost all $y \in \mathbb{R}^N$ and for all $x \in \overline{Q}$,

(H_2) There exist three constants $p > 1$ and c_1, $c_2 > 0$ such that

$$c_1 |\lambda|^p \leq f(x,y,\lambda) \leq c_2(1 + |\lambda|^p) \tag{5.2}$$

for all $(x,\lambda) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and for almost all $y \in \mathbb{R}^N$.

From the above hypotheses, for any fixed $\varepsilon > 0$ and for $v \in L^p(Q;\mathbb{R}^{nN})$, the function $x \mapsto f\left(x, x/\varepsilon, v(x)\right)$ of Q into $\mathbb{R}_+$ (denoted by $f^\varepsilon(\cdot,\cdot,v)$), is well defined and lies in $L^1(Q)$, with

$$c_1 \|v\|^p_{L^p(Q)^{nN}} \leq \|f^\varepsilon(\cdot,\cdot,v)\|_{L^1(Q)} \leq c_2' \left(1 + \|v\|^p_{L^p(Q)^{nN}}\right)$$

where $c_2' = c_2 \max(1, |Q|)$ with $|Q| = \int_Q dx$. Hence (5.1) makes sense and, by classical arguments, there exists [15, 21] (for each fixed $\varepsilon > 0$) a unique $u_\varepsilon \in W_0^{1,p}(Q;\mathbb{R}^n)$ that realizes the infimum of F_ε on $L^p(Q;\mathbb{R}^{nN})$, i.e.,

$$F_\varepsilon(u_\varepsilon) = \min_{v \in W_0^{1,p}(Q;\mathbb{R}^n)} F_\varepsilon(v). \tag{5.3}$$

Our objective here amounts to find, under the assumption that

$$f(x,\cdot,\lambda) \in B_A^1 \text{ for all } x \in \overline{Q}, \lambda \in \mathbb{R}^{nN}, \tag{5.4}$$

(where A is an algebra wmv) a homogenized functional $\overline{F}$ such that the sequence of minimizers u_ε converges to a limit $\mathbf{u}$, which is precisely the minimizer of $\overline{F}$.

This issue has already been addressed in many papers (see in particular [2, 5, 22, 29]). In [29] the general deterministic homogenization of (5.1) is addressed,

but in separable ergodic algebras wmv using the Σ-convergence method. Here no ergodicity assumption is made on the algebra A and moreover, we use the Young measures theory to solve the problem. This reduces considerably the length of this section in contrast to what has been done so far; see e.g. [29]. So we mean here to provide by means of Young measures generated by an algebra wmv, a full study of the functional F_ε in the general framework of algebras wmv.

5.2. **Homogenization result.** Let A be an algebra wmv. Using (5.4) and some well-known results (see e.g. [29, Proposition 2.3], [48, Proposition 3.1]) one can easily define the function $f(\cdot,\cdot,\mathbf{w}) : (x,y) \to f(x,y,\mathbf{w}(x,y))$ (for $w \in L^p(Q;(B_A^p)^{nN})$), as an element of $L^1(Q;B_A^1)$ with

$$\widehat{f}(x,s,\widehat{\mathbf{w}}(x,s)) = \mathcal{G}(f(x,\cdot,\mathbf{w}(x,\cdot)))(s) \text{ a.e. in } (x,s) \in Q \times \Delta(A).$$

Now, let

$$\mathbb{F}_0^{1,p} = W_0^{1,p}(Q;\mathbb{R}^n) \times L^p(Q;(\mathcal{B}^{1,p}_{\#A})^n).$$

$\mathbb{F}_0^{1,p}$ is a Banach space under the norm

$$\|\mathbf{u}\|_{\mathbb{F}_0^{1,p}} = \left(\|u_0\|^p_{W_0^{1,p}(Q)^n} + \|u_1\|^p_{L^p(Q;(\mathcal{B}^{1,p}_{\#A})^n)} \right)^{\frac{1}{p}} \quad (\mathbf{u} = (u_0,u_1) \in \mathbb{F}_0^{1,p}),$$

(where $\|u_1\|_{L^p(Q;(\mathcal{B}^{1,p}_{\#A})^n)} = (\sum_{i=1}^n \sum_{j=1}^N \left\|\overline{\partial} u_{1,i}/\partial y_j\right\|^p_{L^p(Q;\mathcal{B}^p_A)})^{1/p}$ for $u_1 = (u_{1,i})_{1\le i\le n}$) admitting $F_0^\infty = \mathcal{D}(Q)^n \times [\mathcal{D}(Q) \otimes (\varrho(A))^n]$ as a dense subspace.

We can now state and prove the main result of this subsection.

Theorem 5.1. *Let A be a separable algebra wmv such that (5.4) holds. For each $\varepsilon > 0$, let u_ε be the unique solution of (5.3). Then, as $\varepsilon \to 0$ we have*

$$u_\varepsilon \to u_0 \text{ in } W_0^{1,p}(Q)^n\text{-weak} \tag{5.5}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \text{ in } L^p(Q)^n\text{-weak } \Sigma \ (1 \le i \le N) \tag{5.6}$$

where $\mathbf{u} = (u_0,u_1) \in \mathbb{F}_0^{1,p}$ is the unique solution of the variational minimization problem

$$F(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbb{F}_0^{1,p}} F(\mathbf{v}) \tag{5.7}$$

with the functional F defined on $L^p(Q;\mathbb{R}^n) \times L^p(Q;(\mathcal{B}^{1,p}_{\#A})^n)$ by

$$F(\mathbf{v}) = \begin{cases} \int_Q \int_{\Delta(A)} \widehat{f}(x,s,Dv_0 + \partial\widehat{v}_1) d\beta dx \text{ for } \mathbf{v} \in \mathbb{F}_0^{1,p} \\ +\infty \text{ elsewhere} \end{cases}$$

and $\partial\widehat{v}_1 = \widehat{\overline{D}_y v_1}$.

Proof. First of all we see that Eq. (5.7) possesses a unique solution since the function $\widehat{f} : \overline{Q} \times \Delta(A) \times \mathbb{R}^{nN} \to [0,\infty)$, $\widehat{f}(x,s,\lambda) = \mathcal{G}(f(x,\cdot,\lambda))(s)$ is a Carathéodory's type function which is strictly convex in λ.

Now, in view of the growth condition (5.2), the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(Q)^n$ and so the sequence $f^\varepsilon(\cdot,\cdot,Du_\varepsilon)$ is bounded in $L^1(Q)$. Thus given an arbitrary fundamental sequence E, there exist a subsequence E' from E and a couple $\mathbf{u}=(u_0,u_1)\in\mathbb{F}_0^{1,p}$ such that (5.5)-(5.6) hold whenever $E'\ni\varepsilon\to 0$. If we show that $\mathbf{u}$ solves (5.7) then thanks to the uniqueness of the solution to (5.7), the convergence results (5.5)-(5.6) will hold for $\varepsilon\to 0$. Thus our only concern here is to check that $\mathbf{u}$ solves (5.7). To this end, let $(\nu_{x,s})_{x\in Q,s\in\Delta(A)}$ be the Young measure associated with $(Du_\varepsilon)_{\varepsilon\in E'}$ at length scale ε. Thanks to [part (i) of] Theorem 4.5 we have

$$\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^{nN}}\widehat{f}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx\le\liminf_{E'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx.$$

But due to Jensen's inequality one has

$$\int_Q\int_{\Delta(A)}\int_{\mathbb{R}^{nN}}\widehat{f}(x,s,\lambda)d\nu_{x,s}(\lambda)d\beta(s)dx\ge\int_Q\int_{\Delta(A)}\widehat{f}\left(x,s,\int_{\mathbb{R}^{nN}}\lambda d\nu_{x,s}(\lambda)\right)d\beta(s)dx,$$

and by Corollary 4.3,

$$\int_Q\int_{\Delta(A)}\widehat{f}(x,s,Du_0+\widehat{\overline{D}_yu_1})d\beta dx\le\liminf_{E'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx$$

since $\int_{\mathbb{R}^{nN}}\lambda d\nu_{x,s}(\lambda)=Du_0(x)+\widehat{\overline{D}_yu_1(x)}(s)=\mathcal{G}_1(Du_0(x)+\overline{D}_yu_1(x))(s)$ where $u_1(x)=u_1(x,\cdot)$. So, let E_1' be a subsequence from E' such that

$$\liminf_{E'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx=\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx.$$

We then have

$$\int_Q\int_{\Delta(A)}\widehat{f}(x,s,Du_0+\widehat{\overline{D}_yu_1})d\beta dx\le\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx.\tag{5.8}$$

Let us establish an upper bound for $\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,x/\varepsilon,Du_\varepsilon\right)dx$. To do that, let $\Phi=(\psi_0,\varrho^n(\psi_1))\in F_0^\infty$ with $\psi_0\in\mathcal{D}(Q)^n$, $\psi_1=(\psi_{1,i})_{1\le i\le n}\in[\mathcal{D}(Q)\otimes(\varrho(A^\infty))^n]$, $\varrho^n(\psi_1)=(\varrho(\psi_{1,i}))_{1\le i\le n}$. We define Φ_ε as follows: $\Phi_\varepsilon=\psi_0+\varepsilon\psi_1^\varepsilon$, that is, $\Phi_\varepsilon(x)=\psi_0(x)+\varepsilon\psi_1(x,x/\varepsilon)$ for $x\in Q$. Then $\Phi_\varepsilon\in W_0^{1,p}(Q)^n$, and, since u_ε is the minimizer, one has

$$\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon\right)dx\le\int_Q f\left(x,\frac{x}{\varepsilon},D\Phi_\varepsilon(x)\right)dx.$$

Set $v_\varepsilon(x)=f\left(x,x/\varepsilon,Du_\varepsilon\right)$ $(x\in Q)$, $\varepsilon\in E_1'$. Then $(v_\varepsilon)_{\varepsilon\in E_1'}$ is uniformly integrable; indeed let $\varphi(t)=t^2$ $(t\ge 0)$; then φ is inf-compact, $\varphi(t)/t\to+\infty$ as $t\to+\infty$, and further

$$\begin{aligned}\int_Q\varphi(v_\varepsilon(x))dx&\le c_2^2\int_Q(1+|D\Phi_\varepsilon|^p)^2dx\\&\le c_2^2\,|Q|\,(1+\|D\psi_0\|_\infty+\|D\psi_1\|_\infty+\|D_y\psi_1\|_\infty)^{2p}\\&<\infty\end{aligned}$$

where $|Q|$ denote the Lebesgue measure of Q. The sequence $(D\Phi_\varepsilon)_{\varepsilon\in E_1'}$ being bounded in $L^p(Q;\mathbb{R}^{nN})$ let $(\mu_{x,s})_{x\in Q,s\in\Delta(A)}$ be the Young measure associated with $(D\Phi_\varepsilon)_{\varepsilon\in E_1'}$ at length scale ε. Since f is Carathéodory and $v_\varepsilon = f(\cdot,\cdot/\varepsilon,Du_\varepsilon)$ is uniformly integrable, we deduce by [part (ii) of] Theorem 4.5 that, as $E_1'\ni\varepsilon\to 0$,

$$\int_Q f\left(x,\frac{x}{\varepsilon},D\Phi_\varepsilon(x)\right)dx \to \int_Q\int_{\Delta(A)}\int_{\mathbb{R}^{nN}} \widehat{f}(x,s,\lambda)d\mu_{x,s}(\lambda)d\beta(s)dx.$$

But $D\Phi_\varepsilon - (D\psi_0 + (D_y\psi_1)^\varepsilon) = \varepsilon(D\psi_1)^\varepsilon$ and so

$$\|D\Phi_\varepsilon - (D\psi_0 + (D_y\psi_1)^\varepsilon)\|_{L^1(Q)} \to 0 \text{ as } E_1'\ni\varepsilon\to 0.$$

This yields by Proposition 4.6 that $\mu_{x,s} = \delta_{D\psi_0+\widehat{D_y\psi}_1}$, in such a way that

$$\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},D\Phi_\varepsilon(x)\right)dx = \int_Q\int_{\Delta(A)}\widehat{f}(x,s,D\psi_0+\widehat{D_y\psi}_1)d\beta dx.$$

Thus

$$\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon(x)\right)dx \le \int_Q\int_{\Delta(A)}\widehat{f}(x,s,D\psi_0+\widehat{D_y\psi}_1)d\beta dx$$

for any $\Phi = (\psi_0,\varrho^n(\psi_1))\in F_0^\infty$, and by a density argument, for all $\Phi\in\mathbb{F}_0^{1,p}$. Whence

$$\lim_{E_1'\ni\varepsilon\to 0}\int_Q f\left(x,\frac{x}{\varepsilon},Du_\varepsilon(x)\right)dx \le \inf_{\mathbf{v}\in\mathbb{F}_0^{1,p}}\int_Q\int_{\Delta(A)}\widehat{f}(x,s,Dv_0+\widehat{\overline{D}_y v}_1)d\beta dx. \quad (5.9)$$

The inequalities (5.8) and (5.9) yield (5.7). This completes the proof. □

5.3. **Some applications of Theorem 5.1.** We can consider the homogenization problem for (5.3) under a variety of assumptions as in the following examples.

