INEQUALITIES FOR INTERPOLATION FUNCTIONS

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Abstract. In this paper, in relation with interpolation functions we study some generalized Powers-\textit{Størmer}'s type inequalities and monotonicity inequality of indefinite type which generalizes a result of Ando.

1. Introduction and preliminaries

Throughout this paper, $M_n$ stands for the algebra of all $n \times n$ matrices. Denote by $M_n^+$ the set of all positive semi-definite matrices. A continuous function $f$ on $I \subset \mathbb{R}$ is called \textit{matrix convex of order} $n$ (or \textit{n-convex}) if the inequality

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$$

holds for all self-adjoint matrices $A, B \in M_n$ with $\sigma(A), \sigma(B) \subset I$ and for all $\lambda \in [0, 1]$, where $\sigma(A)$ stands for the spectrum of $A$. Also, $f$ is called a $n$-concave on $I$ if $-f$ is $n$-convex on $I$.

A continuous function $f$ on $I$ is called \textit{matrix monotone of order} $n$ or \textit{n-monotone}, if

$$A \leq B \implies f(A) \leq f(B)$$

for any pair of self-adjoint matrices $A, B \in M_n$ with $\sigma(A), \sigma(B) \subset I$. We call a function $f$ \textit{operator convex} (resp. \textit{operator concave}) if $f$ is $k$-convex (resp. $k$-concave) for any $k \in \mathbb{N}$, and \textit{operator monotone} if $f$ is $k$-monotone for any $k \in \mathbb{N}$.

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A function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) (where \( \mathbb{R}_+ = (0, \infty) \)) is called an interpolation function of order \( n \) if for any \( T, A \in M_n \) with \( A > 0 \) and \( T^* T \leq 1 \),
\[
T^* AT \leq A \quad \implies \quad T^* f(A) T \leq f(A).
\]

We denote by \( C_n \) the class of all interpolation functions of order \( n \).

Let \( \mathcal{P}(\mathbb{R}_+) \) be the set of all Pick functions on \( \mathbb{R}_+ \), and \( \mathcal{P}' \) the set of all positive Pick functions on \( \mathbb{R}_+ \), i.e., functions of the form
\[
h(s) = \int_{[0,\infty]} \frac{(1 + t)s}{s + t} d\rho(t), \quad s > 0,
\]
where \( \rho \) is some positive Radon measure on \([0, \infty]\).

Denote by \( \mathcal{P}'_n \) the set of all strictly positive \( n \)-monotone functions on \((0, \infty)\).

Theorem 1.1. ([2, Corollary 2.4]) A function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) belongs to \( C_n \) if and only if for every \( n \)-set \( \{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+ \) there exists a function \( h \) from \( \mathcal{P}' \) such that
\[
f(\lambda_i) = h(\lambda_i) \quad \text{for} \quad i = 1, \ldots, n.
\]

Corollary 1.2. Let \( A \) be a positive definite matrix in \( M_n \) and \( f \in C_n \). Then there exists a positive Radon measure \( \rho \) on \([0, \infty]\) such that
\[
f(A) = \int_{[0,\infty]} A(1 + s)(A + s)^{-1} d\rho(s).
\]

Remark 1.3.

(i) \( \mathcal{P}' = \cap_{n=1}^\infty \mathcal{P}'_n \) \([13]\), \( \mathcal{P}' = \cap_{n=1}^\infty C_n \) \([7]\);
(ii) \( C_{n+1} \subseteq C_n \);
(iii) \( \mathcal{P}'_{n+1} \subseteq C_{2n+1} \subseteq C_{2n} \subseteq \mathcal{P}'_n \), \( \mathcal{P}'_n \nsubseteq C_n \) \([2]\);
(iv) \( C_{2n} \nsubseteq \mathcal{P}'_n \) \([14]\);
(v) \( C_n \cap C_n \subseteq C_n \);
(vi) A function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) belongs to \( C_n \) if and only if \( \frac{t}{f(t)} \) belongs to \( C_n \).

It is not known whether \( \mathcal{P}'_{n+1} \subseteq C_{2n+1} \) or not.

