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DILATION OF DUAL G-FRAMES TO DUAL G-RIESZ BASES

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Abstract. In this paper, we study disjoint, strongly disjoint and weakly disjoint g-frames in Hilbert spaces and we provide necessary and sufficient conditions for disjointness, strongly disjointness and weakly disjointness of g-frames. Also, by using the orthogonal projections in Hilbert spaces, we prove that dual g-frames for a Hilbert space can be dilated to a g-Riesz basis for some larger Hilbert space and its dual g-Riesz basis.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable Hilbert space. We call a sequence $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}$ a frame for $\mathcal{H}$ if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

If in (1.1), $A = B = 1$ we say that $F = \{f_i\}_{i\in I}$ is a Parseval frame for $\mathcal{H}$. Let $F = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$, then the operator

$$T_F : l_2(I) \to \mathcal{H}, \quad T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$$

is well define and onto, also its adjoint is

$$T_F^* : \mathcal{H} \to l_2(I), \quad T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}.$$

The operators T_F and T_F^* are called the synthesis and analysis operators of frame $F = \{f_i\}_{i\in I}$. The operator $S_F = T_F T_F^*$ is called the frame operator of frame

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$F = \{f_i\}_{i\in I}$ which is bounded, invertible and positive. Also, for each $f \in \mathcal{H}$, we have

$$f = \sum_{i\in I} \langle f, f_i\rangle S_F^{-1} f_i = \sum_{i\in I} \langle f, S_F^{-1} f_i\rangle f_i. \tag{1.2}$$

We recall that if $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ are frame for a Hilbert space $\mathcal{H}$, then G is called a dual frame of F if

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i, \quad f \in \mathcal{H}.$$

In this case, we say that F, G are dual frames for $\mathcal{H}$. Let $F = \{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$ and $\tilde{f}_i = S_F^{-1} f_i$, for all $i \in I$, then $\widetilde{F} = \{\tilde{f}_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and by (1.2), $\widetilde{F}$ is a dual frame of F. We call $\widetilde{F}$ the canonical dual of F. A sequence $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a Riesz basis for $\mathcal{H}$, if $\overline{\text{span}}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist constants $0 < A \le B < \infty$ such that for every finite scalar sequence $\{c_i\}$ one has

$$A \sum_i |c_i|^2 \le \Big\| \sum_i c_i f_i \Big\|^2 \le B \sum_i |c_i|^2.$$

The concepts of disjoint frames and strongly disjoint frames introduced by Han and Larson [5]. These notions generalized to frames in Banach spaces by Casazza, Han and Larson [4].

Definition 1.1. [5] Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We say that

(1) F and G are disjoint, if $\{f_i \oplus g_i\}_{i\in I}$ is a frame for $\mathcal{H} \oplus K$.
(2) F and G are strongly disjoint, if there are invertible operators $T_1 \in B(\mathcal{H})$ and $T_2 \in B(K)$ such that $\{T_1 f_i\}_{i\in I}$, $\{T_2 g_i\}_{i\in I}$ and $\{T_1 f_i \oplus T_2 g_i\}_{i\in I}$ are respective Parseval frames for $\mathcal{H}$, K and $\mathcal{H} \oplus K$.

Theorem 2.9 of [5] implies the following result.

Proposition 1.2. *Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then*

(1) *F and G are disjoint if and only if $RangeT_F^* \cap RangeT_G^* = \{0\}$ and $RangeT_F^* + RangeT_G^*$ is a closed subspace of $l_2(I)$.*
(2) *F and G are strongly disjoint if and only if $RangeT_F^*$ and $RangeT_G^*$ are orthogonal.*

In 2006, Sun [10] introduced g-frames as a generalization of ordinary frames. Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\langle ., .\rangle_{\mathcal{H}}$ and $\langle ., .\rangle_{\mathcal{K}}$ denote the inner product of $\mathcal{H}$ and $\mathcal{K}$, respectively. Also, $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces and $\|.\|_i$ and $\langle ., .\rangle_i$ denote the norm and inner product of $\mathcal{H}_i$, for all $i \in I$.

Definition 1.3. A sequence of bounded operators $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A_Λ and B_Λ such that

$$A_\Lambda \|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|_i^2 \le B_\Lambda \|f\|^2, \quad f \in \mathcal{H}. \tag{1.3}$$

We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a tight g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if $A_\Lambda = B_\Lambda$ and Parseval g-frame, if $A_\Lambda = B_\Lambda = 1$. A_Λ and B_Λ are called the lower and upper g-frame bounds, respectively. If the right hand inequality in (1.3) holds for all $f \in \mathcal{H}$, then $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. If there is no confusion, we will use the phrase *"g-frame for $\mathcal{H}$"* instead of *"g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$"*.

Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ be given for all $i \in I$. Let us define the space

$$\widehat{\mathcal{H}} = \left\{ \{f_i\}_{i\in I} : f_i \in \mathcal{H}_i, \sum_{i\in I} \|f_i\|_i^2 < \infty \right\}$$

with the inner product given by $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle_{\widehat{\mathcal{H}}} = \sum_{i\in I}\langle f_i, g_i\rangle_i$. It is clear that $\widehat{\mathcal{H}}$ is a Hilbert space with respect to the point wise operations. It is proved in [9], $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$ if and only if the operator

$$T_\Lambda : \widehat{\mathcal{H}} \to \mathcal{H}, \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i \tag{1.4}$$

is well defined and bounded. In this case, the adjoint of T_Λ is

$$T_\Lambda^* : \mathcal{H} \to \widehat{\mathcal{H}}, \quad T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}.$$

Also, a sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ if and only if the operator T_Λ defined by (1.4) is bounded and onto. We call the operators T_Λ and T_Λ^*, the synthesis and analysis operators of Λ, respectively. If $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$, then

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

is a bounded invertible positive operator [10] and every $f \in \mathcal{H}$ has the following representation

$$f = \sum_{i\in I} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in I} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f. \tag{1.5}$$

The operator S_Λ is called the g-frame operator of Λ.
Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ and let $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ for all $i \in I$. Then $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ [10]. We can refer to [1, 2, 8, 11], for some properties of g-frames in Hilbert spaces.