Example 5.2 (*Homogenization in ergodic algebras*)**.** We assume that the algebra A is ergodic. This allows us to solve the following deterministic homogenization problems:

- $(\mathrm{P})_1$ The function f is periodic in y;
- $(\mathrm{P})_2$ The function f is almost periodic in y [9, 11];
- $(\mathrm{P})_3$

$$f(x,\cdot,\lambda)\in L^1_{\infty,AP}(\mathbb{R}^N) \text{ for all } x\in\overline{Q} \text{ and all } \lambda\in\mathbb{R}^{nN}$$

where $L^1_{\infty,AP}(\mathbb{R}^N)$ denotes the closure with respect to the seminorm $\|\cdot\|_1$ (defined in Section 2) of the space of finite sums

$$\sum_{\text{finite}}\varphi_i u_i \quad \text{with } \varphi_i\in\mathcal{B}_\infty(\mathbb{R}^N),\ u_i\in AP(\mathbb{R}^N),$$

$AP(\mathbb{R}^N)$ being the space of all continuous real-valued almost periodic functions on $\mathbb{R}^N$ and $\mathcal{B}_\infty(\mathbb{R}^N)$ the space of continuous real-valued functions on $\mathbb{R}^N$ that converge at infinity.

Example 5.3 (*Homogenization in non ergodic algebra*)**.** We assume here that $N = 1$. Let A be the algebra generated by the function $f(z) = \cos \sqrt[3]{z}$ $(z \in \mathbb{R})$ and all its translates $f(\cdot + a)$, $a \in \mathbb{R}$. It is known that A is an algebra with mean value which is not ergodic; see [25, p. 243] for details. Since A satisfies all the requirements of Theorem 4.1, the conclusion of Theorem 5.1 holds under the hypothesis

(H) $f(x, \cdot, \lambda) \in B^1_A$ for all $(x, \lambda) \in \overline{Q} \times \mathbb{R}^n$.

The homogenization problem solved here is new. One can also consider other homogenization problems in the present setting of non-ergodic algebras.

6. Homogenization of a stochastic Ladyzhenskaya model for incompressible viscous flow

We assume in this section that all vector spaces are real vector spaces, and all scalar functions are real-valued. Let A_y and A_τ be two algebras wmv on $\mathbb{R}^N_y$ and $\mathbb{R}_\tau$, respectively. We set $A = A_y \odot A_\tau$, the product algebra wmv defined as in [28, 31, 49]. Obviously, no ergodicity assumption is required neither on A_y, nor on A_τ.

6.1. **Statement of the problem and a priori estimates.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On $(\Omega, \mathcal{F}, \mathbb{P})$ we define a prescribed m-dimensional standard Wiener process W. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $(\mathcal{F}^t)$ of W. We therefore aim at studying the asymptotics of the following stochastic generalized Navier-Stokes type equations

$$\begin{cases} d\mathbf{u}_\varepsilon + \left(-\operatorname{div}\left[a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)\nabla \mathbf{u}_\varepsilon + b\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)|\nabla \mathbf{u}_\varepsilon|^{p-2}\nabla \mathbf{u}_\varepsilon\right] + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon + \nabla q_\varepsilon\right) dt \\ \qquad = \mathbf{f} dt + g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \mathbf{u}_\varepsilon\right) dW \text{ in } Q_T \\ \operatorname{div} \mathbf{u}_\varepsilon = 0 \text{ in } Q_T \\ \mathbf{u}_\varepsilon = 0 \text{ on } \partial Q \times (0, T) \\ \mathbf{u}_\varepsilon(x, 0) = \mathbf{u}^0(x) \text{ in } Q. \end{cases} \tag{6.1}$$

In order that (6.1) becomes meaningful, we need to precise the data. Let Q be a smooth bounded open set in $\mathbb{R}^N_x$ ($N = 2$ or 3), and let T be a positive real number. In $Q_T = Q \times (0, T)$ we consider the partial differential operator (where ∇ and div denote respectively the gradient operator and divergence operator in Q)

$$P^\varepsilon = -\operatorname{div}\left[a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)\nabla \cdot\right] := -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right)$$

where the function $a = (a_{ij})_{1 \le i,j \le N} \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}_\tau)^{N \times N}$ satisfies the following assumptions:

$$a_{ij} = a_{ji} \tag{6.2}$$

and there exists a constant $\nu_0 > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(y, \tau)\lambda_i \lambda_j \ge \nu_0 |\lambda|^2 \text{ for all } \lambda = (\lambda_i) \in \mathbb{R}^N \text{ and a.e. } (y, \tau) \in \mathbb{R}^{N+1}_{y,\tau}. \tag{6.3}$$

The operator P^ε above defined is assumed to act on vector functions as follows: for $\mathbf{u} = (u^i)_{1\le i\le N} \in (W^{1,p}(Q))^N$ we have $P^\varepsilon \mathbf{u} = (P^\varepsilon u^i)_{1\le i\le N}$. The function $b \in L^\infty(\mathbb{R}^{N+1}_{y,\tau})$ and verifies $c_1 \le b(y,\tau) \le c_1^{-1}$ a.e. $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$ where c_1 is a positive constant. So, putting $b(y,\tau,\lambda) = b(y,\tau)\,|\lambda|^{p-2}\,\lambda$, the function $b : (y,\tau,\lambda) \mapsto b(y,\tau,\lambda)$, from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{N\times N}$ into $\mathbb{R}^{N\times N}$ satisfies:

$$\text{For each fixed } \lambda \in \mathbb{R}^N,\ b(\cdot,\cdot,\lambda) \text{ is measurable;} \tag{6.4}$$

$$b(y,\tau,0) = 0 \text{ a.e. } (y,\tau) \in \mathbb{R}^N \times \mathbb{R}; \tag{6.5}$$

$$\begin{aligned}&\text{There are two positive constants } \nu_1 \text{ and } \nu_2 \text{ such that}\\ &\text{(i) } (b(y,\tau,\lambda) - b(y,\tau,\mu))\cdot(\lambda-\mu) \ge \nu_1\,|\lambda-\mu|^p\\ &\text{(ii) } |b(y,\tau,\lambda) - b(y,\tau,\mu)| \le \nu_2\,(|\lambda|+|\mu|)^{p-2}\,|\lambda-\mu|\\ &\text{for all } \lambda,\mu \in \mathbb{R}^{N\times N} \text{ and for a.e. } (y,\tau) \in \mathbb{R}^N \times \mathbb{R},\end{aligned} \tag{6.6}$$

where $p \ge 3$ is a real number, the dot denotes the usual Euclidean inner product in $\mathbb{R}^{N\times N}$, and $|\cdot|$ the associated norm. Next, the mapping $(y,\tau,u) \mapsto g(y,\tau,u)$ from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^m$ (integer $m \ge 1$) satisfies the assumption that there exist positive constants c_0 and c_1 such that

$$\begin{aligned}&\text{(i) } g(\cdot,\cdot,u) \text{ is measurable for any } u \in \mathbb{R}^N;\\ &\text{(ii) } |g(y,\tau,u)| \le c_0(1+|u|);\\ &\text{(iii) } |g(y,\tau,u_1) - g(y,\tau,u_2)| \le c_1\,|u_1-u_2|\\ &\text{for all } u,u_1,u_2 \in \mathbb{R}^N \text{ and for a.e. } (y,\tau) \in \mathbb{R}^N \times \mathbb{R}.\end{aligned} \tag{6.7}$$

The first issue to be discussed is related to the existence and uniqueness of the solution of (6.1). Prior to that, we introduce the following spaces [27, 41]

$$\begin{aligned}&\mathcal{V} = \{\varphi \in \mathcal{C}_0^\infty(Q)^N : \operatorname{div}\varphi = 0\};\\ &V = \text{ closure of } \mathcal{V} \text{ in } W^{1,p}(Q)^N = \{\mathbf{u} \in W_0^{1,p}(Q)^N : \operatorname{div}\mathbf{u} = 0\};\\ &H = \text{ closure of } \mathcal{V} \text{ in } L^2(Q)^N.\end{aligned}$$

We endow V with the $W_0^{1,p}(Q)^N$-norm (the gradient norm), which gives a reflexive Banach space. The space H is equipped with the $L^2(Q)^N$-norm which makes it a Hilbert space. For $\mathbf{u} \in L^p(0,T;V)$ the question for the existence of the trace function $(x,t) \mapsto b(x/\varepsilon, t/\varepsilon, \nabla\mathbf{u}(x,t))$ can be discussed in the same way as in [30]. Also the function $(x,t) \mapsto a(x/\varepsilon, t/\varepsilon)$ is well defined. With this in mind, let

$$a^\varepsilon(x,t) = a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$$

and

$$b^\varepsilon(\cdot,\cdot,\nabla\mathbf{u})(x,t) = b\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \nabla\mathbf{u}(x,t)\right) := b\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)|\nabla\mathbf{u}(x,t)|^{p-2}\,\nabla\mathbf{u}(x,t)$$

for $(x,t) \in Q_T$. We introduce the functionals

$$a_I(\mathbf{u},\mathbf{v}) = \int_Q (a^\varepsilon \nabla\mathbf{u})\cdot\nabla\mathbf{v}dx + \int_Q b^\varepsilon(\cdot,\cdot,\nabla\mathbf{u})\cdot\nabla\mathbf{v}dx \ (\mathbf{u},\mathbf{v} \in W_0^{1,p}(Q)^N);$$

$$b_I(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,k=1}^{N} \int_Q u^i \frac{\partial v^k}{\partial x_i} w^k dx \quad (\mathbf{u}=(u^i),\mathbf{v},\mathbf{w} \in W_0^{1,p}(Q)^N).$$

Then the following estimates hold:

$$|a_I(\mathbf{u},\mathbf{v})| \leq \|a\|_\infty \|\nabla \mathbf{u}\|_{L^2(Q)} \|\nabla \mathbf{v}\|_{L^2(Q)} + \nu_2 \|\nabla \mathbf{u}\|_{L^p(Q)}^{p-1} \|\nabla \mathbf{v}\|_{L^p(Q)}; \tag{6.8}$$

$$a_I(\mathbf{v},\mathbf{v}) \geq \nu_0 \|\nabla \mathbf{v}\|_{L^2(Q)}^2 + \nu_1 \|\nabla \mathbf{v}\|_{L^p(Q)}^p \tag{6.9}$$

for all $\mathbf{u},\mathbf{v} \in W_0^{1,p}(Q)^N$. From the above estimate (6.8) we infer by the Riesz representation theorem the existence of an operator $\mathcal{A}^\varepsilon : V \to V'$ such that

$$a_I(\mathbf{u},\mathbf{v}) = \langle P^\varepsilon \mathbf{u},\mathbf{v}\rangle + \langle \mathcal{A}^\varepsilon \mathbf{u},\mathbf{v}\rangle \text{ for all } \mathbf{u},\mathbf{v} \in V.$$