In this paper, we consider some inequalities with interpolation functions. More precisely, in Section 2, we extend Petz’s trace inequality \([15\), Theorem 11.18\) (Theorem 2.1) to the class of interpolation functions and give a new trace inequality (Theorem 2.5) which might play an important role in the quantum information theory. Moreover, in Section 3 we extend an Ando’s monotonicity inequality of indefinite type. We show that for \( f \in C_{2n} \) and any pair of \( J \)-selfadjoint matrices \( A, B \in M_n \) such that \( \sigma(A), \sigma(B) \subset (0, \infty) \),
\[
A \leq^J B \quad \implies \quad f(A) \leq^J f(B),
\]
where \( J \) is a selfadjoint involution and \( A \leq^J B \) means that \( JA^* J = A, JB^* J = B \), and \( JA \leq JB \).
**Theorem 1.4.** Let $f \in C_{2n}$. For positive definite matrices $K$ and $L$ in $M_n$, let $Q$ be the projection onto the range of $(K - L)_+$. We have, then,
\[
\text{Tr}(QL(f(K) - f(L))) \geq 0. \tag{1.1}
\]

**Proof.** Let $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$ be sets of eigenvalues of $K$ and $L$, respectively. Then by Theorem 1.1 there exists an interpolation function $h \in \mathcal{P}'$ such that $f(\lambda) = h(\lambda)$ for $\lambda \in \{\lambda_i\}_{i=1}^n \cup \{\mu_i\}_{i=1}^n$. By Corollary 1.2 there is some positive Radon measure $\rho$ on $[0, \infty]$ such that

\[
f(K) - f(L) = \int_{[0, \infty]} K(1 + s)(K + s)^{-1}d\rho(s) - \int_{[0, \infty]} L(1 + s)(L + s)^{-1}d\rho(s)
= \int_{[0, \infty]} [(1 + s)(K + s)^{-1}K - L(1 + s)(L + s)^{-1}]d\rho(s)
= \int_{[0, \infty]} (1 + s)s(K + s)^{-1}(K - L)(L + s)^{-1}d\rho(s).
\]

Hence
\[
\text{Tr}(QL(f(K) - f(L))) = \int_{[0, \infty]} (1 + s)s \text{Tr}(QL(K + s)^{-1}(K - L)(L + s)^{-1})d\rho(s)
\]

Repeat the same steps in [15, Theorem 11.18], we get the conclusion. \hfill \square

**Corollary 1.5.** Let $f \in \mathcal{P}_{n+1}'$. For positive definite matrices $K$ and $L$ in $M_n$, let $Q$ be the projection onto the range of $(K - L)_+$. We have, then,
\[
\text{Tr}(QL(f(K) - f(L))) \geq 0.
\]

**Proof.** It is suffices to mention that $\mathcal{P}_{n+1}' \subset C_{2n}$ by Remark 1.3. The conclusion follows from Theorem 1.4. \hfill \square

Using Theorem 1.4 we get a generalized Powers-Størmer’s type inequality. Another generalization of Powers-Størmer inequality can be found in [12]. We need the following lemmas.

**Lemma 1.6.** Let $h : (0, \infty) \to (0, \infty)$ be a function such that the function $th(t)$ is operator monotone. Then the inverse of $\frac{t}{h(t)}$ is operator monotone.

**Proof.** Since $th(t)$ is operator monotone, the function $\frac{1}{h(t)} = \frac{t}{th(t)}$ is operator monotone by [11, Corollary 2.6]. Hence the inverse of $\frac{t}{h(t)}$ is operator monotone from by [3, Lemma 5]. \hfill \square

**Lemma 1.7.** Let $f$ be a function from $(0, \infty)$ into itself such that $tf(t) \in C_{2n}$. Then the inverse of $g(t) = \frac{t}{f(t)}$ ($t > 0$) belongs to $C_{2n}|_{g((0, \infty))}$.

**Proof.** Indeed, for any set $T \subset g((0, \infty))$ with $|T| = 2n$ we can write

\[
T = \{g(t_1), g(t_2), \ldots, g(t_{2n})\},
\]
where \( t_i \in (0, \infty) \) for \( 1 \leq i \leq 2n \). Since \( tf(t) \in \mathcal{C}_{2n} \), there is an interpolation map \( k_T \in \mathcal{P}^\prime \) such that \( t_i f(t_i) = k_T(t_i) \) for \( 1 \leq i \leq 2n \). Then we have

\[
g(t_i) = \frac{t_i}{f(t_i)} = t_i \frac{t_i}{k_T(t_i)} \quad (1 \leq i \leq 2n).
\]

Consequently,

\[
g^{-1}(g(t_i)) = t_i = \left( \frac{t^2}{k_T(t)} \right)^{-1} (g(t_i)) \quad (1 \leq i \leq 2n).
\]

From the above argument, it is clear that \( \left( \frac{t^2}{k_T(t)} \right)^{-1} \) is operator monotone. From (1.2) we conclude that the inverse \( g^{-1} \) of \( g \) belongs to \( C_{2n} |_{g((0,\infty))} \). \( \square \)

The main theorem of this section is as follows.

