ON THE MEASURE OF THE SPECTRUM OF DIRECT INTEGRALS

ANTON A. KUTSENKO

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ABSTRACT. We obtain the estimate of the Lebesgue measure of the spectrum for the direct integral of matrix-valued functions. These estimates are applicable for a wide class of discrete periodic operators. For example these results give new and sharp spectral bounds for 1D periodic Jacobi matrices and 2D discrete periodic Schrodinger operators.

1. INTRODUCTION

The direct integral plays important role in the theory of periodic operators of mathematical physics [6]. For example the Schrodinger operator, the discrete approximations of elastic wave equations, etc. are unitarily equivalent to the direct integrals of some analytic families of finite matrices. So the spectrum of discrete periodic operators coincides with the spectrum of corresponding direct integrals. The main goal of this paper is to provide simple and efficient estimates of the total measure of spectra for such direct integrals. In particular, the spectral estimates for Jacobi matrices in terms of its components, presented in this paper improve well-known estimates obtained by different way (by using monodromy matrix approach and special polynomials).

We start from some well-known definitions, see e.g. [6]. Let K be a compact metric space with a regular Borel measure μ, which satisfies $0 < \mu(G) < +\infty$ for any open subset $G \subset K$. Define the Hilbert space $L_N^2 = \oplus_{n=1}^N L^2(K)$ of all

quadratic-summable vector-valued functions $f : K \to \mathbb{C}^N$. Let $\mathcal{A} : L_N^2 \to L_N^2$ be some bounded self-adjoint operator given by

$$\mathcal{A}(f)(k) = A(k)f(k), \quad \forall k \in K, \tag{1.1}$$

where $A : K \to \mathbb{C}^{N \times N}$ is some continuous matrix-valued function. The matrices $A(k)$ are self-adjoint with dimension $N \times N$ for any $k \in K$. It is well known (see e.g. [6]) that the spectrum of the operator $\mathcal{A}$ consists of spectra of $A(k)$

$$\sigma(\mathcal{A}) = \bigcup_{k \in K} \sigma(A(k)).$$

Denoting eigenvalues of $A(k)$ as $\lambda_1(k) \leqslant \cdots \leqslant \lambda_N(k)$ we obtain

$$\sigma(\mathcal{A}) = \bigcup_{n=1}^N \lambda_n(K). \tag{1.2}$$

Note that all $\lambda_n : K \to \mathbb{R}$ are continuous functions, since A is a continuous function on K. Our goal is to obtain estimates of the Lebesgue measure of the spectrum $\mathrm{mes}(\sigma(\mathcal{A}))$ in terms of $A(k)$, but without calculating eigenvalues $\lambda_n(k)$. Simple estimate immediately gives us

$$\mathrm{mes}(\sigma(\mathcal{A})) \leqslant 2\|\mathcal{A}\| = 2 \max_{k \in K} \|A(k)\|, \tag{1.3}$$

but usually this estimate is not very accurate. For example if $A(k) = A_0 = \mathrm{const}$, $k \in K$ then (1.3) gives us $\mathrm{mes}(\sigma(\mathcal{A})) \leqslant 2\|A_0\|$ but in fact $\mathrm{mes}(\sigma(\mathcal{A})) = 0$.

We restrict our consideration to the case in which the matrices $A(k)$ are of the form

$$A(k) = \sum_{j=1}^M (\varphi_j(k)A_j + \overline{\varphi_j(k)}A_j^*), \tag{1.4}$$

where $*$ denotes hermitian conjugate, $\overline{z}$ denotes complex conjugate, A_m are constant matrices (not necessarily self-adjoint) and $\varphi_m : K \to \mathbb{C}$ are some continuous functions. For any subset $S \subset \mathbb{C}$ we denote

$$\mathrm{diam}(S) = \sup\{|z_1 - z_2|, \ z_1, z_2 \in S\}.$$

Theorem 1.1. *For the operator $\mathcal{A}$ (1.1) with $A(k)$ satisfied (1.4) the following estimate for the Lebesgue measure of the spectrum is fulfilled*

$$\mathrm{mes}(\sigma(\mathcal{A})) \leqslant 2 \sum_{j=1}^M \mathrm{diam}(\varphi_j(K)) \, \mathrm{Tr}(A_j^* A_j)^{\frac{1}{2}}. \tag{1.5}$$

This Theorem can be extended to the case of "piecewise" continuous functions φ_m. The bound (1.5) can be improved when some terms in (1.4) have overlapping spectra. To find absolute gap in the spectrum of $\mathcal{A}$ we can apply estimates of spectral curves $\lambda_n^- \leqslant \lambda_n(k) \leqslant \lambda_n^+$ (2.4), where $\lambda_n^{\pm}$ do not depend on k.