It is worth noting that since $p \geq 3$ (hence $p \geq 2$) we have $P^\varepsilon \mathbf{u} \in V'$ for $\mathbf{u} \in V$. Moreover the operator $\mathcal{A}^\varepsilon$ (for fixed $\varepsilon > 0$) is maximal monotone, surjective and hemicontinuous [27, Chap. 2, Section 2]. As far as the trilinear form b_I is concerned, we have that [27, 41]

$$b_I(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \text{ for all } \mathbf{u} \in V \text{ and } \mathbf{v} \in W_0^{1,p}(Q)^N;$$

$$b_I(\mathbf{u},\mathbf{u},\mathbf{v}) = b_I(\mathbf{u},\mathbf{v},\mathbf{u}) \text{ for } \mathbf{u} \in V \text{ and } \mathbf{v} \in W_0^{1,p}(Q)^N.$$

Furthermore, since $W^{1,p}(Q) \subset L^r(Q)$ [1] for any $r > 1$ (indeed for $N = 2,3$ and $p \geq 3$ we have $\frac{1}{p} - \frac{1}{N} \leq 0$, so that, by the Sobolev embedding, the above embedding holds true). So choosing $r > 1$ in such a way that $\frac{2}{r} + \frac{1}{p} = 1$, we have by Hölder's inequality that

$$|b_I(\mathbf{u},\mathbf{v},\mathbf{u})| \leq c \|\mathbf{u}\|_{L^r(Q)}^2 \|\nabla \mathbf{v}\|_{L^p(Q)} \text{ for all } \mathbf{u},\mathbf{v} \in W_0^{1,p}(Q)^N. \tag{6.10}$$

We therefore infer the existence of an element $B(\mathbf{u}) \in V'$ such that

$$\langle B(\mathbf{u}),\mathbf{v}\rangle = b_I(\mathbf{u},\mathbf{u},\mathbf{v}) \text{ for all } \mathbf{u},\mathbf{v} \in V. \tag{6.11}$$

Equation (6.11) defines a bounded operator $B : V \to V'$ with the further property that if $\mathbf{u} \in L^p(0,T;V)$ then $B(\mathbf{u}) \in L^{p'}(0,T;V')$. In fact, from (6.10)-(6.11) we have by Hölder's inequality (for $\mathbf{u} \in L^p(0,T;V)$),

$$\|B(\mathbf{u})\|_{L^{p'}(0,T;V')} \leq \left(\int_0^T \|\mathbf{u}(t)\|_{L^r(Q)}^{2p'} dt \right)^{1/p'}.$$

But $W^{1,p}(Q) \subset L^r(Q)$ (with continuous embedding), hence there is a positive constant c independent of $\mathbf{u}$, such that

$$\|B(\mathbf{u})\|_{L^{p'}(0,T;V')} \leq c \left(\int_0^T \|\mathbf{u}(t)\|_V^{2p'} dt \right)^{1/p'}.$$

Also, as $p \geq 3$, we have $2p' \leq p$, so that using once again Hölder's inequality with exponent $p/2p' \geq 1$, we get

$$\left(\int_0^T \|\mathbf{u}(t)\|_V^{2p'} dt \right)^{1/p'} \leq c \left(\int_0^T \|\mathbf{u}(t)\|_V^p dt \right)^{2/p}.$$

We therefore deduce

$$\|B(\mathbf{u})\|_{L^{p'}(0,T;V')} \leq c\left(\int_0^T \|\mathbf{u}(t)\|_V^p\, dt\right)^{2/p}. \tag{6.12}$$

The above inequality will be useful in the sequel. Finally, for the sake of completeness, we choose $\mathbf{f} \in L^{p'}(0,T;V')$ and $\mathbf{u}^0 \in H$. We are now in a position to state the existence and uniqueness result for (6.1). Before we can do that, however, we need to take the projection of (6.1) on V'; we get the following abstract form in V':

$$\begin{cases} d\mathbf{u}_\varepsilon + (P^\varepsilon \mathbf{u}_\varepsilon + \mathcal{A}^\varepsilon \mathbf{u}_\varepsilon + B(\mathbf{u}_\varepsilon))dt = \mathbf{f}dt + g^\varepsilon(\mathbf{u}_\varepsilon)dW,\ 0 < t < T \\ \mathbf{u}_\varepsilon(0) = \mathbf{u}^0. \end{cases} \tag{6.13}$$

With all the properties of the operator $\mathcal{A}^\varepsilon$ (among which the strict monotonicity, the maximality and the hemicontinuity) the existence and uniqueness of a martingale solution (and hence from the uniqueness, the strong) solution to (6.13) follows exactly the way of proceeding as in [40], and we can formulate the following result without proof.

Theorem 6.1. *Let the hypotheses be as above. Let $\varepsilon > 0$ be freely fixed and let $1 < r < \infty$. There exists an $\mathcal{F}^t$-progressively measurable process $\mathbf{u}_\varepsilon \in L^r(\Omega, \mathcal{F}, \mathbb{P}; L^p(0,T;V) \cap L^\infty(0,T;H))$ such that*

$$\begin{aligned} (\mathbf{u}_\varepsilon(t), \mathbf{v}) + \int_0^t (P^\varepsilon \mathbf{u}_\varepsilon(s) + \mathcal{A}^\varepsilon \mathbf{u}_\varepsilon(s) + B(\mathbf{u}_\varepsilon(s)), \mathbf{v})\, ds = (\mathbf{u}^0, \mathbf{v}) \\ + \int_0^t (\mathbf{f}(s), \mathbf{v})\, ds + \int_0^t (g^\varepsilon(\mathbf{u}_\varepsilon(s), \mathbf{v})\, dW(s) \end{aligned} \tag{6.14}$$

for all $\mathbf{v} \in V$ and for almost all $(\omega, t) \in \Omega \times [0,T]$.
Moreover $\mathbf{u}_\varepsilon \in L^r(\Omega; \mathcal{F}, \mathbb{P}; \mathcal{C}([0,T];H))$ and is unique in the sense that if $\mathbf{u}_\varepsilon$ and $\overline{\mathbf{u}}_\varepsilon$ satisfy (6.14) then $\mathbb{P}(\omega : \mathbf{u}_\varepsilon(t) = \overline{\mathbf{u}}_\varepsilon(t)$ in V' for all $t \in [0,T]) = 1$.

Remark 6.2. (1) For the existence result in the above theorem, we only need to have $p \geq 1 + \frac{2N}{N+2}$, and for the uniqueness, the more restricted assumption $p \geq 1 + \frac{N}{2}$ is required; see [27]. We have taken $p \geq 3$ only for the sake of simplicity. We might take $p \geq 1 + \frac{N}{2}$ for both the existence and uniqueness. (2) Since $f \in L^{p'}(0,T;V')$ the existence of the pressure q_ε is out of reach; see e.g., [38, Proposition 3] (see also [39]). That is why, in the sequel we are mainly concern with the asymptotics of the velocity field $\mathbf{u}_\varepsilon$ defined in Theorem 6.1. Accordingly, throughout the remainder of this section, we will only refer to problem (6.13) instead of (6.1).

It is very important to note that very few results are available as regards the homogenization of SPDEs. We may cite [6, 23, 24, 34, 46, 47] in that framework. In the just mentioned work, the homogenization of SPDEs is studied under the periodicity assumption on the coefficients of the equations considered. In addition, the convergence method used is either the G-convergence method [6, 23, 24] or the two-scale convergence method [46, 47]. In view of the study of the qualitative properties of the solutions of SPDEs, it is more convenient to use an appropriate method taking into account both the random and deterministic behaviours of these solutions; see Subsection 3.2. As for the homogenization of SPDEs beyond

the periodic setting, to the best of our knowledge, the only results available so far in the literature are [32, 33].

Before we can proceed with the a priori estimates, let us set a convention. The letter C will throughout denote a positive constant whose value may change from line to line. The dependence of constants on the parameters will be written explicitly only when necessary. With this in mind, the following a priori estimates hold.

Proposition 6.3. *For each fixed $\varepsilon > 0$, let $\mathbf{u}_\varepsilon$ be the unique solution of (6.13). Then for any $1 < r < \infty$ we have*

$$\mathbb{E} \sup_{0\le t\le T} \|\mathbf{u}_\varepsilon(t)\|^r_{L^2(Q)} \le C; \tag{6.15}$$

$$\mathbb{E} \int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(Q)^N}\, dt \le C \tag{6.16}$$

and

$$\mathbb{E} \int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{V}\, dt \le C \tag{6.17}$$

for any $\varepsilon > 0$, where C is a positive constant independent of ε.

Proof. Applying Itô's formula to $\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)}$ gives

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)} + 2\int_0^t \langle P^\varepsilon \mathbf{u}_\varepsilon(s) + A^\varepsilon \mathbf{u}_\varepsilon(s) + B(\mathbf{u}_\varepsilon(s)), \mathbf{u}_\varepsilon(s)\rangle\, ds \\ = \;& \|\mathbf{u}^0\|^2_{L^2(Q)} + 2\int_0^t \langle \mathbf{f}(s), \mathbf{u}_\varepsilon(s)\rangle\, ds + \int_0^t |g^\varepsilon(\mathbf{u}_\varepsilon(s))|^2\, ds \\ &+2\int_0^t (g^\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{u}_\varepsilon(s))\, dW(s). \end{aligned}$$

By using (6.3), [part (i) of] (6.6) and (6.7) we get

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)} + 2\nu_0 \textstyle\int_0^t \|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(Q)^N}\, ds + 2\nu_1 \int_0^t \|\mathbf{u}_\varepsilon(t)\|^p_V\, ds \\ &\le \|\mathbf{u}^0\|^2_{L^2(Q)} + 2\textstyle\int_0^t \|\mathbf{f}(s)\|_{V'} \|\mathbf{u}_\varepsilon(s)\|_V\, ds + C\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\, ds \\ &+C + 2\textstyle\int_0^t (g^\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{u}_\varepsilon(s))\, dW(s). \end{aligned} \tag{6.18}$$

By Young's inequality applied to the first integral on the right-hand side of (6.18),

$$\begin{aligned} 2\int_0^t \|\mathbf{f}(s)\|_{V'} \|\mathbf{u}_\varepsilon(s)\|_V\, ds &\le C(\nu_1)\int_0^t \|\mathbf{f}(s)\|^{p'}_{V'}\, ds + \nu_1 \int_0^t \|\mathbf{u}_\varepsilon(s)\|^p_V\, ds \\ &\le C + \nu_1 \int_0^t \|\mathbf{u}_\varepsilon(s)\|^p_V\, ds. \end{aligned}$$

Taking into account the above inequality in (6.18) we are led to

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)} + 2\nu_0 \int_0^t \|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(Q)^N}\, ds + \nu_1 \int_0^t \|\mathbf{u}_\varepsilon(t)\|^p_V\, ds \\ \le\;& \|\mathbf{u}^0\|^2_{L^2(Q)} + C + C\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\, ds + 2\int_0^t (g^\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{u}_\varepsilon(s))\, dW(s). \end{aligned}$$

Taking first the supremum over $0 \leq s \leq t$ (for all $0 \leq t \leq T$) and next the mathematical expectation in the above inequality,

$$\begin{aligned}&\mathbb{E}\sup_{0\leq s\leq t}\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}+2\nu_0\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(Q)^N}\,ds+\nu_1\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(t)\|^p_V\,ds\\&\leq\|\mathbf{u}^0\|^2_{L^2(Q)}+C+C\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\,ds\\&\quad+2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left(g^\varepsilon(\mathbf{u}_\varepsilon(\tau)),\mathbf{u}_\varepsilon(\tau)\right)dW(\tau)\right|.\end{aligned}\tag{6.19}$$