**Theorem 1.8.** Let \( f \) be a function from \((0, \infty)\) into itself such that \( t f(t) \in \mathcal{C}_{2n} \). Then for any pair of positive definite matrices \( A, B \in M_n \),

\[
\text{Tr}(A^2) + \text{Tr}(B^2) - \text{Tr}(|A^2 - B^2|) \leq 2 \text{Tr}(Af(A)^{1/2} g(B) f(A)^{1/2}),
\]

where \( g(t) = \frac{t}{f(t)}, \; t \in (0, \infty) \).

**Proof.** Let \( A, B \) be positive definite matrices and \( e(t) = tf(t) \) for \( t \in (0, \infty) \). Let \( Q \) be the projection on the range of \((g(A) - g(B))_+ \) and \( L = g(B) \).

Let \( S \) be the set of eigenvalues of \( g(A) \) and \( g(B) \). Since \( e \in \mathcal{C}_{2n} \), there is an interpolation map \( h \in \mathcal{P}^\prime \) such that \( e(\lambda) = h(\lambda) \) for \( \lambda \in S \). Since \( t(h(t)/t) = h(t) \) is operator monotone, the inverse of \( t^2/h(t) \) is operator monotone by Lemma 1.6. By Lemma 1.7 the inverse of \( g \) belongs to \( C_{2n} |_{g((0,\infty))} \). Consequently, \( e \circ g^{-1} \in C_{2n} |_{g((0,\infty))} \) by Remark 1.3(v).

Apply Theorem 1.4 for the function \( e \circ g^{-1} \), we get

\[
0 \leq \text{Tr}(Qg(B))(e \circ g^{-1})(g(A)) - (e \circ g^{-1})(g(B)) = \text{Tr}(Qg(B)(Af(A) - Bf(B))) = \text{Tr}(Qg(B)Af(A)) - \text{Tr}(QB^2).
\]

On the contrary,

\[
\text{Tr}(Q(A^2 - B^2)) - \text{Tr}(Af(A)Q(g(A) - g(B))) = \text{Tr}(QA^2) - \text{Tr}(QB^2) - \text{Tr}(Af(A)Qg(A)) + \text{Tr}(Af(A)Qg(B)) \quad (1.4)
\]

\[
= \text{Tr}(Qg(B)Af(A)) - \text{Tr}(QB^2) \geq 0.
\]

Hence we have

\[
\text{Tr}(Af(A)Q(g(A) - g(B))) \leq \text{Tr}(Q(A^2 - B^2)) \leq \text{Tr}((A^2 - B^2)_+). \quad (1.5)
\]

Therefore, from (1.4) and (1.5) we have
\[
\text{Tr}(Af(A)(g(A) - g(B))) \leq \text{Tr}(Af(A)(g(A) - g(B))_+)
\]
\[
= \text{Tr}(Af(A)Q(g(A) - g(B)))
\]
\[
\leq \text{Tr}((A^2 - B^2)_+)
\]
\[
= \frac{1}{2} \text{Tr}((A^2 - B^2) + |A^2 - B^2|),
\]
and
\[
\text{Tr}(A^2 + B^2 - |A^2 - B^2|) \leq 2 \text{Tr}(Af(A)g(B)).
\]

\[\square\]

**Corollary 1.9.** Let \( f \) be a function from \((0, \infty)\) into itself such that \( tf(t) \in \mathcal{P}'_{n+1} \). Then for any pair of positive definite matrices \( A, B \in M_n \),
\[
\text{Tr}(A^2) + \text{Tr}(B^2) - \text{Tr}(|A^2 - B^2|) \leq 2 \text{Tr}(Af(A)^{1/2}g(B)f(A)^{1/2}),
\]
where \( g(t) = \frac{1}{t(f(t))} \) for \( t \in (0, \infty) \).

**Corollary 1.10 ([5]).** Let \( A, B \) be positive definite matrices, then for all \( 0 \leq s \leq 1 \)
\[
\text{Tr}(A + B - |A - B|) \leq 2 \text{Tr}(A^{1-s}B^s).
\]

**Proof.** By adding \( \varepsilon > 0 \) to \( A \) and \( B \), we may assume that \( A \) and \( B \) are positive invertible matrices.