Definition 1.4. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-frames for $\mathcal{H}$ such that

$$f = \sum_{i\in I} \Theta_i^* \Lambda_i f, \quad f \in \mathcal{H},$$

then we say that Θ is a dual g-frame for Λ or Λ and Θ are dual g-frames for $\mathcal{H}$.

By (1.5), $\widetilde{\Lambda}$ is a dual g-frame for Λ, which is called the canonical dual of Λ.

Definition 1.5. We say a sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is

(1) a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A and B such that for any finite subset $F \subseteq I$ one has

$$A\sum_{i\in F}\|g_i\|_i^2 \leq \Big\|\sum_{i\in F}\Lambda_i^* g_i\Big\|_{\mathcal{H}}^2 \leq B\sum_{i\in F}\|g_i\|_i^2, \quad g_i\in\mathcal{H}_i,$$

and Λ is g-complete, i.e.,

$$\{f|\Lambda_i f=0, i\in I\}=\{0\}.$$

(2) a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if for all $f\in\mathcal{H}$, $\sum_{i\in I}\|\Lambda_i f\|_i^2=\|f\|^2$, and

$$\langle\Lambda_i^* g_i,\Lambda_j^* g_j\rangle_{\mathcal{H}}=\delta_{ij}\langle g_i,g_j\rangle, \quad g_i\in\mathcal{H}_i,\, g_j\in\mathcal{H}_j, \quad i,j\in I.$$

2. Disjointness of g-frames

In this section we study disjointness, weakly disjointness and strongly disjointness of g-frames. First of all, we define these notions and related topics.

Definition 2.1. Let $\Lambda=\{\Lambda_i\in B(\mathcal{H},\mathcal{H}_i): i\in I\}$ and $\Theta=\{\Theta_i\in B(\mathcal{K},\mathcal{H}_i): i\in I\}$ be g-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then Λ and Θ are called

(1) disjoint, if $RangeT_\Lambda^*\cap RangeT_\Theta^*=\{0\}$ and $RangeT_\Lambda^*+RangeT_\Theta^*$ is a closed subspace of $\widehat{\mathcal{H}}$.
(2) strongly disjoint, if $RangeT_\Lambda^*\perp RangeT_\Theta^*$.
(3) complementary pair, if $RangeT_\Lambda^*\cap RangeT_\Theta^*=\{0\}$ and

$$RangeT_\Lambda^*+RangeT_\Theta^*=\widehat{\mathcal{H}}.$$

(4) strong complementary pair, if

$$RangeT_\Lambda^*\oplus RangeT_\Theta^*=\widehat{\mathcal{H}}.$$

(5) weakly disjoint if $RangeT_\Lambda^*\cap RangeT_\Theta^*=\{0\}$.

Proposition 2.2. *Let $\Lambda=\{\Lambda_i\in B(\mathcal{H},\mathcal{H}_i): i\in I\}$ and $\Theta=\{\Theta_i\in B(\mathcal{K},\mathcal{H}_i): i\in I\}$ be g-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then Λ and Θ are strongly disjoint if and only if there exist invertible operators $T_1\in B(\mathcal{H})$ and $T_2\in B(\mathcal{K})$ such that $\{\Lambda_i T_1\in B(\mathcal{H},\mathcal{H}_i): i\in I\}$, $\{\Theta_i T_2\in B(\mathcal{K},\mathcal{H}_i): i\in I\}$ and $\{\Delta_i\in B(\mathcal{H}\oplus\mathcal{K},\mathcal{H}_i): i\in I\}$ are respective Parseval g-frames for $\mathcal{H}$, K and $\mathcal{H}\oplus\mathcal{K}$, where*

$$\Delta_i:\mathcal{H}\oplus\mathcal{K}\to\mathcal{H}_i, \quad \Delta_i(f\oplus g)=\Lambda_i T_1 f+\Theta_i T_2 g,$$

for all $i\in I$.

Proof. Let us consider $T_1=S_\Lambda^{-\frac{1}{2}}$ and $T_2=S_\Theta^{-\frac{1}{2}}$. Then $\Lambda_1=\{\Lambda_i T_1\in B(\mathcal{H},\mathcal{H}_i): i\in I\}$ and $\Theta_1=\{\Theta_i T_2\in B(\mathcal{K},\mathcal{H}_i): i\in I\}$ are Parseval g-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Also,

$$RangeT_\Lambda^*=RangeT_{\Lambda_1}^*, \quad RangeT_\Theta^*=RangeT_{\Theta_1}^*.$$

For $f \in \mathcal{H}$ and $g \in \mathcal{K}$ we have

$$\begin{aligned}\sum_{i\in I}\|\Delta_i(f\oplus g)\|_i^2 =&\sum_{i\in I}\|\Lambda_i T_1 f+\Theta_i T_2 g\|_i^2\\ =&\sum_{i\in I}\|\Lambda_i T_1 f\|_i^2+\sum_{i\in I}\|\Theta_i T_2 g\|_i^2\\ &+2Re\sum_{i\in I}\langle\Lambda_i T_1 f,\Theta_i T_2 g\rangle_i \qquad (2.1)\\ =&\sum_{i\in I}\|\Lambda_i T_1 f\|_i^2+\sum_{i\in I}\|\Theta_i T_2 g\|_i^2\\ =&\|f\|^2+\|g\|^2=\|f\oplus g\|^2.\end{aligned}$$