Making use of the Burkhölder–Davis–Gundy's inequality applied to the last term in the right-hand side of (6.19),

$$\begin{aligned}&2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left(g^\varepsilon(\mathbf{u}_\varepsilon(\tau)),\mathbf{u}_\varepsilon(\tau)\right)dW(\tau)\right|\\\leq\;&C\mathbb{E}\left(\int_0^t\left|\left(g^\varepsilon(\mathbf{u}_\varepsilon(s)),\mathbf{u}_\varepsilon(s)\right)\right|^2ds\right)^{1/2}\\\leq\;&C\mathbb{E}\left[\sup_{0\leq s\leq t}\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\left(\int_0^t\|g^\varepsilon(\mathbf{u}_\varepsilon(s))\|^2_{L^2(Q)}\,ds\right)^{1/2}\right],\end{aligned}$$

and by Cauchy-Schwartz's inequality,

$$\begin{aligned}2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left(g^\varepsilon(\mathbf{u}_\varepsilon(\tau)),\mathbf{u}_\varepsilon(\tau)\right)dW(\tau)\right|&\leq\tfrac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\\&\quad+C\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\,ds.\end{aligned}$$

Putting this in (6.19) we derive

$$\begin{aligned}&\tfrac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}+2\nu_0\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(Q)^N}\,ds+\nu_1\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(s)\|^p_V\,ds\\&\leq C+C\mathbb{E}\int_0^t\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\,ds.\end{aligned}\tag{6.20}$$

It therefore follows from Gronwall's inequality that

$$\mathbb{E}\sup_{0\leq t\leq T}\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)}\leq C\tag{6.21}$$

where C is independent of ε. It also follows from (6.20) and (6.21) that

$$\mathbb{E}\int_0^T\|\mathbf{u}_\varepsilon(t)\|^p_V\,dt\leq C\text{ and }\mathbb{E}\int_0^T\|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(Q)^N}\,dt\leq C$$

where C is also independent of ε, hence (6.16) and (6.17).

Now, as far as (6.15) is concerned, we start again from the Itô's formula which reads in this case as: if

$$X_t=X_0+\int_0^t\phi(s)ds+N_t$$

where $X_t=\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(Q)}$, $X_0=\|\mathbf{u}^0\|^2_{L^2(Q)}$, $N_t=2\int_0^t\left(g^\varepsilon(\mathbf{u}_\varepsilon(s),\mathbf{u}_\varepsilon(s))\right)dW(s)$ and $\phi(s)=-2\left\langle P^\varepsilon\mathbf{u}_\varepsilon(s)+\mathcal{A}^\varepsilon\mathbf{u}_\varepsilon(s),\mathbf{u}_\varepsilon(s)\right\rangle+2\left\langle\mathbf{f}(s),\mathbf{u}_\varepsilon(s)\right\rangle+\|g^\varepsilon(\mathbf{u}_\varepsilon(s))\|^2_{L^2(Q)}$, then

$$X_t^l=X_0^l+l\int_0^tX_s^{l-1}\phi(s)ds+l\int_0^tX_s^{l-1}dN_s+\frac{1}{2}l(l-1)\int_0^tX_s^{l-2}d\left\langle N_s\right\rangle$$

for any $1 \leq l < \infty$. We apply the above formula with $l = r/2$ $(r > 2)$ and we get

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon(t)\|^r_{L^2(Q)} + r\int_0^t \|\mathbf{u}_\varepsilon(s)\|^{r-2}_{L^2(Q)} \left[\nu_0 \|\mathbf{u}_\varepsilon(s)\|^2_{H^1_0(Q)^N} + \nu_1 \|\mathbf{u}_\varepsilon(s)\|^p_V\right] ds \\
\leq\ & \|\mathbf{u}^0\|^r_{L^2(Q)} + \frac{r}{2}\int_0^t \|\mathbf{u}_\varepsilon(s)\|^{r-2}_{L^2(Q)} \|g^\varepsilon(\mathbf{u}_\varepsilon(s))\|^2_{L^2(Q)}\, ds \\
&+ r\left(\frac{r}{2} - 1\right)\int_0^t \|\mathbf{u}_\varepsilon(s)\|^{r-2}_{L^2(Q)} \|g^\varepsilon(\mathbf{u}_\varepsilon(s))\|^2_{L^2(Q)}\, ds \\
&+ \frac{r}{2}\int_0^t \|\mathbf{u}_\varepsilon(s)\|^{r-2}_{L^2(Q)} \left(g^\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{u}_\varepsilon(s)\right) dW(s).
\end{aligned}$$

We can therefore follow the same way of reasoning as before (see also [40]) to get (6.15). The proof is completed. □

The estimate (6.15) (for $r > 2$) is concerned with the higher integrability of the sequence $(\mathbf{u}_\varepsilon)_\varepsilon$ so that we can make use of it together with the Vitali's theorem. However this will become precise in the next subsection. Now, in order to prove the tightness property of the sequence of probability laws of $(\mathbf{u}_\varepsilon)_\varepsilon$, we will also need the

Proposition 6.4. *Assuming that the function $t \mapsto \mathbf{u}_\varepsilon(t)$ is extended by zero outside the interval $[0, T]$, there exists a positive constant C such that*

$$\mathbb{E}\sup_{|\theta|\leq\delta}\int_0^T \|\mathbf{u}_\varepsilon(t+\theta) - \mathbf{u}_\varepsilon(t)\|^{p'}_{V'}\, dt \leq C\delta^{\frac{1}{p-1}}$$

for each $\varepsilon > 0$ and $0 < \delta < 1$.

Proof. Assume $\theta \geq 0$. The same way of reasoning will apply for $\theta < 0$. We have

$$\begin{aligned}
\mathbf{u}_\varepsilon(t+\theta) - \mathbf{u}_\varepsilon(t) &= \int_t^{t+\theta} d\mathbf{u}_\varepsilon(s) \\
&= \int_t^{t+\theta} [-P^\varepsilon \mathbf{u}_\varepsilon(s) - \mathcal{A}^\varepsilon \mathbf{u}_\varepsilon(s) - B(\mathbf{u}_\varepsilon(s)) + \mathbf{f}(s)] ds \\
&\quad + \int_t^{t+\theta} g^\varepsilon(\mathbf{u}_\varepsilon(s)) dW(s),
\end{aligned}$$

hence

$$\begin{aligned}
\|\mathbf{u}_\varepsilon(t+\theta) - \mathbf{u}_\varepsilon(t)\|^{p'}_{V'} &\leq C\left\|\int_t^{t+\theta} P^\varepsilon \mathbf{u}_\varepsilon(s) ds\right\|^{p'}_{V'} + C\left\|\int_t^{t+\theta} \mathcal{A}^\varepsilon \mathbf{u}_\varepsilon(s) ds\right\|^{p'}_{V'} \\
&\quad + C\left\|\int_t^{t+\theta} B(\mathbf{u}_\varepsilon(s)) ds\right\|^{p'}_{V'} + C\left\|\int_t^{t+\theta} \mathbf{f}(s)) ds\right\|^{p'}_{V'} \\
&\quad + C\left\|\int_t^{t+\theta}\int_t^{t+\theta} g^\varepsilon(\mathbf{u}_\varepsilon(s)) dW(s)\right\|^{p'}_{V'} \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

But

$$I_1 \leq C \left(\int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|_{V'} \, ds \right)^{p'}$$

and

$$\begin{aligned} \int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|_{V'} \, ds &\leq C \int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|_{H^{-1}(Q)^N} \, ds \\ &\leq C\theta^{\frac{1}{p}} \left(\int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|^{p'}_{H^{-1}(Q)^N} \, ds \right)^{\frac{1}{p'}}, \end{aligned}$$

hence

$$\begin{aligned} I_1 &\leq C\theta^{\frac{p'}{p}} \int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|^{p'}_{H^{-1}(Q)^N} \, ds \\ &\leq C\theta^{\frac{p'}{p}} \int_t^{t+\theta} \left(1 + \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|^2_{H^{-1}(Q)^N} \right) ds \text{ since } p' < 2 \\ &\leq C\theta^{\frac{p'}{p}} \int_t^{t+\theta} \|P^\varepsilon \mathbf{u}_\varepsilon(s)\|^2_{H^{-1}(Q)^N} \, ds \\ &\leq C\delta^{\frac{p'}{p}} \int_t^{t+\theta} \|\mathbf{u}_\varepsilon(s)\|^2_{H^1_0(Q)^N} \, ds. \end{aligned}$$

We infer from (6.16) that

$$\mathbb{E} \sup_{0\leq\theta\leq\delta} \int_0^T I_1 dt \leq C\delta^{\frac{p'}{p}} \mathbb{E} \int_0^T \left(\int_t^{t+\delta} \|\mathbf{u}_\varepsilon(s)\|^2_{H^1_0(Q)^N} \, ds \right) dt \leq C\delta^{\frac{p'}{p}}.$$

As for I_2 we have, as above,

$$I_2 \leq C\theta^{\frac{p'}{p}} \int_t^{t+\theta} \|\mathcal{A}^\varepsilon \mathbf{u}_\varepsilon(s)\|^{p'}_{V'} \, ds,$$

hence

$$\begin{aligned} \mathbb{E} \sup_{0\leq\theta\leq\delta} \int_0^T I_2 dt &\leq C\delta^{\frac{p'}{p}} \mathbb{E} \int_0^T \left(\int_t^{t+\delta} \|\nabla \mathbf{u}_\varepsilon(s)\|^p_{L^p(Q)} \, ds \right) dt \\ &\leq C\delta^{\frac{p'}{p}} \text{ by (6.17)}. \end{aligned}$$

Now, dealing with I_3 we have

$$\int_t^{t+\theta} \|B(\mathbf{u}_\varepsilon(s))\|_{V'} \, ds \leq \theta^{\frac{1}{p}} \int_t^{t+\theta} \|B(\mathbf{u}_\varepsilon(s))\|^{p'}_{V'} \, ds,$$

and proceeding as above (taking into account (6.12) and (6.17)) we get

$$\mathbb{E} \sup_{0\leq\theta\leq\delta} \int_0^T I_3 dt \leq C\delta^{\frac{p'}{p}}.$$

We also have $\mathbb{E}\sup_{0\leq\theta\leq\delta}\int_0^T I_4 dt \leq C\delta^{\frac{p'}{p}}$. For the last integral, by using the Burkhölder–Davis–Gundy's inequality we have

$$\begin{aligned}
\mathbb{E}\sup_{0\leq\theta\leq\delta}\int_0^T I_5 dt &\leq C\int_0^T \mathbb{E}\left(\int_t^{t+\theta}\|g^\varepsilon(\mathbf{u}_\varepsilon(s))\|^2_{L^2(Q)}\,ds\right)^{\frac{p'}{2}} dt\\
&\leq C\int_0^T \mathbb{E}\left(\int_t^{t+\theta}\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(Q)}\,ds\right)^{\frac{p'}{2}} dt \quad (\text{by } (6.7))\\
&\leq C\delta^{\frac{p'}{2}}\int_0^T \mathbb{E}\sup_{0\leq s\leq T}\|\mathbf{u}_\varepsilon(s)\|^{p'}_{L^2(Q)}\,dt\\
&\leq C\delta^{\frac{p'}{2}} \text{ by } (6.15).
\end{aligned}$$

Combining all the above estimates leads at once at

$$E\sup_{0\leq\theta\leq\delta}\int_0^T \|\mathbf{u}_\varepsilon(t+\theta)-\mathbf{u}_\varepsilon(t)\|^{p'}_{V'}\,dt \leq C\delta^{\frac{p'}{p}}$$

since $\delta^{\frac{p'}{2}} \leq \delta^{\frac{p'}{p}}$ (recall that $p\geq 3$ and $0<\delta<1$). As the same inequality obviously holds for $\theta<0$, the proof is completed. □

6.2. **Tightness property of the probability laws of $(\mathbf{u}_\varepsilon)_\varepsilon$.** We are now able to prove the tightness of the law of $(\mathbf{u}_\varepsilon, W)$. We shall for this aim, follow the lead of Bensoussan [7, Proposition 3.1] and Debussche et al. [16]. Before we can proceed any further, we need the following important result.

