Adv. Oper. Theory 3 (2018), no. 4, 829–836
https://doi.org/10.15352/aot.1806-1382
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

COMPACT AND "COMPACT" OPERATORS ON STANDARD HILBERT MODULES OVER C^*-ALGEBRAS

ZLATKO LAZOVIĆ

Communicated by E. Troitsky

Abstract. We construct a topology on the standard Hilbert module $H_{\mathcal{A}}$ over a unital C^*-algebra and topology on $H_{\mathcal{A}}^{\#}$ (the extension of the module $H_{\mathcal{A}}$ by the algebra $\mathcal{A}^{**}$) such that any "compact" operator (i.e. any operator in the norm closure of the linear span of the operators of the form $z \mapsto x\langle y, z\rangle$, $x, y \in H_{\mathcal{A}}$ (or $x, y \in H_{\mathcal{A}}^{\#}$)) maps bounded sets into totally bounded sets.

1. Introduction and preliminaries

In the paper [2] proved that on standard Hilbert module $H_{\mathcal{A}}$, where $\mathcal{A}$ a unital W^*-algebra, there is locally convex topology such that any "compact" operator (i.e. any operator in the norm closure of the linear span of the operators of the form $z \mapsto x\langle y, z\rangle$, $x, y \in H_{\mathcal{A}}$) is compact (in the sense that it maps bounded sets into totally bounded sets).

In this note we find a topology on $H_{\mathcal{A}}$, where $\mathcal{A}$ a unital C^*-algebra, and a topology on $H_{\mathcal{A}}^{\#}$ (the extension of the module $H_{\mathcal{A}}$ by the algebra $\mathcal{A}^{**}$) such that for them holds the same property.

Let $\mathcal{A}$ be a C^*-algebra, A^{**} be the enveloping W^*-algebra of $\mathcal{A}$, and M be a Hilbert $\mathcal{A}$-module. Consider the algebraic tensor product $M \otimes \mathcal{A}^{**}$ (over $\mathbb{C}$). One can equip this tensor product with the structure of a right $\mathcal{A}^{**}$-module by the formula $(x \otimes a) \cdot b := x \otimes ab$, $x \in M$, a, $b \in \mathcal{A}^{**}$. Define the inner product

$$[\cdot, \cdot]\colon M \otimes \mathcal{A}^{**} \times M \otimes \mathcal{A}^{**} \to \mathcal{A}^{**}$$

Date: Received: Jun. 12, 2018; Accepted: Jun. 29, 2018.
2010 *Mathematics Subject Classification.* Primary 46L08; Secondary 47B07.
Key words and phrases. Hilbert module, compact operator, locally convex topology.

by the formula

$$\left[\sum_{i=1}^{n} x_1 \otimes a_i, \sum_{j=1}^{m} y_j \otimes b_j\right] = \sum_{i,j} a_i^* \langle x_i, y_j \rangle b_j,$$

where x_i, $y_j \in M$, $a_i, b_j \in \mathcal{A}^{**}$. The sesquilinearity and the properties $[z,w]^* = [w,z]$ and $[z, w\cdot a] = [z,w]a$ clearly hold. This inner product is positive [6, Lemma 5.1.2].

Denote by $M^{\#}$ the Hilbert $\mathcal{A}^{**}$-module obtained by completion of $M \otimes \mathcal{A}^{**}/N$ with respect to the norm given by the inner product $[\cdot,\cdot]$, where $N = \{z \in M \otimes \mathcal{A}^{**} \;:\; [z,z] = 0\}$. We call this module the extension of the module M by the algebra $\mathcal{A}^{**}$. The W^*-algebra $\mathcal{A}^{**}$ always contains the unit element, and, for any $x \in M$ and $a \in \mathcal{A}$, we have $(x\cdot a)\otimes 1 - x \otimes a \in N$. Therefore the $\mathcal{A}$-module map $i\colon M \mapsto M^{\#}$, $i(x) = x \times 1 + N$ is well defined. Since $[x\times 1 + N, y \times 1 + N] = \langle x, y\rangle$, this map is an isometric inclusion.

Denote by $\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^{**})$ the set of all bounded $\mathcal{A}$-linear maps from M to $\mathcal{A}^{**}$. Introduce on this set the structure of a vector space over $\mathbb{C}$ by the formula $(\lambda\phi)(x) = \overline{\lambda}\phi(x)$, where $\lambda \in \mathbb{C}$, $x \in M$, $\phi \in \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^{**})$, and also with the structure of a right $\mathcal{A}^{**}$-module by the formula, $(\phi \cdot b)(x) = b^*\phi(x)$, $b \in \mathcal{A}^{**}$. For a bounded $\mathcal{A}$-linear functional $f \in (M^{\#})'$, we can define the map $f_R \in \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^{**})$ as the restriction of f onto M; that is, $f_R(x) = f(x \otimes 1 + N)$. Obviously, $||f_R|| \leqslant ||f||$. For any C^*-algebra $\mathcal{A}$ and for any Hilbert $\mathcal{A}$-module M, the map $f \mapsto f_R$ is an isometry of $(M^{\#})'$ onto $\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^{**})$.