For the converse implication, we assume that the operators $T_1 \in B(\mathcal{H})$ and $T_2 \in B(\mathcal{K})$ are invertible and $\Lambda_1 = \{\Lambda_i T_1 \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta_1 = \{\Theta_i T_2 \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ and $\{\Delta_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I\}$ are Parseval g-frames. From (2.1), we have

$$Re\sum_{i\in I}\langle\Lambda_i T_1 f,\Theta_i T_2 g\rangle_i=0,\quad f\in\mathcal{H},\quad g\in\mathcal{K}. \qquad (2.2)$$

If we replace g by ig in (2.2), then $Re\sum_{i\in I}\langle\Lambda_i T_1 f, i\Theta_i T_2 g\rangle_i = 0$ and therefore

$$Im\sum_{i\in I}\langle\Lambda_i T_1 f,\Theta_i T_2 g\rangle_i=0,\quad f\in\mathcal{H},\quad g\in\mathcal{K}.$$

Hence $RangeT^*_{\Lambda_1} \perp RangeT^*_{\Theta_1}$, consequently $RangeT^*_{\Lambda} \perp RangeT^*_{\Theta}$. □

Proposition 2.3. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be g-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Consider the operators*

$$\Gamma_i:\mathcal{H}\oplus\mathcal{K}\to\mathcal{H}_i,\quad \Gamma_i(f\oplus g)=\Lambda_i f+\Theta_i g,$$

for all $i \in I$. Then Λ and Θ are

1. *disjoint if and only if $\{\Gamma_i\}_{i\in I}$ is a g-frame for $\mathcal{H} \oplus \mathcal{K}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$,*
2. *complementary pair if and only if $\{\Gamma_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H} \oplus \mathcal{K}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$,*
3. *strong complementary pair if and only if Λ and Θ are strongly disjoint and $\{\Gamma_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H} \oplus \mathcal{K}$,*
4. *weakly disjoint if and only if*

$$\{f\oplus g:\Gamma_i(f\oplus g)=0,\forall i\in I\}=\{0\}.$$

Proof. It is easy and we omit the proof. □

Here we intend to state some examples about several kind of disjointness of g-frames and related topics.

Example 2.4. Let $\{e_i\}_{i\in\mathbb{N}}$ and $\{h_i\}_{i\in\mathbb{N}}$ be orthonormal bases for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $\mathcal{H}_i = \mathbb{C}^2$, for all $i \in \mathbb{N}$. We define the operators

$$\Lambda_i:\mathcal{H}\to\mathbb{C}^2,\quad \Lambda_i f=\Big(\langle f,e_i\rangle_{\mathcal{H}},\langle f,e_{i+1}\rangle_{\mathcal{H}}\Big),$$

and

$$\Theta_i : \mathcal{K} \to \mathbb{C}^2, \quad \Theta_i g = \Big(\langle g, h_i\rangle_{\mathcal{K}}, \langle g, h_{i+1}\rangle_{\mathcal{K}}\Big),$$

for all $i \in \mathbb{N}$. Then $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}^2) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathbb{C}^2) : i \in \mathbb{N}\}$ are g-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. For fixed $j \in \mathbb{N}$ we have

$$\{\Lambda_i e_j\}_{i\in\mathbb{N}} = \{\Theta_i h_j\}_{i\in\mathbb{N}} = \{\delta_{ij}(1, 0)\}_{i\in\mathbb{N}},$$

where δ_{ij} is the Kronecker delta. Therefore, $RangeT^*_\Lambda \cap RangeT^*_\Theta \neq \{0\}$ and Λ and Θ are not weakly disjoint. From the other hand, if

$$\Gamma_i : \mathcal{H} \oplus \mathcal{K} \to \mathbb{C}^2, \quad \Gamma_i(f \oplus g) = \Big(\langle f, e_i\rangle_{\mathcal{H}} + \langle g, h_i\rangle_{\mathcal{K}}, \langle f, e_{i+1}\rangle_{\mathcal{H}} + \langle g, h_{i+1}\rangle_{\mathcal{K}}\Big),$$

for all $i \in \mathbb{N}$. Then $\Gamma = \{\Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathbb{C}^2) : i \in \mathbb{N}\}$ is not a g-frame for $\mathcal{H} \oplus \mathcal{K}$. Since for fixed $j \in \mathbb{N}$, $\Gamma_i(-e_j \oplus h_j) = 0$, for all $i \in \mathbb{N}$, but $-e_j \oplus h_j \neq 0$.

Example 2.5. Let $\{f_i\}_{i\in\mathbb{N}}$ and $\{g_i\}_{i\in\mathbb{N}}$ be frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $n > 1$ and let $\mathcal{H}_i = \mathbb{C}^n$, for all $i \in \mathbb{N}$. We define the operators

$$\Lambda_i : \mathcal{H} \to \mathbb{C}^n, \quad \Lambda_i f = \Big(\langle f, f_i\rangle_{\mathcal{H}}, 0, ..., 0\Big),$$

and

$$\Theta_i : \mathcal{K} \to \mathbb{C}^n, \quad \Theta_i g = \frac{\langle g, g_i\rangle_{\mathcal{K}}}{\sqrt{n-1}}(0, 1, 1, ..., 1),$$

for all $i \in \mathbb{N}$. Then $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}^n) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathbb{C}^n) : i \in \mathbb{N}\}$ are g-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Λ and Θ are strongly disjoint while Λ and Θ are not complementary pair, since $\{\delta_{i2}(0, 1, 2, 0, ..., 0)\}_{i\in\mathbb{N}} \in \widehat{\mathcal{H}}$ but $\{\delta_{i2}(0, 1, 2, 0, ..., 0)\}_{i\in\mathbb{N}}$ does not belong to $RangeT^*_\Lambda + RangeT^*_\Theta$.