Lemma 6.5 ([7, Proposition 3.1])**.** *Let $(\mu_n)_n$ and $(\nu_n)_n$ be two ordinary sequences of positive real numbers such that $\mu_n, \nu_n \to 0$ as $n\to\infty$. For the three positive constants K, L and M, the set*

$$\begin{aligned}
Z = \{\mathbf{u} : &\int_0^T \|\mathbf{u}\|^p_V\,dt \leq L,\ \|\mathbf{u}(t)\|^2_H \leq K \ \text{a.e. } t,\\
&\sup_{|\theta|\leq\mu_n}\int_0^T \|\mathbf{u}(t+\theta)-\mathbf{u}(t)\|^{p'}_{V'}\,dt \leq \nu_n M \text{ for all } n\in\mathbb{N}\}
\end{aligned}$$

is a compact subset of $L^2(0,T;H)$.

This being so, set $\mathfrak{S} = L^2(0,T;H)\times\mathcal{C}(0,T;\mathbb{R}^m)$, a metric space equipped with its Borel σ-algebra $\mathcal{B}(\mathfrak{S})$. For $0<\varepsilon<1$, let Ψ_ε be the measurable $\mathfrak{S}$-valued mapping defined on $(\Omega,\mathcal{F},\mathbb{P})$ as

$$\Psi_\varepsilon(\omega) = (\mathbf{u}_\varepsilon(\cdot,\omega), W(\cdot,\omega))\ (\omega\in\Omega).$$

We introduce the image of $\mathbb{P}$ under Ψ_ε defined by

$$\pi_\varepsilon(S) = \mathbb{P}(\Psi_\varepsilon^{-1}(S))\ \ (S\in\mathcal{B}(\mathfrak{S})),$$

which defines a sequence of probability measures on $\mathfrak{S}$. The following result holds.

Theorem 6.6. *The sequence $(\pi_\varepsilon)_{0<\varepsilon<1}$ is tight in $(\mathfrak{S},\mathcal{B}(\mathfrak{S}))$.*

Proof. Let $\delta > 0$ and let L_δ, K_δ, M_δ be positive constants depending only on δ (to be fixed later). We have by Lemma 6.5 that

$$Z_\delta = \left\{ \mathbf{u} : \int_0^T \|\mathbf{u}\|_V^p \, dt \le L_\delta, \ \|\mathbf{u}(t)\|_H^2 \le K_\delta \text{ a.e. } t, \ \sup_{|\theta|\le\mu_n} \int_0^T \|\mathbf{u}(t+\theta) - \mathbf{u}(t)\|_{V'}^{p'} \le \nu_n M_\delta \right\}$$

is a compact subset of $L^p(0,T;H)$ for any $\delta > 0$. Here we choose the sequence $(\mu_n)_n$ and $(\nu_n)_n$ so that $\sum \frac{1}{\nu_n}(\mu_n)^{\frac{1}{p-1}} < \infty$. Then we have

$$\mathbb{P}\left(\mathbf{u}_\varepsilon \notin Z_\delta\right) \le \mathbb{P}\left(\int_0^T \|\mathbf{u}_\varepsilon\|_V^p \, dt \ge L_\delta \right) + \mathbb{P}\left(\sup_{t\in[0,T]} \|\mathbf{u}_\varepsilon(t)\|_H^2 \ge K_\delta \right)$$
$$+ \mathbb{P}\left(\sup_{|\theta|\le\mu_n} \int_0^T \|\mathbf{u}_\varepsilon(t+\theta) - \mathbf{u}_\varepsilon(t)\|_{V'}^{p'} \, dt \ge \nu_n M_\delta \right).$$

In view of Tchebychev's inequality we have

$$\mathbb{P}(\mathbf{u}_\varepsilon \notin Z_\delta) \le \frac{1}{L_\delta}\mathbb{E}\int_0^T \|\mathbf{u}_\varepsilon(t)\|_V^p \, dt + \frac{1}{K_\delta}\mathbb{E}\sup_{t\in[0,T]} \|\mathbf{u}_\varepsilon(t)\|_H^2$$
$$+ \sum \frac{1}{\nu_n M_\delta}\mathbb{E}\sup_{|\theta|\le\mu_n} \int_0^T \|\mathbf{u}_\varepsilon(t+\theta) - \mathbf{u}_\varepsilon(t)\|_{V'}^{p'} \, dt.$$

From Propositions 6.3 and 6.4 it follows that

$$\mathbb{P}(\mathbf{u}_\varepsilon \in Z_\delta) \le \frac{C}{L_\delta} + \frac{C}{K_\delta} + \frac{C}{M_\delta}\sum_n \frac{1}{\nu_n}(\mu_n)^{\frac{1}{p-1}}.$$

So if we choose

$$K_\delta = L_\delta = \frac{6C}{\delta} \text{ and } M_\delta = \frac{6C\left(\sum_n \frac{1}{\nu_n}(\mu_n)^{\frac{1}{p-1}}\right)}{\delta},$$

then we have are led to

$$\mathbb{P}\left(\mathbf{u}_\varepsilon \notin Z_\delta\right) \le \frac{\delta}{2}. \tag{6.22}$$

Next, considering the sequence of probability measures $\pi_2^\varepsilon(A) := \mathbb{P}(W \in A)$ $(A \in \mathcal{B}(\mathcal{C}(0,T;\mathbb{R}^m)))$, it consists of only one element, hence it is weakly compact. As $\mathcal{C}(0,T;\mathbb{R}^m)$ is a Polish space, any weakly compact sequence of probability measure is tight, so that, given $\delta > 0$ there is a compact subset C_δ of $\mathcal{C}(0,T;\mathbb{R}^m)$ such that $\mathbb{P}(W \in C_\delta) \ge 1 - \delta/2$. We infer from this together with (6.21) that

$$\mathbb{P}\left((\mathbf{u}_\varepsilon, W) \in Z_\delta \times C_\delta\right) \ge 1 - \delta.$$

So we have just checked that for any $\delta > 0$ there is a compact $Z_\delta \times C_\delta \subset \mathfrak{S}$ such that

$$\pi_\varepsilon(Z_\delta \times C_\delta) \ge 1 - \delta,$$

by this proving the tightness of the family π_ε in $\mathfrak{S} = L^p(0,T;H)\times\mathcal{C}(0,T;\mathbb{R}^m)$. □

It follows from Theorem 6.6 and Prokhorov's theorem that there is a subsequence $(\pi_{\varepsilon_n})_n$ of $(\pi_\varepsilon)_{0<\varepsilon<1}$ converging weakly (in the sense of measure) to a probability measure Π. It emerges from Skorokhod's theorem that we can find a new probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and random variables $(\mathbf{u}_{\varepsilon_n}, W^{\varepsilon_n})$, $(\mathbf{u}_0, \bar W)$ defined

on this new probability space and taking values in $\mathfrak{S} = L^p(0,T;H)\times\mathcal{C}(0,T;\mathbb{R}^m)$ such that:

(i) The probability law of $(\mathbf{u}_{\varepsilon_n}, W^{\varepsilon_n})$ is π_{ε_n};
(ii) The probability law of $(\mathbf{u}_0, \bar{W})$ is Π;
(iii) As $n\to\infty$,

$$W^{\varepsilon_n}\to\bar{W} \text{ in } \mathcal{C}(0,T;\mathbb{R}^m)\ \bar{\mathbb{P}}\text{-a.s.} \tag{6.23}$$

and
(iv) As $n\to\infty$,

$$\mathbf{u}_{\varepsilon_n}\to\mathbf{u}_0 \text{ in } L^2(0,T;H)\ \bar{\mathbb{P}}\text{-a.s.} \tag{6.24}$$

We can see that $\{W^{\varepsilon_n}\}$ is a sequence of m-dimensional standard Brownian motions. Let $\bar{\mathcal{F}}^t$ be the σ-algebra generated by $(\bar{W}(s),\mathbf{u}_0(s))$ $(0\le s\le t)$ and the null sets of $\bar{\mathcal{F}}$. Arguing as in [7, Proof of Theorem 1.1] we can show that $\bar{W}$ is an $\bar{\mathcal{F}}^t$-adapted standard $\mathbb{R}^m$-valued Wiener process. Also by the same argument as in [8, pp. 281-283] we can show that, for all $\mathbf{v}\in V$ and for almost every $(\omega,t)\in\bar{\Omega}\times[0,T]$ the following holds true

$$\begin{aligned}(\mathbf{u}_{\varepsilon_n}(t),\mathbf{v}) + \textstyle\int_0^t (P^\varepsilon\mathbf{u}_{\varepsilon_n}(s) + \mathcal{A}^\varepsilon\mathbf{u}_{\varepsilon_n}(s) + B(\mathbf{u}_{\varepsilon_n}(s)),\mathbf{v})\,ds = (\mathbf{u}^0,\mathbf{v})\\ + \textstyle\int_0^t (\mathbf{f}(s),\mathbf{v})\,ds + \int_0^t (g^\varepsilon(\mathbf{u}_{\varepsilon_n}(s),\mathbf{v})\,dW^{\varepsilon_n}(s).\end{aligned} \tag{6.25}$$

6.3. **Homogenization results.**

6.3.1. ***Abstract formulation of the problem and preliminary results***. We begin this subsection by stating some important preliminary results necessary to the homogenization process. The notations are those of the preceding sections. It is worth noting that property (3.3) in Definition 3.8 still valid for $f\in B(\Omega;\mathcal{C}(\overline{Q}_T;B_A^{p',\infty}))$ where $B_A^{p',\infty}=B_A^{p'}\cap L^\infty(\mathbb{R}^{N+1}_{y,\tau})$ and as usual, $p'=p/(p-1)$.

Bearing this in mind, the question of homogenization of (6.13) will naturally arise from the following important assumption:

$$\begin{cases} b\in B_A^\infty \text{ and } a_{ij},\ g_k(\cdot,\cdot,\mu)\in B_A^2, 1\le i,j\le N, 1\le k\le m\\ \text{for any } \mu\in\mathbb{R}^N\end{cases} \tag{6.26}$$

where $g=(g_k)_{1\le k\le m}$.

The above hypothesis, which depends on the algebra wmv A, is crucial in homogenization theory. It gives the structure of the coefficients of the operator under consideration, and therefore allows one to pass to the limit. Without such a hypothesis, one cannot perform the homogenization since the convergence process relies heavily on the latter. The commonly assumption used is the periodicity (obtained by taking the algebra to be the continuous periodic functions). Hypothesis (6.26) includes a variety of behaviours, ranging from the periodicity to the weak almost periodicity (as far as the ergodic algebras are concerned), and also encompassing all the non ergodic algebras wmv.

Let $\Psi\in B(\Omega;\mathcal{C}(\overline{Q}_T;(A)^{N\times N}))$. Suppose that (6.26) is satisfied. It can be shown (as in [35, Proposition 4.5]) that the function

$$(x,t,y,\tau,\omega)\mapsto b(y,\tau,\Psi(x,t,y,\tau,\omega))$$

denoted below by $b(\cdot,\Psi)$, belongs to $B(\Omega;\mathcal{C}(\overline{Q}_T;B_A^{p',\infty}))^{N\times N}$; assumption (6.26) is crucially used in order to obtain the above result. Likewise, the function $(x,t,y,\tau,\omega)\mapsto g_k(y,\tau,\psi_0(x,t,\omega))$ (for $\psi_0\in B(\Omega;\mathcal{C}(\overline{Q}_T))^N$) denoted by $g_k(\cdot,\psi_0)$, is an element of $B(\Omega;\mathcal{C}(\overline{Q}_T;B_A^{2,\infty}))$. We may then define their traces

$$(x,t,\omega)\mapsto b\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\Psi\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right)$$

and

$$(x,t,\omega)\mapsto g_k\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\psi_0\left(x,t,\omega\right)\right)$$

from $Q_T\times\Omega$ into $\mathbb{R}$, denoted respectively by $b^\varepsilon(\cdot,\Psi^\varepsilon)$ and $g_k^\varepsilon(\cdot,\psi_0)$, as elements of $L^\infty(Q_T\times\Omega)$. The following result and its corollary can be proven exactly as their homologues in [30] (see especially Proposition 3.1 therein).