Firstly, we consider the case \( s \in [\frac{1}{2}, 1] \). Let \( f(t) = t^{1-2s} \). Then \( tf(t) = t^{2-2s} \)
is operator monotone on \((0, \infty)\). Substitute \( X = A^{1/2} \) and \( Y = B^{1/2} \) into the inequality (1.3) in Theorem 1.8, we get
\[
\text{Tr}(A + B - |A - B|) \leq 2 \text{Tr}(A^{1-s}B^s).
\]
The remaining case \( 0 \leq s \leq \frac{1}{2} \) obviously follows by interchanging the roles of \( A \) and \( B \). \( \square \)

**Remark 1.11.** In Lemma 1.6 and Lemma 1.7 operator monotonicity and \( C_{2n}\)-property of inverse functions were considered. There exists counterexample that the inverse of a \( n \)-matrix function may not be \( n \)-matrix. Indeed, it is well-known that \( f_s(t) = t^s(0 \leq s \leq 1) \) is operator monotone, but the inverse \( f_s^{-1}(t) = t^{1/s} \) of \( f_s \) is not \( 2 \)-monotone. A similar picture for \( C_n \)-functions is still not clear.

Inequality (1.3) in Theorem 1.8 is different to generalized Powers-Størmer inequality in [12]. The proof of (1.3) is based on the fact that \( (tf) \circ g^{-1} \in \mathcal{C}_{2n}|_{g((0, \infty))} \). If we have the condition \( f \circ g^{-1} \in \mathcal{C}_{2n}|_{g((0, \infty))} \), then by similar arguments above we can get the generalized Powers-Størmer inequality as in [12]. More precisely, we have the following theorem.

**Theorem 1.12.** Let \( f \) be a function in \( C_{2n} \) such that \( f \circ g^{-1} \in \mathcal{C}_{2n}|_{g((0, \infty))} \), where \( g(t) = \frac{t}{f(t)} \), \( t \in (0, \infty) \). Then for any pair of positive definite matrices \( A, B \in M_n \),
\[
\text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2 \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}).
\]
Since the proof of this theorem is done by the same steps in Theorem 1.8, the detail is left to reader.

2. Matrix monotonicity inequality of indefinite type

Let \( J (\neq I_n \text{ -- unit in } M_n) \) be a selfadjoint involution different to identity, that means, \( J = J^*, J^2 = I_n \). For a matrix \( A \) its \( J \)-adjoint \( A^J \) is defined as follows: \( A^J = JA^*J \). A matrix \( A \) is said to be \( J \)-selfadjoint if \( A = A^J \), or, \( JA = A^*J \).

For a pair of \( J \)-selfadjoint matrices \( A, B \), we define an indefinite order relation \( A \leq^J B \) as follows:
\[
A \leq^J B \text{ if } JA \leq JB.
\]

It is known as a result of Potapov-Ginzburg (see [6, Chapter 2, Section 4]) that \( \sigma(JA^*JA) \subset [0, +\infty) \) for any \( A \). If \( A \) is a \( J \)-selfadjoint operator with \( \sigma(A) \subset (0, \infty) \), then for any function \( f(t) \in C \alpha \) the matrix \( f(A) \) is well-defined by Corollary 1.2. Note that \( f(A) \) is \( J \)-selfadjoint.

It is well-known that any operator monotone function on \((-1, 1)\) has an integral representation
\[
f(t) = f(0) + \int_{-1}^{1} \frac{t}{1-t}\lambda d\mu(\lambda),
\]
where \( d\mu(\cdot) \) is a positive measure on \([-1, 1]\). T. Ando [4] used this fact to study operator monotonicity inequality of indefinite type.

**Theorem 2.1 ([4], Theorem 4).** Let \( J \) be a selfadjoint involution, and \( A, B \) be \( J \)-selfadjoint matrices with spectra in \((\alpha, \beta)\). Then
\[
A \leq^J B \implies f(A) \leq^J f(B)
\]
for any operator monotone function \( f(t) \) on \((\alpha, \beta)\).

For \( n \)-monotone functions his proof is not applicable, since an integral representation of \( n \)-monotone functions is not clear in general. Fortunately, we can extend Ando’s result to class \( C_2n \), with a help of Corollary 1.2.

The assertions of the following lemma were obtained in [4]. But for convenience of readers we give a proof.