Example 2.6. Let $\{e_i\}_{i\in\mathbb{N}}$ and $\{h_i\}_{i\in\mathbb{N}}$ be orthonormal bases for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $\mathcal{H}_i = \mathbb{C}^4$, for all $i \in \mathbb{N}$. Let us consider the operators

$$\Lambda_i : \mathcal{H} \to \mathbb{C}^4, \quad \Lambda_i f = \Big(\langle f, e_{2i}\rangle_{\mathcal{H}}, \langle f, e_{2i-1}\rangle_{\mathcal{H}}, 0, 0\Big),$$

and

$$\Theta_i : \mathcal{K} \to \mathbb{C}^4, \quad \Theta_i f = \Big(0, 0, \langle g, h_{2i}\rangle_{\mathcal{K}}, \langle g, h_{2i-1}\rangle_{\mathcal{K}}\Big),$$

for all $i \in \mathbb{N}$. Then $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}^4) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathbb{C}^4) : i \in \mathbb{N}\}$ are Parseval g-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Also, $RangeT^*_\Lambda \perp RangeT^*_\Theta$. If $\{(c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})\}_{i\in\mathbb{N}} \in \widehat{\mathcal{H}}$ and

$$f = \sum_{i\in\mathbb{N}} c_{1,i}e_{2,i} + c_{2,i}e_{2i-1}, \quad g = \sum_{i\in\mathbb{N}} c_{3,i}h_{2,i} + c_{4,i}h_{2i-1},$$

then

$$\{\Lambda_i f\}_{i\in\mathbb{N}} = \{(c_{1,i}, c_{2,i}, 0, 0)\}_{i\in\mathbb{N}}, \quad \{\Theta_i g\}_{i\in\mathbb{N}} = \{(0, 0, c_{3,i}, c_{4,i})\}_{i\in\mathbb{N}}.$$

So

$$\{\Lambda_i f\}_{i\in\mathbb{N}} + \{\Theta_i g\}_{i\in\mathbb{N}} = \{(c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})\}_{i\in\mathbb{N}}.$$

This implies that $RangeT^*_\Lambda + RangeT^*_\Theta = \widehat{\mathcal{H}}$. Consequently, Λ and Θ are strongly complementary pair. Also, if we consider

$$\Gamma_i : \mathcal{H} \oplus \mathcal{K} \to \mathbb{C}^4, \quad \Gamma_i(f \oplus g) = \Big(\langle f, e_{2i}\rangle_{\mathcal{H}}, \langle f, e_{2i-1}\rangle_{\mathcal{H}}, \langle g, h_{2i}\rangle_{\mathcal{K}}, \langle g, h_{2i-1}\rangle_{\mathcal{K}}\Big),$$

for all $i \in \mathbb{N}$. Then $\Gamma = \{\Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathbb{C}^2) : i \in \mathbb{N}\}$ is a g-orthonormal basis for $\mathcal{H} \oplus \mathcal{K}$.

3. Dilation of dual g-frames

It is proved in [4], dual frames in a Hilbert space can be dilated to a Riesz basis for a larger Hilbert space and its dual Riesz basis. Also, the authors of [6] showed that the mentioned dilation theorem is valid for Hilbert C^*-module dual frame pairs. Following the section 7 of [4], we intend to answer this dilation question: if $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are dual g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, is there a Hilbert space $\mathcal{H} \subset M$ and a g-Riesz basis $\Gamma = \{\Gamma_i \in B(M, \mathcal{H}_i) : i \in I\}$ for M with $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ and $\Theta_i = \widetilde{\Gamma}_i P_{\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}}$ is the orthogonal projection from M onto $\mathcal{H}$ and $\widetilde{\Gamma} = \{\widetilde{\Gamma}_i \in B(M, \mathcal{H}_i) : i \in I\}$ is the canonical dual g-frame of Γ?
We first prove the next proposition, which has important role in the proof of Theorem 3.4 of this section. By proving Theorem 3.4, we answer pervious question affirmatively.

Proposition 3.1. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be dual g-frames for $\mathcal{H}$. Let $\Psi = \{\Psi_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ and $\Phi = \{\Phi_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be dual g-frames for $\mathcal{K}$. If Λ and Θ are are strongly disjoint with Φ and Ψ, respectively, then $\Gamma = \{\Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I\}$ and $\Delta = \{\Delta_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I\}$ are dual g-frames for $\mathcal{H} \oplus \mathcal{K}$, where*

$$\Gamma_i(f \oplus g) = \Lambda_i f + \Psi_i g, \quad \Delta_i(f \oplus g) = \Theta_i f + \Phi_i g,$$

for all $i \in I$. Moreover, if Γ is a g-Riesz basis for $\mathcal{H} \oplus \mathcal{K}$ then Δ is a g-Riesz basis for $\mathcal{H} \oplus \mathcal{K}$.

Proof. It is clear that Γ and Δ are g-Bessel sequences for $\mathcal{H}\oplus\mathcal{K}$. Let $f \oplus g \in \mathcal{H}\oplus\mathcal{K}$, then

$$\begin{aligned}
\sum_{i\in I} \Gamma_i^* \Delta_i(f \oplus g) =& \sum_{i\in I} \Gamma_i^*(\Theta_i f + \Phi_i g) \\
=& \sum_{i\in I} \Lambda_i^*(\Theta_i f + \Phi_i g) \oplus \Psi_i^*(\Theta_i f + \Phi_i g) \\
=& \Big(\sum_{i\in I} \Lambda_i^* \Theta_i f + \sum_{i\in I} \Lambda_i^* \Phi_i g \Big) \oplus \Big(\sum_{i\in I} \Psi_i^* \Theta_i f + \sum_{i\in I} \Psi_i^* \Phi_i g \Big) \\
=& f \oplus g.
\end{aligned}$$

Therefore Γ and Δ are g-frames for $\mathcal{H} \oplus \mathcal{K}$ (see [9]). Now, if Γ is a g-Riesz basis for $\mathcal{H}\oplus\mathcal{K}$ then by Proposition 12 of [1], Γ has only one dual g-frame and this dual g-frame is the canonical dual of Γ. Theorem 5 of [1] implies that the canonical dual of a g-Riesz basis, also is a g-Riesz basis. Therefore Δ is a g-Riesz basis for $\mathcal{H} \oplus \mathcal{K}$. □