Proposition 6.7. *Let $3\le p<\infty$. Suppose that (6.26) holds. For $\Psi\in B(\Omega;\mathcal{C}(\overline{Q}_T;(A)^{N\times N}))$ we have*

$$b^\varepsilon(\cdot,\Psi^\varepsilon)\to b(\cdot,\Psi)\ \text{in}\ L^{p'}(Q_T\times\Omega)^{N\times N}\text{-weak}\ \Sigma\ \text{as}\ \varepsilon\to 0.$$

The mapping $\Psi\mapsto b(\cdot,\Psi)$ of $B(\Omega;\mathcal{C}(\overline{Q}_T;(A)^{N\times N}))$ into $L^{p'}(Q_T\times\Omega;B_A^{p'})^{N\times N}$ extends by continuity to a unique mapping still denoted by b, of $L^p(Q_T\times\Omega;(B_A^p)^{N\times N})$ into $L^{p'}(Q_T\times\Omega;B_A^{p'})^{N\times N}$ such that

$$(b(\cdot,\mathbf{v})-b(\cdot,\mathbf{w}))\cdot(\mathbf{v}-\mathbf{w})\ge\nu_1|\mathbf{v}-\mathbf{w}|^p\ \ a.e.\ \text{in}\ Q_T\times\Omega\times\mathbb{R}_y^N\times\mathbb{R}_\tau$$

$$\begin{aligned}&\|b(\cdot,\mathbf{v})-b(\cdot,\mathbf{w})\|_{L^{p'}(Q_T\times\Omega;B_A^{p'})^{N\times N}}\\&\quad\le\nu_2\left\||\mathbf{v}|+|\mathbf{w}|\right\|^{p-2}_{L^p(Q_T\times\Omega;B_A^p)}\|\mathbf{v}-\mathbf{w}\|_{L^p(Q_T\times\Omega;(B_A^p)^{N\times N})}\\&\quad b(\cdot,0)=0\ \ a.e.\ \text{in}\ \mathbb{R}_y^N\times\mathbb{R}_\tau\end{aligned}$$

for all $\mathbf{v},\mathbf{w}\in L^p(Q_T\times\Omega;(B_A^p)^{N\times N})$.

Corollary 6.8. *Let $\psi_0\in(B(\Omega)\otimes\mathcal{C}_0^\infty(Q_T))^N$ and $\psi_1\in(B(\Omega)\otimes\mathcal{C}_0^\infty(Q_T)\otimes A^\infty)^N$. For $\varepsilon>0$, let*

$$\Phi_\varepsilon=\psi_0+\varepsilon\psi_1^\varepsilon,\tag{6.27}$$

i.e., $\Phi_\varepsilon(x,t,\omega)=\psi_0(x,t,\omega)+\varepsilon\psi_1(x,t,x/\varepsilon,t/\varepsilon,\omega)$ for $(x,t,\omega)\in Q_T\times\Omega$. Let $(v_\varepsilon)_{\varepsilon\in E}$ is a sequence in $L^p(Q_T\times\Omega)^{N\times N}$ such that $v_\varepsilon\to v_0$ in $L^p(Q_T\times\Omega)^{N\times N}$-weak Σ as $E\ni\varepsilon\to 0$ where $\mathbf{v}_0\in L^p(Q_T\times\Omega;\mathcal{B}_A^p)^{N\times N}$, then, as $E\ni\varepsilon\to 0$,

$$\int_{Q_T\times\Omega}b^\varepsilon(\cdot,\nabla\Phi_\varepsilon)\cdot v_\varepsilon dxdtd\mathbb{P}\to\iint_{Q_T\times\Omega\times\Delta(A)}\widehat{b}(\cdot,\nabla\psi_0+\partial\widehat{\psi}_1)\cdot\widehat{v}_0dxdtd\mathbb{P}d\beta.$$

Also we will need the following important result in order to pass to the limit in the stochastic term.

Lemma 6.9. *Let $(\mathbf{u}_\varepsilon)_\varepsilon$ be a sequence in $L^2(Q_T\times\Omega)^N$ such that $\mathbf{u}_\varepsilon\to\mathbf{u}_0$ in $L^2(Q_T\times\Omega)^N$ as $\varepsilon\to 0$ where $\mathbf{u}_0\in L^2(Q_T\times\Omega)^N$. Then for each $1\le k\le m$ we have,*

$$g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon)\to g_k(\cdot,\mathbf{u}_0)\ \text{in}\ L^2(Q_T\times\Omega)\text{-weak}\ \Sigma\ \text{as}\ \varepsilon\to 0.$$

Proof. First of all, let $\mathbf{u} \in B(\Omega; \mathcal{C}(\overline{Q}_T))^N$; as seen above, the function $(x,t,y,\tau,\omega) \mapsto g_k(y,\tau,\mathbf{u}(x,t,\omega))$ lies in $B(\Omega; \mathcal{C}(\overline{Q}_T; B_A^{2,\infty}))$, so that we have $g_k^\varepsilon(\cdot,\mathbf{u}) \to g_k(\cdot,\mathbf{u})$ in $L^2(Q_T\times\Omega)$-weak Σ as $\varepsilon \to 0$. Next, since $B(\Omega; \mathcal{C}(\overline{Q}_T))$ is dense in $L^2(Q_T\times\Omega)$, it can be easily shown that

$$g_k^\varepsilon(\cdot,\mathbf{u}_0) \to g_k(\cdot,\mathbf{u}_0) \text{ in } L^2(Q_T\times\Omega)\text{-weak } \Sigma \text{ as } \varepsilon \to 0. \tag{6.28}$$

Now, let $f \in L^2(\Omega; L^2(Q_T; A))$; then

$$\begin{aligned}
&\int_{Q_T\times\Omega} g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon) f^\varepsilon dxdtd\mathbb{P} - \iint_{Q_T\times\Omega\times\Delta(A)} \widehat{g}_k(\cdot,\mathbf{u}_0)\widehat{f} dxdtd\mathbb{P}d\beta \\
= &\int_{Q_T\times\Omega} (g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon) - g_k^\varepsilon(\cdot,\mathbf{u}_0)) f^\varepsilon dxdtd\mathbb{P} + \int_{Q_T\times\Omega} g_k^\varepsilon(\cdot,\mathbf{u}_0) f^\varepsilon dxdtd\mathbb{P} \\
&- \iint_{Q_T\times\Omega\times\Delta(A)} \widehat{g}_k(\cdot,\mathbf{u}_0)\widehat{f} dxdtd\mathbb{P}d\beta.
\end{aligned}$$

Using the inequality

$$\left| \int_{Q_T\times\Omega} (g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon) - g_k^\varepsilon(\cdot,\mathbf{u}_0)) f^\varepsilon dxdtd\mathbb{P} \right| \le C \left\|\mathbf{u}_\varepsilon - \mathbf{u}_0\right\|_{L^2(Q_T\times\Omega)^N} \left\|f^\varepsilon\right\|_{L^2(Q_T\times\Omega)}$$

in conjunction with (6.28) leads at once to the result. □

Remark 6.10. In view of the Lipschitz property on the function g_k we may get more information on the limit of the sequence $g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon)$. Indeed, because of $|g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon) - g_k^\varepsilon(\cdot,\mathbf{u}_0)| \le C\,|\mathbf{u}_\varepsilon - \mathbf{u}_0|$, we deduce the following convergence result:

$$g_k^\varepsilon(\cdot,\mathbf{u}_\varepsilon) \to g_k(\cdot,\mathbf{u}_0) \text{ in } L^2(Q_T\times\Omega)\text{-strong } \Sigma \text{ as } \varepsilon \to 0.$$

We end this subsection by collecting here below some function spaces that we will make use in the sequel. We begin by noting that the space $B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(0,T)\otimes\mathcal{V}$ is dense in $L^p(\bar{\Omega}; L^p(0,T;V))$. Next, let the space

$$\mathcal{B}^{1,p}_{\text{div}} = \{\mathbf{u} \in (\mathcal{B}^{1,p}_{\#A_y})^N : \ \overline{\text{div}}_y\mathbf{u} = 0\}$$

where $\overline{\text{div}}_y\mathbf{u} = \sum_{i=1}^N \overline{\partial} u^i/\partial y_i$, and its smooth counterpart

$$A^\infty_{y,\text{div}} = \{\mathbf{u} \in (\mathcal{D}_{A_y}(\mathbb{R}^N)/I^p_{A_y})^N : \ \overline{\text{div}}_y\mathbf{u} = 0\}.$$

The following result holds.

Lemma 6.11. *The space $\varrho_y^N(A^\infty_{y,\text{div}})$ is dense in $\mathcal{B}^{1,p}_{\text{div}}$ where, for $\mathbf{u} = (u^i)_{1\le i\le N} \in (A_y^\infty)^N$ we have $\varrho_y^N(\mathbf{u}) = (\varrho_y(u^i))_{1\le i\le N}$, ϱ_y being the canonical mapping of $B^p_{A_y}$ into its separated completion $\mathcal{B}^p_{A_y}$.*

Proof. This follows exactly in a same way as the proof of [50, Lemma 2.3]. □

Now, let

$$\mathbb{F}^{1,p}_0 = L^p(\bar{\Omega}\times(0,T);V) \times L^p(Q_T\times\bar{\Omega}; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\text{div}}))$$

and

$$\mathcal{F}_0^\infty = [B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(0,T)\otimes\mathcal{V}] \times \left[B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(Q_T)\otimes\left(\mathcal{D}_{A_\tau}(\mathbb{R}_\tau)\otimes\varrho_y^N(A^\infty_{y,\text{div}})\right)\right].$$

Thanks to Lemma 6.11 we have the density of $\mathcal{F}_0^\infty$ in $\mathbb{F}^{1,p}_0$.

6.3.2. ***Homogenized problem.*** Let $(\mathbf{u}_{\varepsilon_n})_n$ be the sequence determined in the Subsection 6.2 and satisfying Eq. (6.25). Because of (6.25) the sequence $(\mathbf{u}_{\varepsilon_n})_n$ also satisfies the a priori estimates (6.15), (6.16) and (6.17). Therefore, owing to the estimate (6.15) (which yields the uniform integrability of the sequence $(\mathbf{u}_{\varepsilon_n})_n$ with respect to ω) and the Vitali's theorem, we deduce from (6.24) that, as $n \to \infty$,

$$\mathbf{u}_{\varepsilon_n} \to \mathbf{u}_0 \text{ in } L^2(\bar{\Omega}; L^2(0,T;H))$$

and hence

$$\mathbf{u}_{\varepsilon_n} \to \mathbf{u}_0 \text{ in } L^2(Q_T \times \bar{\Omega})^N \text{ as } n \to \infty. \tag{6.29}$$