**Lemma 2.2.** Let \( A, B \) be \( J \)-selfadjoint matrices in \( M_n \) such that \( \sigma(A), \sigma(B) \subset (0, +\infty) \). Then
\[
A \leq^J B \implies B^{-1} \leq^J A^{-1}.
\]

**Proof.** Mention that for any matrix \( C \in M_n \),
\[
JC^JBC - JC^JAC = C^\ast(JB - JA)C \geq 0, \text{ i.e. } C^JAC \leq^J C^JBC.
\]

Since \( \sigma(A) \subset (0, +\infty) \) and the function \( f(t) = t^{1/2} \) is operator monotone on \((0, \infty)\), the \( J \)-selfadjoint square root \( A^{1/2} \) is well defined and its reverse \( A^{-1/2} \) is also \( J \)-selfadjoint. In the case \( B = I_n \), we have
\[
A^{-1} - I_n = A^{-1/2}(I_n - A)A^{-1/2} \geq^J 0.
\]
(2.1)

In general case,
\[
I_n = B^{-1/2}BB^{-1/2} \geq^J B^{-1/2}AB^{-1/2} = [A^{1/2}B^{-1/2}]^J A^{1/2}B^{-1/2}.
\]
On account of a result of Potapov-Ginzburg mentioned, and since $B^{-1/2}AB^{-1/2}$ is invertible, the latter implies that $\sigma(B^{-1/2}AB^{-1/2}) \subset (0, +\infty)$. By (2.1), we obtain

$$I_n \leq^J (B^{-1/2}AB^{-1/2})^{-1} = B^{1/2}A^{-1}B^{1/2},$$

which equivalent to $A^{-1} \geq^J B^{-1}$.

**Theorem 2.3.** Let $f \in C_{2n}$. Then for any pair of $J$-selfadjoint matrices $A \leq^J B$ in $M_n$ such that $\sigma(A), \sigma(B) \subset (0, \infty)$,

$$f(A) \leq^J f(B). \quad (2.2)$$

**Proof.** Let $\lambda_i \ (1 \leq i \leq n)$ and $\mu_j \ (1 \leq j \leq n)$ be the sets of eigenvalues of $A$ and $B$, respectively.

Then there is an interpolation function $h \in C_{2n}$ such that $f(\lambda) = h(\lambda)$ for $\lambda \in \{\lambda_i, \mu_j\}_{1 \leq i, j \leq n}$. By Corollary 1.4, there is a positive Radon measure $\rho$ on $[0, \infty]$ such that

$$f(\alpha) = \int_{[0,\infty]} \frac{\alpha(1+s)}{s+\alpha} d\rho(s) \quad (\alpha \in \{\lambda_i, \mu_j\}_{1 \leq i, j \leq n}).$$

Then inequality (2.2) is equivalent to the following:

$$\int_{[0,\infty]} A(1+s)(s+A)^{-1}d\rho(s) \leq^J \int_{[0,\infty]} B(1+s)(s+B)^{-1}d\rho(s).$$

Therefore, it suffices to prove that

$$A(s+A)^{-1} \leq^J B(s+B)^{-1} \quad (s > 0),$$

or equivalently,

$$(s+A)^{-1} \geq^J (s+B)^{-1} \quad (s > 0). \quad (2.3)$$

From $A \leq^J B$ it follows that $s+A \leq^J s+B \quad (s > 0)$. On the other hand, $\sigma(s+A), \sigma(s+B) \subset (s, \infty) \subset (0, \infty)$. On account of Lemma 2.2 we obtain (2.3). □

**Remark 2.4.** A similar conclusion for matrix convex functions on $[0, \infty)$ is wrong. Indeed, it is well-known that the function $f(t) = t^2 \ (t \in (0, \infty))$ is operator convex. Let $A$ be an arbitrary $J$-positive matrix (that means, $JA$ is positive) with spectrum in $(2, \infty)$. Put $B = A + J$. It is clear that $A \leq^J B$ and $\sigma(B) \subset (0, \infty)$. We have

$$f\left(\frac{A}{2} + \frac{B}{2}\right) \not\leq^J \frac{1}{2}f(A) + \frac{1}{2}f(B),$$

that is,

$$\frac{1}{2}(A^2 + B^2) - \left(\frac{A + B}{2}\right)^2 = \frac{1}{4}(B - A)^2 = \frac{1}{4}J^2 = \frac{I}{4} \not\leq^J 0.$$

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