Example 3.2. Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $\mathcal{H}_i = \mathbb{C}^2$, for all $i \in I$. We define the operators

$$\Lambda_i : \mathcal{H} \to \mathbb{C}^2, \quad \Lambda_i f = \Big(\langle f, f_i\rangle_{\mathcal{H}}, 0\Big),$$

and

$$\Theta_i : \mathcal{H} \to \mathbb{C}^2, \quad \Theta_i f = \Big(\langle S_F^{-1} f, f_i\rangle_{\mathcal{H}}, 0\Big),$$

for all $i \in I$. Then $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}^2) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathbb{C}^2) : i \in I\}$ are g-frames for $\mathcal{H}$. Also

$$\sum_{i\in I} \Lambda_i^* \Theta_i f = \sum_{i\in I} \Lambda_i^* \Big(\langle S_F^{-1} f, f_i\rangle_{\mathcal{H}}, 0\Big) = \sum_{i\in I} \langle S_F^{-1} f, f_i\rangle_{\mathcal{H}} f_i = f, \quad f \in \mathcal{H}.$$

Therefor, Λ and Θ are dual g-frames for $\mathcal{H}$. We define

$$\psi_i : \mathcal{K} \to \mathbb{C}^2, \quad \psi_i g = \Big(0, \langle g, g_i\rangle_{\mathcal{K}}\Big),$$

and

$$\phi_i : \mathcal{K} \to \mathbb{C}^2, \quad \phi_i g = \Big(0, \langle S_G^{-1} g, g_i\rangle_{\mathcal{K}}\Big),$$

for all $i \in I$. Then $\psi = \{\psi_i \in B(\mathcal{K}, \mathbb{C}^2) : i \in I\}$ and $\phi = \{\phi_i \in B(\mathcal{K}, \mathbb{C}^2) : i \in I\}$ are g-frames for $\mathcal{K}$. Also

$$\sum_{i\in I} \psi_i^* \phi_i g = \sum_{i\in I} \psi_i^* \Big(0, \langle S_G^{-1} g, g_i\rangle_{\mathcal{K}}\Big) = \sum_{i\in I} \langle S_G^{-1} g, g_i\rangle_{\mathcal{K}} g_i = g, \quad g \in \mathcal{K}.$$

Therefor, ψ and ϕ are dual g-frames for $\mathcal{K}$. From the other hand, Λ and Θ are strongly disjoint with ϕ and ψ, respectively. Let us consider

$$\Gamma_i : \mathcal{H} \oplus \mathcal{K} \to \mathbb{C}^2, \quad \Gamma_i(f \oplus g) = \Big(\langle f, f_i\rangle_{\mathcal{H}}, \langle g, g_i\rangle_{\mathcal{K}}\Big),$$

and

$$\Delta_i : \mathcal{H} \oplus \mathcal{K} \to \mathbb{C}^2, \quad \Theta_i(f \oplus g) = \Big(\langle S_F^{-1} f, f_i\rangle_{\mathcal{H}}, \langle S_G^{-1} g, g_i\rangle_{\mathcal{K}}\Big),$$

for all $i \in I$. Then

$$\Gamma_i^*(c_1, c_2) = c_1 f_i \oplus c_2 g_i, \quad i \in I,$$

and a simple computation shows that

$$\sum_{i\in I} \Gamma_i^* \Delta_i (f \oplus g) = f \oplus g, \quad f \oplus g \in \mathcal{H} \oplus \mathcal{K},$$

which Proposition 3.1 also confirm this fact.

Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. It is proved in [9], $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\widehat{\mathcal{H}}$, where

$$(E_{ij})_k = \begin{cases} e_{ij}, & i = k \\ 0, & i \neq k. \end{cases} \tag{3.1}$$

Let M and N be closed subspaces of a Hilbert space $\mathcal{H}$. Let P and Q be orthogonal projections from $\mathcal{H}$ onto M and N, respectively. It is proved in [3] that

$$\|P - Q\| = \max\Big\{ \sup_{g\in M, \|g\|=1} \|Q^{\perp} g\|, \sup_{h\in N, \|h\|=1} \|P^{\perp} h\| \Big\}. \tag{3.2}$$

We use above facts in the rest of this paper.
We mention that if $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ are dual frames for a Hilbert space $\mathcal{H}$ and P, Q are the respective orthogonal projections of $l_2(I)$ onto $Range T_F^*$ and $Range T_G^*$, then the followings hold [4, Proposition 7.2]:

(1) for all $i \in I$, $T_F^* S_F^{-1} f_i = Pe_i$.
(2) $PT_F^* = T_F^* S_F^{-1}$. Therefore, $P|_{Q(l_2(I))} : Q(l_2(I)) \to P(l_2(I))$ is an onto isomorphism. Similarly, $P^{\perp}|_{Q^{\perp}(l_2(I))} : Q^{\perp}(l_2(I)) \to P^{\perp}(l_2(I))$ is an onto isomorphism .

In the next proposition, we generalize this result to g-frames.

Proposition 3.3. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be dual g-frames for $\mathcal{H}$. Let P and Q be the orthogonal projections from $\widehat{\mathcal{H}}$ onto $RangeT_\Lambda^*$ and $RangeT_\Theta^*$, respectively. Then we have*

(1) *For all $i \in I$ and $j \in J_i$, $T_\Theta^* S_\Theta^{-1} \Theta_i^* e_{ij} = QE_{ij}$.*
(2) *$QT_\Lambda^* = T_\Theta^* S_\Theta^{-1}$.*

Therefore, $Q|_{P(\widehat{\mathcal{H}})} : P(\widehat{\mathcal{H}}) \to Q(\widehat{\mathcal{H}})$ is an onto isomorphism. Similarly, $Q^{\perp}|_{P^{\perp}(\widehat{\mathcal{H}})} : P^{\perp}(\widehat{\mathcal{H}}) \to Q^{\perp}(\widehat{\mathcal{H}})$ is an onto isomorphism.