In view of (6.17) and by the diagonal process, one can find a subsequence of $(\mathbf{u}_{\varepsilon_n})_n$ (not relabeled) which weakly converges in $L^p(\bar{\Omega}; L^p(0,T;V))$ to the function $\mathbf{u}_0$ (this means that $\mathbf{u}_0 \in L^p(\bar{\Omega}; L^p(0,T;V))$). From Theorem 3.10, we infer the existence of a function $\mathbf{u}_1 = (u_1^k)_{1\le k\le N} \in L^p(Q_T \times \bar{\Omega}; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#A_y})^N)$ such that the convergence result

$$\frac{\partial \mathbf{u}_{\varepsilon_n}}{\partial x_i} \to \frac{\partial \mathbf{u}_0}{\partial x_i} + \frac{\overline{\partial} \mathbf{u}_1}{\partial y_i} \text{ in } L^p(Q_T \times \bar{\Omega})^N\text{-weak } \Sigma \ (1 \le i \le N) \tag{6.30}$$

holds when $\varepsilon_n \to 0$. We recall that $\frac{\partial \mathbf{u}_0}{\partial x_i} = \left(\frac{\partial u_0^k}{\partial x_i}\right)_{1\le k\le N}$ ($\mathbf{u}_0 = (u_0^k)_{1\le k\le N}$) and $\frac{\overline{\partial} \mathbf{u}_1}{\partial y_i} = \left(\frac{\overline{\partial} u_1^k}{\partial y_i}\right)_{1\le k\le N}$. Now, let us consider the following functionals:

$$\begin{aligned}\widehat{a}_I(\mathbf{u},\mathbf{v}) &= \sum_{i,j,k=1}^N \iint_{Q_T\times\bar{\Omega}\times\Delta(A)} \widehat{a}_{ij}(s,s_0)\mathbb{D}_j u^k \mathbb{D}_i v^k dx dt d\bar{\mathbb{P}} d\beta \\ &\quad + \iint_{Q_T\times\bar{\Omega}\times\Delta(A)} \widehat{b}(s,s_0,\mathbb{D}\mathbf{u})\cdot \mathbb{D}\mathbf{v} dx dt d\bar{\mathbb{P}} d\beta\end{aligned}$$

where $\mathbb{D}_j u^k = \frac{\partial u_0^k}{\partial x_j} + \partial_j \widehat{u}_1^k$ ($\partial_j \widehat{u}_1^k = \mathcal{G}_1\left(\frac{\overline{\partial} u_1^k}{\partial y_j}\right)$, and the same definition for $\mathbb{D}_i v^k$) and $\mathbb{D}\mathbf{u} = (\mathbb{D}_j \mathbf{u})_{1\le j\le N}$ with $\mathbb{D}_j \mathbf{u} = (\mathbb{D}_j u^k)_{1\le k\le N}$;

$$\widehat{b}_I(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,k=1}^N \iint_{Q_T\times\bar{\Omega}} u_0^i \frac{\partial v_0^k}{\partial x_i} w_0^k dx dt d\bar{\mathbb{P}}$$

for $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1), \mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1), \mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1) \in \mathbb{F}_0^{1,p}$. The functionals $\widehat{a}_I$ and $\widehat{b}_I$ are well-defined. Next, associated to these functionals is the variational problem

$$\begin{cases} \mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}_0^{1,p} : \\ -\int_{Q_T\times\bar{\Omega}} \mathbf{u}_0 \cdot \psi_0' dx dt d\bar{\mathbb{P}} + \widehat{a}_I(\mathbf{u},\mathbf{\Phi}) + \widehat{b}_I(\mathbf{u},\mathbf{u},\mathbf{\Phi}) \\ \quad = \int_{\bar{\Omega}} \int_0^T (\mathbf{f}(t), \psi_0(t,\omega))\, dt d\bar{\mathbb{P}} + \int_{\bar{\Omega}} \int_0^T (\widetilde{g}(\mathbf{u}_0), \psi_0)\, d\bar{W} d\bar{\mathbb{P}} \\ \text{for all } \Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty. \end{cases} \tag{6.31}$$

The following *global* homogenization result holds.

Theorem 6.12. *The couple* $(\mathbf{u}_0, \mathbf{u}_1)$ *determined by* (6.29)-(6.30) *solves problem* (6.31).

Proof. In what follows, we drop the index n from the sequence ε_n. So we will merely write ε for ε_n. Now, from the equality $\operatorname{div} \mathbf{u}_\varepsilon = 0$ we easily obtain that $\operatorname{div} \mathbf{u}_0 = 0$ and $\overline{\operatorname{div}}_y \mathbf{u}_1 = 0$, hence $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}_0^{1,p}$. It remains to show that $\mathbf{u}$ solves (6.31). For that, let $\mathbf{\Phi} = (\psi_0, \varrho^N(\psi_1)) \in \mathcal{F}_0^\infty$; define $\mathbf{\Phi}_\varepsilon$ as in Corollary 6.8 (see (6.27) therein), that is, as follows:

$$\mathbf{\Phi}_\varepsilon(x,t,\omega) = \psi_0(x,t,\omega) + \varepsilon \psi_1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) \text{ for } (x,t,\omega) \in Q_T \times \bar{\Omega}.$$

Then we have $\mathbf{\Phi}_\varepsilon \in (B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T))^N$ and, using $\mathbf{\Phi}_\varepsilon$ as a test function in the variational formulation of (6.25) we get

$$\begin{aligned} &-\int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \frac{\partial \mathbf{\Phi}_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} a^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{\Phi}_\varepsilon dxdtd\bar{\mathbb{P}} \\ &+ \int_{Q_T\times\bar{\Omega}} b^\varepsilon(\cdot, \nabla \mathbf{u}_\varepsilon) \cdot \nabla \mathbf{\Phi}_\varepsilon dxdtd\bar{\mathbb{P}} + \int_{\bar{\Omega}} \int_0^T b_I(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{\Phi}_\varepsilon) dtd\bar{\mathbb{P}} \\ = &\int_0^T \int_{\bar{\Omega}} (\mathbf{f}(t), \mathbf{\Phi}_\varepsilon)\, dtd\bar{\mathbb{P}} + \int_0^T \int_{\bar{\Omega}} (g^\varepsilon(\cdot, \mathbf{u}_\varepsilon), \mathbf{\Phi}_\varepsilon)\, dW^\varepsilon d\bar{\mathbb{P}}. \end{aligned} \tag{6.32}$$

We pass to the limit in (6.32) by considering each term separately. First we have

$$\begin{aligned} \int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \frac{\partial \mathbf{\Phi}_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} &= \int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \frac{\partial \psi_0}{\partial t} dxdtd\bar{\mathbb{P}} + \varepsilon \int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \left(\frac{\partial \psi_1}{\partial t}\right)^\varepsilon dxdtd\bar{\mathbb{P}} \\ &\quad + \int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \left(\frac{\partial \psi_1}{\partial \tau}\right)^\varepsilon dxdtd\bar{\mathbb{P}}. \end{aligned}$$

In view of (6.29) coupling with the convergence result $(\partial\psi_1/\partial\tau)^\varepsilon \to M(\partial\psi_1/\partial\tau) = 0$ in $L^2(Q_T \times \bar{\Omega})^N$-weak, we obtain

$$\int_{Q_T\times\bar{\Omega}} \mathbf{u}_\varepsilon \cdot \frac{\partial \mathbf{\Phi}_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} \to \int_{Q_T\times\bar{\Omega}} \mathbf{u}_0 \cdot \frac{\partial \psi_0}{\partial t} dxdtd\bar{\mathbb{P}}.$$

Next, it is an usual well known fact that, using the convergence result (6.29) together with the weak Σ-convergence of the sequence $(\nabla \mathbf{\Phi}_\varepsilon)_\varepsilon$ to $\mathbb{D}\mathbf{\Phi}$, we get

$$\int_{Q_T\times\bar{\Omega}} a^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{\Phi}_\varepsilon dxdtd\bar{\mathbb{P}} \to \widehat{a}_I(\mathbf{u}, \Phi).$$

Considering the next term, we use the monotonicity property to have

$$\int_{Q_T\times\bar{\Omega}} (b^\varepsilon(\cdot, \nabla \mathbf{u}_\varepsilon) - b^\varepsilon(\cdot, \nabla \mathbf{\Phi}_\varepsilon)) \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{\Phi}_\varepsilon) dxdtd\bar{\mathbb{P}} \geq 0. \tag{6.33}$$

Owing to the estimate (6.17) (denoting by $\overline{\mathbb{E}}$ the mathematical expectation on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$) we infer that

$$\sup_{\varepsilon>0} \overline{\mathbb{E}} \, \|b^\varepsilon(\cdot, \nabla \mathbf{u}_\varepsilon)\|^{p'}_{L^{p'}(Q_T)^{N\times N}} < \infty,$$

so that, from Theorem 3.9, there exist a function $\chi \in L^{p'}(Q_T \times \bar{\Omega}; \mathcal{B}_A^{p'})^{N\times N}$ and a subsequence of ε not relabeled, such that $b^\varepsilon(\cdot, \nabla \mathbf{u}_\varepsilon) \to \chi$ in $L^{p'}(Q_T \times \bar{\Omega})^{N\times N}$-weak

Σ as $\varepsilon \to 0$. We therefore pass to the limit in (6.33) (as $\varepsilon \to 0$) using Corollary 6.8 to get

$$\iint_{Q_T\times\bar{\Omega}\times\Delta(A)} (\widehat{\chi} - \widehat{b}(\cdot, \mathbb{D}\Phi)) \cdot (\mathbb{D}\mathbf{u} - \mathbb{D}\mathbf{\Phi}) dxdtd\bar{\mathbb{P}}d\beta \geq 0 \tag{6.34}$$

for any $\Phi \in \mathcal{F}_0^\infty$ where, as above, $\mathbb{D}\mathbf{u} = \nabla \mathbf{u}_0 + \partial \widehat{\mathbf{u}}_1$ ($\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$) and $\mathbb{D}\mathbf{\Phi} = \nabla\psi_0 + \partial\widehat{\psi}_1$. By the density of $\mathcal{F}_0^\infty$ in $\mathbb{F}_0^{1,p}$ and by a continuity argument, (6.34) still holds for $\mathbf{\Phi} \in \mathbb{F}_0^{1,p}$. Hence by taking $\mathbf{\Phi} = \mathbf{u} + \lambda\mathbf{v}$ for $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{F}_0^{1,p}$ and $\lambda > 0$ arbitrarily fixed, we get

$$\lambda \iint_{Q_T\times\bar{\Omega}\times\Delta(A)} (\widehat{\chi} - \widehat{b}(\cdot, \mathbb{D}\mathbf{u} + \lambda\mathbb{D}\mathbf{v})) \cdot \mathbb{D}\mathbf{v} dxdtd\bar{\mathbb{P}}d\beta \geq 0 \text{ for all } \mathbf{v} \in \mathbb{F}_0^{1,p}.$$

Therefore by a mere routine, we deduce that $\chi = b(\cdot, \nabla\mathbf{u}_0 + \overline{\nabla}_y\mathbf{u}_1)$.

The next point to check is to compute the $\lim_{\varepsilon\to 0} \int_{\bar{\Omega}} \int_0^T b_I(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{\Phi}_\varepsilon) dtd\bar{\mathbb{P}}$. We claim that, as $\varepsilon \to 0$,

$$\int_{\bar{\Omega}} \int_0^T b_I(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{\Phi}_\varepsilon) dtd\bar{\mathbb{P}} \to \widehat{b}_I(\mathbf{u}, \mathbf{u}, \mathbf{\Phi}).$$

Indeed, by the strong convergence result (6.29) in conjunction with the Theorem 3.5, our claim is justified.

We obviously have that

$$\int_0^T \int_{\bar{\Omega}} (\mathbf{f}(t), \mathbf{\Phi}_\varepsilon(t,\omega))\, dtd\bar{\mathbb{P}} \to \int_0^T \int_{\bar{\Omega}} (\mathbf{f}(t), \psi_0(t,\omega))\, dtd\bar{\mathbb{P}}.$$

The last point is concerned with the stochastic part $\int_0^T \int_{\bar{\Omega}} (g^\varepsilon(\cdot, \mathbf{u}_\varepsilon), \mathbf{\Phi}_\varepsilon)\, dW^\varepsilon d\bar{\mathbb{P}}$.