Usually, the discrete Schrödinger operators on periodic graphs are unitarily equivalent to the direct integrals of matrices $A(k)$ of the form (1.4). This allows us to obtain efficient bounds of the total length of the spectral bands of such operators. Now we consider some of the most common discrete periodic operators.

1D Jacobi matrices with matrix valued periodic coefficients. Consider 1D periodic Jacobi matrix $\mathcal{J} : \ell_m^2(\mathbb{Z}) \to \ell_m^2(\mathbb{Z})$ given by

$$\mathcal{J}y = a_{n-1}^* y_{n-1} + b_n y_n + a_n y_{n+1}, \quad y_n \in \mathbb{C}^m, \tag{1.6}$$

where matrices $a_n, b_n \in \mathbb{C}^{m \times m}$ satisfy

$$a_{n+p} = a_n, \quad b_{n+p} = b_n, \quad b_n = b_n^*, \quad \text{for all } n \in \mathbb{Z}$$

for some period $p \in \mathbb{N}$. This operator is unitarily equivalent to $\mathcal{A}$ (3.1). Applying Theorem 1.1 we obtain

Theorem 1.2. *The following estimate is fulfilled*

$$\mathrm{mes}(\sigma(\mathcal{J})) \leqslant 4 \min_n \mathrm{Tr}(a_n^* a_n)^{\frac{1}{2}}. \tag{1.7}$$

For $m = 1$ (1.7) improves the well known estimate from [1] (and [5] for the almost periodic case)

$$\mathrm{mes}(\sigma(\mathcal{J})) \leqslant 4|a_1 \cdots a_p|^{\frac{1}{p}}. \tag{1.8}$$

For example if $p > 1$ and all $a_n = T$ except one $a_{n_0} = 1$ inside the period, then (1.7) gives $\mathrm{mes}(\sigma(\mathcal{J})) \leqslant 1$ while (1.8) gives $\mathrm{mes}(\sigma(\mathcal{J})) \leqslant T^{\frac{p-1}{p}} \to \infty$ for $T \to \infty$.

For $m > 1$ we are unaware of existence in the literature of any estimates similar to (1.7). Moreover, estimate (1.7) is sharp. In particular, inequality (3.2) becomes the equality in the case of $a_n = I_m$ (identity matrix) and $b_n = \mathrm{diag}(4k)_1^m$ for all n (see example below (3.2)).

2D discrete periodic Schrodinger operator. Consider the operator

$$\mathcal{J}_1 y_{n,m} = y_{n,m-1} + y_{n,m+1} + y_{n-1,m} + y_{n+1,m} + q_{n,m} y_{n,m}, \quad n, m \in \mathbb{Z}$$

with N, M-periodic sequence $q_{n,m}$, i.e.

$$q_{n+N,m+M} = q_{n,m} \in \mathbb{R} \quad \text{for all } n, m \in \mathbb{Z}.$$

The operator $\mathcal{J}_1$ is unitarily equivalent to the multiplication by the matrix $J_1(k_1, k_2)$ (4.3). Applying Theorem 1.1 to this case leads to

Theorem 1.3. *For any N, M-periodic sequence $q_{n,m} \in \mathbb{R}$ the following estimate is fulfilled*

$$\mathrm{mes}(\sigma(\mathcal{J}_1)) \leqslant 4(N + M). \tag{1.9}$$

For $N = M = 1$ (1.9) reaches equality in the case of $q_{n,m} = \mathrm{const}$.

For arbitrary periods N, M we construct the example (5.1) with $|\sigma(\mathcal{J}_1)| \approx 4\max(N, M)$, which is 2 times worse than (1.9). This example shows us that the power of N, M in (1.9) is precisely 1 but common factor equal to 4 in (1.9) may probably be reduced. For the moment, it can only be claimed that the exact value of this common factor lies between 2 and 4.

2. Proof of Theorem 1.1

For self-adjoint matrices B, C we will write $B \leqslant C$ iff $C - B$ is a positive-semidefinite matrix, i.e. $x^*(C - B)x \geqslant 0$ for all $x \in \mathbb{C}^N$. Also we denote $|B| = (B^* B)^{\frac{1}{2}}$ (see [7]). The matrix $|B|$ is positive-semidefinite and self-adjoint.