Proof. For $f \in \mathcal{H}$ we have

$$\begin{aligned}\langle T_\Theta^* f, T_\Theta^* S_\Theta^{-1} \Theta_i^* e_{ij} \rangle_{\widehat{\mathcal{H}}} &= \langle f, T_\Theta T_\Theta^* S_\Theta^{-1} \Theta_i^* e_{ij} \rangle_{\mathcal{H}} = \langle f, \Theta_i^* e_{ij} \rangle_{\mathcal{H}} \\ &= \langle \Theta_i f, e_{ij} \rangle_i = \langle T_\Theta^* f, E_{ij} \rangle_{\widehat{\mathcal{H}}} = \langle T_\Theta^* f, QE_{ij} \rangle_{\widehat{\mathcal{H}}}.\end{aligned}$$

Therefore, $T_\Theta^* S_\Theta^{-1} \Theta_i^* e_{ij} = QE_{ij}$, for all $i \in I$ and $j \in J_i$. Also, for each $f \in \mathcal{H}$ we have

$$\begin{aligned}QT_\Lambda^* f &= Q\Big(\sum_{i \in I} \sum_{j \in J_i} \langle \{\Lambda_i f\}_{i \in I}, E_{ij} \rangle_{\widehat{\mathcal{H}}} E_{ij} \Big) \\ &= \sum_{i \in I} \sum_{j \in J_i} \langle \{\Lambda_i f\}_{i \in I}, E_{ij} \rangle_{\widehat{\mathcal{H}}} QE_{ij} \\ &= \sum_{i \in I} \sum_{j \in J_i} \langle \Lambda_i f, e_{ij} \rangle_i T_\Theta^* S_\Theta^{-1} \Theta_i^* e_{ij} \\ &= T_\Theta^* S_\Theta^{-1} \Big(\sum_{i \in I} \sum_{j \in J_i} \langle \Lambda_i f, e_{ij} \rangle_i \Theta_i^* e_{ij} \Big).\end{aligned}$$

Sine Λ and Θ are dual g-frames, $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$ and $\{\Theta_i^* e_{ij}\}_{i \in I, j \in J_i}$ are dual frames (see [1, proposition 9]). Hence,

$$QT_\Lambda^* f = T_\Theta^* S_\Theta^{-1} f, \quad f \in \mathcal{H}. \tag{3.3}$$

If $g \in P(\widehat{\mathcal{H}})$ and $Qg = 0$, then $g = T_\Lambda^* f$ for some $f \in \mathcal{H}$ and by (3.3)

$$0 = Qg = QT_\Lambda^* f = T_\Theta^* S_\Theta^{-1} f.$$

Since T_Θ^* and S_Θ^{-1} are injective, $f = 0$ and consequently $g = 0$. This means that $Q|_{P(\widehat{\mathcal{H}})}$ is injective. On the other hand, if $y \in Q(\widehat{\mathcal{H}})$ then $y = T_\Theta^* h_1$ for some $h_1 \in \mathcal{H}$, and $h_1 = S_\Theta^{-1} f$ for some $f \in \mathcal{H}$. Hence,

$$y = T_\Theta^* h_1 = T_\Theta^* S_\Theta^{-1} f = QT_\Lambda^* f.$$

Therefore $Q|_{P(\widehat{\mathcal{H}})}$ is surjective.

Now, we show that $Q^\perp|_{P^\perp(\widehat{\mathcal{H}})}$ is injective. Let us consider $Q^\perp g = 0$ for some $g \in P^\perp(\widehat{\mathcal{H}})$. Then $Qg = g$, and we have

$$0 = \langle g, T_\Lambda^* f\rangle_{\widehat{\mathcal{H}}} = \langle Qg, T_\Lambda^* f\rangle_{\widehat{\mathcal{H}}} = \langle g, QT_\Lambda^* f\rangle_{\widehat{\mathcal{H}}} = \langle g, T_\Theta^* S_\Theta^{-1} f\rangle_{\widehat{\mathcal{H}}},$$

for all $f \in \mathcal{H}$. So $g \in (Q(\widehat{\mathcal{H}}))^\perp$ and $g = Qg = 0$.
Since $Q|_{P(\widehat{\mathcal{H}})} : P(\widehat{\mathcal{H}}) \to Q(\widehat{\mathcal{H}})$ is invertible, there exists $\delta > 0$ such that

$$\delta\|g\| \leq \|Qg\|, \quad g \in P(\widehat{\mathcal{H}}).$$

Now, for $g \in P(\widehat{\mathcal{H}})$ we have

$$\|Q^\perp g\|^2 = \|g\|^2 - \|Qg\|^2 \leq (1-\delta^2)\|g\|^2.$$

Therefore $\sup_{g\in P(\widehat{\mathcal{H}}),\|g\|=1} \|Q^\perp g\| \leq (1-\delta^2)^{\frac{1}{2}} < 1$. Similarly,

$$\sup_{h\in Q(\widehat{\mathcal{H}}),\|h\|=1} \|P^\perp h\| < 1.$$

Consequently by (3.2), $\|P-Q\| < 1$. Since $\|P^\perp - Q^\perp\| = \|P-Q\| < 1$, $Q^\perp|_{P^\perp(\widehat{\mathcal{H}})} : P^\perp(\widehat{\mathcal{H}}) \to Q^\perp(\widehat{\mathcal{H}})$ is onto (see [7]). □

In Theorem 7.3 of [4], the authors proved this dilation result: if $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ are dual frames for $\mathcal{H}$, then there is a Hilbert space $\mathcal{H} \subset M$ and a Riesz basis $H = \{h_i\}_{i\in I}$ for M with $P_{\mathcal{H}} h_i = f_i$ and $P_{\mathcal{H}}\tilde{h}_i = g_i$, where $\widetilde{H} = \{\tilde{h}_i\}_{i\in I}$ is the canonical dual of H and $P_{\mathcal{H}}$ is the orthogonal projection from M onto $\mathcal{H}$. In the next theorem, we generalize mentioned dilation result to g-frames.