But thanks to Remark 6.10 we get at once

$$\int_0^T \int_{\bar{\Omega}} (g^\varepsilon(\cdot, \mathbf{u}_\varepsilon), \mathbf{\Phi}_\varepsilon)\, dW^\varepsilon d\bar{\mathbb{P}} \to \int_{\bar{\Omega}} \int_0^T (\widetilde{g}(\mathbf{u}_0), \psi_0)\, d\bar{W} d\bar{\mathbb{P}}$$

where $\widetilde{g}_k(\mathbf{u}_0)(x,t,\omega) = \int_{\Delta(A)} \widehat{g}_k(s, s_0, \mathbf{u}_0(x,t,\omega)) d\beta$ and $\widetilde{g}(\mathbf{u}_0) = (\widetilde{g}_k(\mathbf{u}_0))_{1\leq k\leq m}$. It emerges from the above study that $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ satisfies (6.31). □

In order to derive the homogenized problem, we need to deal with an equivalent expression of problem (6.31). As we can see, this problem is equivalent to the following system (6.35)-(6.36) reading as

$$\begin{cases} \iint_{Q_T\times\bar{\Omega}\times\Delta(A)} \widehat{a}\mathbb{D}\mathbf{u} \cdot \partial\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta + \iint_{Q_T\times\bar{\Omega}\times\Delta(A)} \widehat{b}(\cdot, \mathbb{D}\mathbf{u}) \cdot \partial\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta = 0 \\ \text{for all } \psi_1 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes [\mathcal{D}_{A_\tau}(\mathbb{R}_\tau) \otimes \varrho_y^N(A^\infty_{y,\text{div}})]; \end{cases} \tag{6.35}$$

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} \mathbf{u}_0 \cdot \psi_0' dxdtd\bar{\mathbb{P}} + \widehat{a}_I(\mathbf{u}, (\psi_0, 0)) + \widehat{b}_I(\mathbf{u}, \mathbf{u}, (\psi_0, 0)) \\ \quad = \int_{\bar{\Omega}} \int_0^T (\mathbf{f}(t), \psi_0(t,\omega))\, dtd\bar{\mathbb{P}} + \int_{\bar{\Omega}} \int_0^T (\widetilde{g}(\mathbf{u}_0), \psi_0)\, d\bar{W} d\bar{\mathbb{P}} \\ \text{for all } \psi_0 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(0,T) \otimes \mathcal{V}. \end{cases} \tag{6.36}$$

It is an easy matter to deal with (6.35). In fact, fix $\xi \in \mathbb{R}^{N\times N}$ and consider the following cell problem:

$$\left\{\begin{array}{l} \pi(\xi) \in \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\rm div}) : \\ \int_{\Delta(A)} \widehat{a}(\xi+\partial\widehat{\pi}(\xi))\cdot\partial\widehat{w}d\beta + \int_{\Delta(A)} \widehat{b}(\cdot,\xi+\partial\widehat{\pi}(\xi))\cdot\partial\widehat{w}d\beta = 0 \\ \text{for all } w \in \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\rm div}). \end{array}\right. \tag{6.37}$$

Due to the properties of the functions a and b, Eq. (6.37) admits at least a solution (see e.g., [27, Chap. 2]). But if $\pi_1 = \pi_1(\xi)$ and $\pi_2 = \pi_2(\xi)$ are two solutions of (6.37) then, setting $\pi = \pi_1 - \pi_2$,

$$\left\{\begin{array}{l} \int_{\Delta(A)} \widehat{a}\partial\widehat{\pi}\cdot\partial\widehat{w}d\beta + \int_{\Delta(A)}(\widehat{b}(\cdot,\xi+\partial\widehat{\pi}_1) - \widehat{b}(\cdot,\xi+\partial\widehat{\pi}_2))\cdot\partial\widehat{w}d\beta = 0 \\ \text{for all } w \in \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\rm div}). \end{array}\right.$$

Taking the particular test function $w = \pi$, we are led to

$$\int_{\Delta(A)} \widehat{a}\partial\widehat{\pi}\cdot\partial\widehat{\pi}d\beta + \int_{\Delta(A)}(\widehat{b}(\cdot,\xi+\partial\widehat{\pi}_1) - \widehat{b}(\cdot,\xi+\partial\widehat{\pi}_2))\cdot\partial\widehat{\pi}d\beta = 0,$$

and using once again the properties of a and b (see in particular the Proposition 6.7), we get

$$\nu_0\int_{\Delta(A)} |\partial\widehat{\pi}|^2\,d\beta + \nu_1\int_{\Delta(A)} |\partial\widehat{\pi}|^p\,d\beta = 0,$$

which gives $\partial\widehat{\pi} = 0$, or equivalently, $\overline{D}_y\pi = 0$. It then follows that $\pi = 0$ since it belong to $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;(\mathcal{B}^{1,p}_{\#A_y})^N)$.

Now, choosing $\psi_1 = \phi\otimes\varphi\otimes w$ in (6.35) with $\phi \in B(\bar{\Omega})$, $\varphi \in \mathcal{C}^\infty_0(Q_T)$ and $w \in [\mathcal{D}_{A_\tau}(\mathbb{R}_\tau)\otimes\varrho^N_y(A^\infty_{y,\rm div})]$, we obtain by disintegration the following equation:

$$\left\{\begin{array}{l} \int_{\Delta(A)} \widehat{a}\mathbb{D}\mathbf{u}\cdot\partial\widehat{w}d\beta + \int_{\Delta(A)} \widehat{b}(\cdot,\mathbb{D}\mathbf{u})\cdot\partial\widehat{w}d\beta = 0 \\ \text{for all } w \in \mathcal{D}_{A_\tau}(\mathbb{R}_\tau)\otimes\varrho^N_y(A^\infty_{y,\rm div}). \end{array}\right. \tag{6.38}$$

Coming back to (6.37) we choose there $\xi = \nabla\mathbf{u}_0(x,t,\omega)$ (for arbitrarily fixed $(x,t,\omega) \in Q_T\times\bar{\Omega}$). Comparing the resulting equation with (6.38) and using the density of $\mathcal{D}_{A_\tau}(\mathbb{R}_\tau)\otimes\varrho^N_y(A^\infty_{y,\rm div})$ in $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\rm div})$ we get by the uniqueness of the solution of (6.37) that $\mathbf{u}_1 = \pi(\nabla\mathbf{u}_0)$, where $\pi(\nabla\mathbf{u}_0)$ stands for the function $(x,t,\omega)\mapsto\pi(\nabla\mathbf{u}_0(x,t,\omega))$ defined from $Q_T\times\bar{\Omega}$ into $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\rm div})$. This shows the uniqueness of the solution of (6.35).

As for (6.36), let, for $\xi \in \mathbb{R}^{N\times N}$,

$$M(\xi) = \int_{\Delta(A)} \widehat{b}(\cdot,\xi+\partial\widehat{\pi}(\xi))d\beta$$

and

$$\mathsf{m}\xi = \int_{\Delta(A)} \widehat{a}(\xi+\partial\widehat{\pi}(\xi))d\beta.$$

Then substituting $\mathbf{u}_1 = \pi(\nabla\mathbf{u}_0)$ in (6.36) and choosing there the special test function $\psi_0(x,t,\omega) = \phi(\omega)\chi(t)\varphi(x)$ with $\phi \in B(\bar{\Omega})$, $\chi \in \mathcal{C}^\infty_0(0,T)$ and $\varphi \in \mathcal{V}$, we

quickly obtain by Itô's formula, the macroscopic homogenized problem (which holds as an equality in V')

$$\begin{cases} d\mathbf{u}_0 + [-\operatorname{div}(\mathsf{m}\nabla\mathbf{u}_0 - M(\nabla\mathbf{u}_0)) + B(\mathbf{u}_0)]dt = \mathbf{f}dt + \widetilde{g}(\mathbf{u}_0)d\bar{W} \\ \mathbf{u}_0(0) = \mathbf{u}^0. \end{cases} \tag{6.39}$$

Since the above problem is of the same type as (6.13) the existence and the uniqueness of its solution is ensured by the same arguments as (6.13). We therefore have the following

Theorem 6.13. *Assume that* (6.2)-(6.7) *hold. Moreover suppose that* (6.26) *holds true. Let* $3 \leq p < \infty$. *For each* $\varepsilon > 0$ *let* $\mathbf{u}_\varepsilon$ *be the unique solution of* (6.13) *on a given stochastic system* $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W$ *defined as in Section* 4. *Then as* $\varepsilon \to 0$, *the whole sequence* $\mathbf{u}_\varepsilon$ *converges in probability to* $\mathbf{u}_0$ *in* $L^2(Q_T)^N$ *(i.e.,* $||\mathbf{u}_\varepsilon - \mathbf{u}_0||_{L^2(Q_T)^N}$ *converges to zero in probability) where* $\mathbf{u}_0$ *is the unique strong probabilistic solution of* (6.39) *with* $\bar{W}$ *replaced by* W.

Proof. The proof of this Theorem is copied on that of [32, Theorem 8]. □

Remark 6.14. The deterministic counterpart of the above result has been recently obtained in [17] in the case of density-dependent Navier-Stokes equations.

6.4. **Some concrete applications of the results of the previous subsection.** A look at the previous subsection reveals that the homogenization process has been made possible because of the assumption (6.26) which was fundamental in the said subsection. This assumption is formulated in a general fashion encompassing a variety of concrete behaviours as regard the coefficients of the operator involved in (6.1). We aim at providing in this subsection some natural situations leading to the homogenization of (6.1).

Example 6.15. The homogenization of (6.1) can be achieved under the periodicity assumption

$(6.26)_1$ The functions $b_i(\cdot,\cdot,\lambda)$, a_{ij} and $g_k(\cdot,\cdot,\mu)$ are both periodic of period 1 in each scalar coordinate.
This leads to (6.26) with $A = \mathcal{C}_{\rm per}(Y\times Z) = \mathcal{C}_{\rm per}(Y)\odot\mathcal{C}_{per}(Z)$ (the product algebra, with $Y = (0,1)^N$ and $Z = (0,1)$), and hence $B^r_A = L^r_{\rm per}(Y\times Z)$ for $1 \leq r \leq \infty$.

Example 6.16. The above functions in $(6.26)_1$ are both Besicovitch almost periodic in (y,τ). This amounts to (6.26) with $A = AP(\mathbb{R}^{N+1}_{y,\tau}) = AP(\mathbb{R}^N_y)\odot AP(\mathbb{R}_\tau)$ ($AP(\mathbb{R}^N_y)$ the Bohr almost periodic functions on $\mathbb{R}^N_y$).

Example 6.17. The homogenization problem for (6.1) can be may be considered under the assumption

$(6.26)_2$ $b_i(\cdot,\cdot,\lambda)$ is weakly almost periodic while the functions a_{ij} and $g_k(\cdot,\cdot,\mu)$ are almost periodic in the Besicovitch sense. This yields (6.26) with $A = WAP(\mathbb{R}^N_y)\odot WAP(\mathbb{R}_\tau)$ ($WAP(\mathbb{R}^N_y)$, the algebra of continuous weakly almost periodic functions on $\mathbb{R}^N_y$; see e.g., [20]).

Example 6.18. Let A_τ be the algebra of Example 5.3 (see Subsection 5.3). It is known that A_τ is not ergodic [25, p. 243]. We may however study the homogenization problem for (6.1) under the assumption that

(6.26)$_3$ $b_i(y,\cdot,\lambda) \in B^{p'}_{A_\tau}$ and $b_i(\cdot,\tau,\lambda)$ is weakly almost periodic; a_{ij} is periodic and $g_k(\cdot,\cdot,\mu)$ is almost periodic.

This assumption is more involved. In fact, let $A_{1,\tau}$ be the algebra generated by $AP(\mathbb{R}_\tau) \cup A_\tau$. It is a fact that $A_{1,\tau}$ is an algebra wmv on $\mathbb{R}_\tau$ which is not ergodic. Next, let $A = WAP(\mathbb{R}^N_y) \odot A_{1,\tau}$. Then, also A is not ergodic, and it can be easily shown that (6.26) is satisfied with the above A.

Many other examples can be considered. We may also consider an example involving only non ergodic algebras by taking for example A to be $N+1$ copies of the A_τ's above: $A = A_\tau \odot \ldots \odot A_\tau$, $N+1$ times, which gives a non ergodic algebra on $\mathbb{R}^{N+1}$.

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