Lemma 2.1. *For any complex matrix B the following inequalities are fulfilled*

$$-|B| - |B^*| \leqslant B + B^* \leqslant |B| + |B^*|. \tag{2.1}$$

Proof. There exists unitary matrix U ($U^{-1} = U^*$) which satisfies $B = U|B|$ (polar decomposition). Then $B^* = |B|U^*$ and $|B^*| = U|B|U^*$. For any $x, y \in \mathbb{C}^N$ we denote $(x, y)_1 \equiv x^*|B|y$, which is a Hermitian form. Note that $(y, y)_1 \geqslant 0$ for any $y \in \mathbb{C}^N$. Substituting $y = U^*x + x$, $x \in \mathbb{C}^N$ into $(y, y)_1 \geqslant 0$ we obtain

$$0 \leqslant (U^*x + x, U^*x + x)_1 = (U^*x, U^*x)_1 + (x, x)_1 + (U^*x, x)_1 + (x, U^*x)_1 =$$

$$x^*U|B|U^*x + x^*|B|x + x^*U|B|x + x^*|B|U^*x = x^*(|B^*| + |B| + B + B^*)x,$$

which gives us the first inequality in (2.1). Analogously substituting $y = U^*x - x$, $x \in \mathbb{C}^N$ into $(y, y)_1 \geqslant 0$ we obtain the second inequality in (2.1). $\qquad\square$

Proof of Theorem 1.1. There exist points $s_j \in \mathbb{C}$, $j = 1, \cdots, M$ for which

$$\frac{1}{2} \operatorname{diam}(\varphi_j(K)) = \max\{|s - s_j|, \ s \in \varphi_j(K)\}. \tag{2.2}$$

Denote

$$B_0 = \sum_{j=1}^{M} (s_j A_j + \overline{s_j} A_j^*).$$

Using (2.1) and (2.2) we deduce that

$$A(k) - B_0 = \sum_{j=1}^{M} ((\varphi_j(k) - s_j)A_j + (\overline{\varphi_j(k) - s_j})A_j^*) \leqslant$$

$$\sum_{j=1}^{M} |\varphi_j(k) - s_j|(|A_j| + |A_j^*|) \leqslant \frac{1}{2} \sum_{j=1}^{M} \operatorname{diam}(\varphi_j(K))(|A_j| + |A_j^*|) \equiv B_1 \tag{2.3}$$

and analogously $-B_1 \leqslant A(k) - B_0$. Then we obtain two-sided inequalities

$$B_0 - B_1 \leqslant A(k) \leqslant B_0 + B_1,$$

where B_0, B_1 do not depend on k. Thus for eigenvalues of $A(k)$ (see above (1.2)) we deduce that

$$\lambda_n^- \leqslant \lambda_n(k) \leqslant \lambda_n^+, \tag{2.4}$$

where $\lambda_1^\pm \leqslant \cdots \leqslant \lambda_N^\pm$ are eigenvalues of $B_0 \pm B_1$ respectively. Identity (1.2) gives us

$$\sigma(\mathcal{A}) = \bigcup_{n=1}^{N} \lambda_n(K) \subset \bigcup_{n=1}^{N} [\lambda_n^-, \lambda_n^+].$$

Since $\lambda_n^\pm$ do not depend on k, we obtain

$$\operatorname{mes}(\sigma(\mathcal{A})) \leqslant \sum_{n=1}^{N} (\lambda_n^+ - \lambda_n^-) = \operatorname{Tr}(B_0 + B_1) - \operatorname{Tr}(B_0 - B_1) = 2\operatorname{Tr} B_1. \tag{2.5}$$

Combining (2.5) with the definition of B_1 (2.3) and with the identity $\operatorname{Tr}|C| = \operatorname{Tr}|C^*|$ for any complex matrix C gives us (1.5).

3. Proof of Theorem 1.2

The operator $\mathcal{J}$ (1.6) is unitarily equivalent to the operator $\mathcal{A} : L^2_{mp} \to L^2_{mp}$ (see e.g. [3], [4]), where $L^2_{mp} = \oplus_{n=1}^{mp} L^2[0, 2\pi]$ and

$$\mathcal{A}(f)(k) = A(k)f(k), \quad f \in L^2_{mN}, \quad A(k) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & e^{-ik}a_0^* \\ a_1^* & b_2 & a_2 & \cdots & 0 \\ 0 & a_2^* & b_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{ik}a_0 & 0 & 0 & \cdots & b_N \end{pmatrix}.$$

(3.1)