Theorem 3.4. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be dual g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then there is a Hilbert space $\mathcal{H} \subset M$ and a g-Riesz basis $\Gamma = \{\Gamma_i \in B(M, \mathcal{H}_i) : i \in I\}$ for M with $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ and $\Theta_i = \widetilde{\Gamma}_i P_{\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}}$ is the orthogonal projection from M onto $\mathcal{H}$ and $\widetilde{\Gamma} = \{\widetilde{\Gamma}_i \in B(M, \mathcal{H}_i) : i \in I\}$ is the canonical dual g-frame of Γ.*

Proof. Let P and Q be the orthogonal projections from $\widehat{\mathcal{H}}$ onto $RangeT_\Lambda^*$ and $RangeT_\Theta^*$, respectively. We consider

$$M = \mathcal{H} \oplus Q^\perp(\widehat{\mathcal{H}}).$$

Let $T = Q^\perp|_{P^\perp(\widehat{\mathcal{H}})}$. Then by Proposition 3.3, T is an isomorphism of $P^\perp(\widehat{\mathcal{H}})$ onto $Q^\perp(\widehat{\mathcal{H}})$. If $S = T^{-1}$, then $Q^\perp S = I_{Q^\perp(\widehat{\mathcal{H}})}$. In fact, if $\{g_i\}_{i\in I} \in Q^\perp(\widehat{\mathcal{H}})$, then there exists $\{h_i\}_{i\in I} \in P^\perp(\widehat{\mathcal{H}})$ such that $\{g_i\}_{i\in I} = T\{h_i\}_{i\in I} = Q^\perp\{h_i\}_{i\in I}$. Therefore

$$Q^\perp S\{g_i\}_{i\in I} = Q^\perp\{h_i\}_{i\in I} = \{g_i\}_{i\in I}.$$

We define the operators

$$\varphi_i : Q^\perp(\widehat{\mathcal{H}}) \to \mathcal{H}_i, \quad \varphi_i(g) = \sum_{j\in J_i} \langle g, Q^\perp E_{ij}\rangle_{\widehat{\mathcal{H}}} e_{ij},$$

for all $i \in I$, where E_{ij} is defined by (3.1). Then $\varphi = \{\varphi_i \in B(Q^{\perp}(\widehat{\mathcal{H}}), \mathcal{H}_i) : i \in I\}$ is a Parseval g-frame for $Q^{\perp}(\widehat{\mathcal{H}})$. In fact, if $g \in Q^{\perp}(\widehat{\mathcal{H}})$ then

$$\begin{aligned}\sum_{i\in I} \|\varphi_i g\|_i^2 &= \sum_{i\in I} \Big\| \sum_{j\in J_i} \langle g, Q^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}} e_{ij} \Big\|_i^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle g, Q^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 \\ &= \sum_{i\in I}\sum_{j\in J_i} |\langle Q^{\perp} g, E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle g, E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 = \|g\|^2.\end{aligned}$$

We claim that Θ and φ are strongly disjoint. Because, for $g \in Q^{\perp}(\widehat{\mathcal{H}})$ we have

$$\begin{aligned}\Big\langle \{\Theta_i f\}_{i\in I}, \{\varphi_i g\}_{i\in I} \Big\rangle_{\widehat{\mathcal{H}}} &= \sum_{i\in I} \langle \Theta_i f, \varphi_i g\rangle_i \\ &= \sum_{i\in I} \Big\langle \sum_{j\in J_i} \langle \Theta_i f, e_{ij}\rangle_i e_{ij}, \sum_{k\in J_i} \langle g, Q^{\perp} E_{ik}\rangle_{\widehat{\mathcal{H}}} e_{ik} \Big\rangle_i \\ &= \sum_{i\in I}\sum_{j\in J_i} \langle \Theta_i f, e_{ij}\rangle_i \overline{\langle g, Q^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}}} \\ &= \Big\langle \sum_{i\in I}\sum_{j\in J_i} \langle \Theta_i f, e_{ij}\rangle_i Q^{\perp} E_{ij}, g \Big\rangle_{\widehat{\mathcal{H}}} \\ &= \langle Q^{\perp} T_{\Theta}^* f, g\rangle_{\widehat{\mathcal{H}}} = 0,\end{aligned}$$

for all $f \in \mathcal{H}$. Now, we consider the bounded operators

$$\psi_i : Q^{\perp}(\widehat{\mathcal{H}}) \to \mathcal{H}_i, \quad \psi_i(g) = \sum_{j\in J_i} \langle g, S^* P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}} e_{ij},$$

for all $i \in I$. Since,

$$\begin{aligned}\sum_{i\in I} \|\psi_i g\|_i^2 &= \sum_{i\in I} \Big\| \sum_{j\in J_i} \langle g, S^* P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}} e_{ij} \Big\|_i^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle g, S^* P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 \\ &= \sum_{i\in I}\sum_{j\in J_i} |\langle Sg, P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle Sg, E_{ij}\rangle_{\widehat{\mathcal{H}}}|^2 = \|Sg\|^2,\end{aligned}$$

for all $g \in Q^{\perp}(\widehat{\mathcal{H}})$, So

$$\frac{1}{\|S^{-1}\|^2}\|g\|^2 \le \sum_{i\in I} \|\psi_i g\|_i^2 \le \|S\|^2 \|g\|^2, \quad g \in Q^{\perp}(\widehat{\mathcal{H}}).$$

Consequentially, $\psi = \{\psi_i \in B(Q^{\perp}(\widehat{\mathcal{H}}), \mathcal{H}_i) : i \in I\}$ is a g-frame for $Q^{\perp}(\widehat{\mathcal{H}})$. Also Λ and ψ are strongly disjoint. In fact,

$$\begin{aligned}\Big\langle \{\Lambda_i f\}_{i\in I}, \{\psi_i g\}_{i\in I}\Big\rangle_{\widehat{\mathcal{H}}} =& \sum_{i\in I}\langle \Lambda_i f, \psi_i g\rangle_i \\ =& \sum_{i\in I}\Big\langle \sum_{j\in J_i}\langle \Lambda_i f, e_{ij}\rangle_i e_{ij}, \sum_{k\in J_i}\langle g, S^* P^{\perp} E_{ik}\rangle_{\widehat{\mathcal{H}}} e_{ik}\Big\rangle_i \\ =& \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i f, e_{ij}\rangle_i \overline{\langle g, S^* P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}}} \\ =& \Big\langle \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i f, e_{ij}\rangle_i P^{\perp} E_{ij}, Sg\Big\rangle_{\widehat{\mathcal{H}}} \\ =& \langle P^{\perp} T_{\Lambda}^* f, Sg\rangle_{\widehat{\mathcal{H}}} = \langle 0, Sg\rangle = 0,\end{aligned}$$

for all $f \in \mathcal{H}$ and $g \in Q^{\perp}(\widehat{\mathcal{H}})$.

We prove that φ and ψ are dual g-frames for $Q^{\perp}(\widehat{\mathcal{H}})$. Let $g \in Q^{\perp}(\widehat{\mathcal{H}})$, then

$$\begin{aligned}\sum_{i\in I}\varphi_i^*\psi_i g =& \sum_{i\in I}\sum_{j\in J_i}\Big\langle \sum_{k\in J_i}\langle g, S^* P^{\perp} E_{ik}\rangle_{\widehat{\mathcal{H}}} e_{ik}, e_{ij}\Big\rangle_i Q^{\perp} E_{ij} \\ =& \sum_{i\in I}\sum_{j\in J_i}\langle g, S^* P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}} Q^{\perp} E_{ij} \\ =& Q^{\perp}\Big(\sum_{i\in I}\sum_{j\in J_i}\langle Sg, P^{\perp} E_{ij}\rangle_{\widehat{\mathcal{H}}} E_{ij}\Big) = Q^{\perp} Sg = g.\end{aligned} \tag{3.4}$$

Let us mention that in the first equality of (3.4), we used the fact,

$$\varphi_i^* g_i = \sum_{j\in J_i}\langle g_i, e_{ij}\rangle_i Q^{\perp} E_{ij}, \quad i \in I, g_i \in \mathcal{H}_i. \tag{3.5}$$

Proposition 3.1 implies that $\Gamma = \{\Gamma_i \in B(M, \mathcal{H}_i) : i \in I\}$ and $\Delta = \{\Delta_i \in B(M, \mathcal{H}_i) : i \in I\}$ are dual g-frames for M, where $\Gamma_i, \Delta_i : M \to \mathcal{H}_i$ are defined by

$$\Gamma_i(f \oplus g) = \Lambda_i f + \varphi_i g, \quad \Delta_i(f \oplus g) = \Theta_i f + \psi_i g,$$

for all $i \in I$. Now, we show that Γ is a g-Riesz basis. It is sufficient to show that T_{Γ} the synthesis operator of Γ is one to one. Let $g = \{g_i\}_{i\in I} \in \widehat{\mathcal{H}}$ and $T_{\Gamma}(\{g_i\}) = 0$. Then

$$T_{\Gamma}(\{g_i\}) = \sum_{i\in I}(\Lambda_i^* g_i \oplus \varphi_i^* g_i) = \Big(\sum_{i\in I}\Lambda_i^* g_i\Big) \oplus \Big(\sum_{i\in I}\varphi_i^* g_i\Big) = 0. \tag{3.6}$$

Since $g = \sum_{i\in I}\sum_{j\in J_i}\langle g, E_{ij}\rangle E_{ij} = \sum_{i\in I}\sum_{j\in J_i}\langle g_i, e_{ij}\rangle E_{ij}$, (3.5) and (3.6) imply that

$$Q^{\perp}(g) = \sum_{i\in I}\sum_{j\in J_i}\langle g_i, e_{ij}\rangle_i Q^{\perp} E_{ij} = 0. \tag{3.7}$$

Also, (3.6) implies that $\sum_{i\in I}\Lambda_i^* g_i = 0$. Therefore,

$$0 = \Big\langle \sum_{i\in I}\Lambda_i^* g_i, f\Big\rangle_{\mathcal{H}} = \sum_{i\in I}\langle g_i, \Lambda_i f\rangle_i = \langle g, \{\Lambda_i f\}_{i\in I}\rangle_{\widehat{\mathcal{H}}}, \quad f \in \mathcal{H}.$$

These means that $g \in P^{\perp}(\widehat{\mathcal{H}})$ or $P^{\perp}g = g$. So by (3.7), $0 = Q^{\perp}g = Q^{\perp}P^{\perp}g$. But by Proposition 3.3, $Q^{\perp}|_{P^{\perp}(\widehat{\mathcal{H}})} : P^{\perp}(\widehat{\mathcal{H}}) \to Q^{\perp}(\widehat{\mathcal{H}})$ is one to one, hence $g = P^{\perp}g = 0$. Therefore $\Gamma = \{\Gamma_i\}_{i\in I}$ is a g-Riesz basis for M and again by Proposition 3.1, $\Delta = \{\Delta_i\}_{i\in I}$ is a g-Riesz basis for M and [1, Proposition 12] implies that $\Delta_i = \widetilde{\Gamma}_i$. It is clear that $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ and $\Theta_i = \widetilde{\Gamma}_i P_{\mathcal{H}}$, for all $i \in I$. □

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