Then $A(k) = A_0 + \varphi_0(k)B_0 + \overline{\varphi}_0(k)B_0^*$, where $\varphi_0 = e^{ik}$ and constant matrices A_0, B_0 are given by

$$A_0 = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1^* & b_2 & a_2 & \cdots & 0 \\ 0 & a_2^* & b_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_N \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Applying Theorem 1.1 to this case we deduce that

$$\mathrm{mes}(\sigma(\mathcal{A})) \leqslant 2\mathrm{diam}(\varphi_0([0, 2\pi]))\,\mathrm{Tr}(B_0^* B_0)^{\frac{1}{2}} = 4\,\mathrm{Tr}(a_0^* a_0)^{\frac{1}{2}}.$$

Since we may shift sequences a_n, b_n, it follows that any element a_n may be chosen instead of a_0 and thus we obtain

$$\mathrm{mes}(\sigma(\mathcal{J})) = \mathrm{mes}(\sigma(\mathcal{A})) \leqslant 4\min_n \mathrm{Tr}(a_n^* a_n)^{\frac{1}{2}},$$

(3.2)

which coincides with (1.7) in Theorem 1.2. Estimate (3.2) is sharp. Let J be the Jacobi matrix with elements $a_n = I_m$ ($m \times m$ identity matrix) and $b_n = \mathrm{diag}(4k)_{k=1}^m$ for any n. Since all a_n and b_n are diagonal matrices, J is unitarily equivalent to the direct sum of scalar Jacobi operators. In our case this is the direct sum of shifted discrete Shrödinger operators $\oplus_{k=1}^m (J^0 + 4kI)$ (J_0 is a scalar Jacobi matrix with $a_n^0 = 1$, $b_n^0 = 0$ and I is an identity operator). Then

$$\sigma(J) = \bigcup_{k=1}^m \sigma(J^0 + 4kI) = \bigcup_{k=1}^m [-2 + 4k, 2 + 4k] = [2, 2 + 4m],$$

which gives us $\mathrm{mes}(\sigma(J)) = 4m = 4\,\mathrm{Tr}(a_n a_n^*)^{\frac{1}{2}}$.

Remark. Now we restrict the operator $\mathcal{A}$ on some interval $[\alpha, \beta] \subset [0, \pi]$, i.e. consider $\mathcal{A}_{\alpha\beta} : \oplus_{n=1}^{mN} L^2[\alpha, \beta] \to \oplus_{n=1}^{mN} L^2[\alpha, \beta]$ given by (3.1). Then Theorem 1.1 gives us

$$\mathrm{mes}(\sigma(\mathcal{A}_{\alpha\beta})) \leqslant 4\sin\frac{\beta - \alpha}{2} \min_n \mathrm{Tr}(a_n^* a_n)^{\frac{1}{2}},$$

since $\mathrm{diam}(\varphi_0([\alpha, \beta])) = 2\sin\frac{\beta - \alpha}{2}$.

4. Proof of the Theorem 1.3

Introduce the following self-adjoint $N \times N$ matrices

$$
S(k_1) = \begin{pmatrix} 0 & 0 & \cdots & e^{-2\pi i k_1} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e^{2\pi i k_1} & 0 & \cdots & 0 \end{pmatrix}, \quad A_m = \begin{pmatrix} q_{1,m} & 1 & \cdots & 0 \\ 1 & q_{2,m} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & q_{N,m} \end{pmatrix}, \quad k_1 \in \mathbb{R},
$$

$$(4.1)$$

where the matrix S contains two non-zero components and A is a tridiagonal matrix. Also introduce the following self-adjoint $NM \times NM$ matrices

$$
R(k_2) = \begin{pmatrix} 0 & 0 & \cdots & e^{-2\pi i k_2}I \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e^{2\pi i k_2}I & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} A_1 & I & \cdots & 0 \\ I & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_M \end{pmatrix}, \quad k_2 \in \mathbb{R},
$$

$$(4.2)$$

$$
C(k_1) = \operatorname{diag}(S(k_1))_1^M, \quad J_1(k_1, k_2) = B + C(k_1) + R(k_2). \tag{4.3}
$$

We apply the standard scheme of rewriting periodic operator into the direct integral of operators with the discrete spectrum (see e.g. [6], XIII.16, p.279). Define the following unitary operator

$$
\mathcal{U} : \ell^2(\mathbb{Z}^2) \to \int_{k_1, k_2 \in [0,1]}^{\oplus} \mathbb{C}^{NM} dk_1 dk_2, \quad \mathcal{U}(u_{n,m}) = (v_{n,m}(k_1, k_2))_{1,1}^{N,M},
$$

where

$$
v_{n,m}(k_1, k_2) = \sum_{n_1, m_1 \in \mathbb{Z}} \exp(2\pi i(n_1 k_1 + m_1 k_2)) u_{n+n_1 N, m+m_1 M}.
$$

It can be shown that the operator $\mathcal{J}_1 : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$ (1.1) is unitarily equivalent to the operator of multiplying by the matrix $J(k_1, k_2)$

$$
\mathcal{U}\mathcal{J}_1\mathcal{U}^{-1}v = J_1(k_1, k_2)v, \quad \forall v = (v_{n,m}(k_1, k_2))_{1,1}^{N,M}.
$$

By the analogy with the Proof of Theorem 1.2, applying the Theorem 1.1 to the operator of multiplication by the matrix $J_1(k_1, k_2)$ (see (4.1)-(4.3)) leads to the results of Theorem 1.3.

5. 2D discrete periodic Schrodinger operator with large spectrum

Without loss of generality we assume $M \geqslant N$. Introduce the potential

$$
q_{n,m} = \varepsilon^{-1}m, \quad n = 1, \cdots, N, \quad m = 1, \cdots, M, \quad \varepsilon > 0. \tag{5.1}
$$

Then the matrix J_1 (4.3) is

$$
J_1(k_1, k_2) = \varepsilon^{-1}(J_0 + \varepsilon D(k_1, k_2)), \quad \text{where} \quad J_0 = \operatorname{diag}(mI)_1^M, \tag{5.2}
$$

$$D(k_1, k_2) = \begin{pmatrix} S_1(k_1) & I & \cdots & e^{-2\pi i k_2} \\ I & S_1(k_1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e^{2\pi i k_2} & 0 & \cdots & S_1(k_1) \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 1 & \cdots & e^{-2\pi i k_1} \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e^{2\pi i k_1} & 0 & \cdots & 0 \end{pmatrix}.$$

For identifying the spectrum of J we will apply the regular perturbation theory with small parameter ε. The eigenvalues of J_0 (5.2) are

$$\lambda_n^{(j)}(0) = j, \quad n = 1, \cdots, N, \quad j = 1, \cdots, M,$$

with the corresponding eigenvectors

$$e_n^{(j)} = e_{n+N(j-1)}, \quad \text{where} \quad e_i = (\delta_{ij})_1^{NM}$$

and δ is a Kronecker symbol. Each eigenvalue j has multiplicity N. The perturbation theory for multiple eigenvalues (see e.g. [2]) tells us that the eigenvalues of J (5.2) satisfy

$$\lambda_n^{(j)}(\varepsilon) = \varepsilon^{-1}(j + \varepsilon\widetilde{\lambda}_n^{(j)} + O(\varepsilon^2)), \tag{5.3}$$

where $\widetilde{\lambda}_n^{(j)} \equiv \widetilde{\lambda}_n^{(j)}(k_1, k_2)$, $n = 1, \cdots, N$ are eigenvalues of the matrix

$$(e_1^{(j)}..e_N^{(j)})^\top D(k_1, k_2)(e_1^{(j)}..e_N^{(j)}) = S_1.$$

It is well known that the spectrum of $\int_{k_1 \in [0,1]}^{\oplus} S_1(k_1)$ coincides with $[-2, 2]$ (since S_1 corresponds to the 1D discrete non-perturbed Schrodinger operator), i.e.

$$\bigcup_{n=1}^{N} \bigcup_{k_1, k_2 \in [0,1]} \{\widetilde{\lambda}_n^{(j)}(k_1, k_2)\} = [-2, 2]. \tag{5.4}$$

So, using (5.3) with (5.4) we deduce that

$$\bigcup_{n,j=1}^{N,M} \bigcup_{k_1, k_2 \in [0,1]} \{\lambda_n^{(j)}(\varepsilon)\} = 4M + O(\varepsilon),$$

since intervals $[-2, 2] + \varepsilon^{-1}j$ do not overlap each other for sufficiently small ε. Then the spectrum $\sigma(\mathcal{J}_1)$ of $\mathcal{J}_1$ with the potential (5.1) has Lebesgue measure

$$\mathrm{mes}(\sigma(\mathcal{J}_1)) = 4M + O(\varepsilon) = 4\max\{N, M\} + O(\varepsilon).$$

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École Nationale Supérieure de Géologie, CRPG-CNRS, Université de Lorraine, 54501 Vandoeuvre-lés-Nancy, France.

E-mail address: aak@nxt.ru