Let $\mathcal{A}$ be a C^*-algebra, and let M be a Hilbert $\mathcal{A}$-module. Then an $\mathcal{A}$-valued inner product on M can be extended up to an $\mathcal{A}^{**}$-valued inner product on the set $\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^{**})$ making this set a self-dual Hilbert $\mathcal{A}^{**}$-module. Let $\mathcal{A}$ be a C^*-algebra, and let M be a self-dual Hilbert $\mathcal{A}$-module. Then the Hilbert $\mathcal{A}^{**}$-module $M^{\#}$ is self-dual too.

Given a unital W^*-algebra $\mathcal{A}$, we consider the standard Hilbert module, denoted by $H_{\mathcal{A}}$, as (the notation $l^2(\mathcal{A})$ is also widespread)

$$H_{\mathcal{A}} = \left\{ x = (\xi_1, \xi_2, \dots) \mid \xi_j \in \mathcal{A}, \sum\nolimits_{j=1}^{+\infty} \xi_j^* \xi_j \text{ converges in the norm topology} \right\},$$

equipped with the $\mathcal{A}$-valued inner product

$$H_{\mathcal{A}} \times H_{\mathcal{A}} \ni (x,y) \mapsto \sum_{j=1}^{+\infty} \xi_j^* \eta_j \in \mathcal{A}, \qquad x = (\xi_1, \xi_2, \dots), \quad y = (\eta_1, \eta_2, \dots).$$

Since an arbitrary $\mathcal{A}$-linear bounded operator on $H_{\mathcal{A}}$ does not need to have an adjoint, the natural algebra of operators is $B^a(H_{\mathcal{A}})$, which is the algebra of all $\mathcal{A}$-linear bounded operators on $H_{\mathcal{A}}$ having an adjoint. It is known that $B^a(H_{\mathcal{A}})$ is a C^*-algebra.

Among all operators in $B^a(H_{\mathcal{A}})$, those that belong to the linear span of the operators of the form $x \mapsto \Theta_{y,z}(x) = y\langle z, x\rangle$ (y, $z \in H_{\mathcal{A}}$) are called finite rank operators. The norm closure of finite rank operators is known as the algebra of all “compact” operators. The quotation marks are usually written in order to emphasize the fact that “compact” operators do not map bounded sets into relatively compact sets, as it is the case in the framework of Hilbert (and also

Banach) spaces, though they share many properties of proper compact operators on a Hilbert space (see [4, 5, 3]).

A subset S of a topological vector space $\{X, \tau\}$ is totally bounded if and only if, given any neighborhood E of the zero element of X, there exists a finite cover of that set S by subsets of X each of which is a translate of a subset of E. In case of topological vector space $\{X, \tau\}$, where τ is a locally convex topology generated by a family of seminorms $p_\alpha,\ \alpha \in I$; a subset S is called totally bounded if, for all $\varepsilon > 0$ and for all $\alpha \in I$, there is a finite set $c_1, c_2, \dots, c_m \in X$ such that sets $B_\alpha(c_j; \varepsilon) = \{y \in X \mid p_\alpha(c_j - y) < \varepsilon\}$ cover S.

On a standard Hilbert module $H_{\mathcal{A}}$, where $\mathcal{A}$ is a unital W^*-algebra, we define a locally convex topology τ by the family of seminorms

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j)|^2}, \tag{1.1}$$

where φ is a normal state, and $y = (\eta_1, \eta_2, \dots)$ is a sequence of elements in $\mathcal{A}$ such that

$$\sup_{j \geqslant 1} \varphi(\eta_j^* \eta_j) = 1.$$

This topology is between the weak Paschke–Frank topology (generated by functionals $x \mapsto \varphi(\langle y, x\rangle)$, $y \in H_{\mathcal{A}}$, φ normal state) and the strong Paschke–Frank topology (generated by seminorms $p(x) = \varphi(\langle x, x\rangle)^{\frac{1}{2}}$, φ normal state) (see [1, 7, 8]).

We say that the operator $T \in B^a(H_{\mathcal{A}})$ is compact if its image of any (norm) bounded set is a totally bounded set in topology τ. For the operator $T \in B^a(H_{\mathcal{A}})$, it is enough to map the unit ball into a totally bounded set to be a compact operator.

Remark 1.1. Totally bounded and relatively compact sets differ in general case (whenever the unit ball is not complete). Also, throughout the literature, there is a certain ambiguity between terms *completely continuous*, *compact*, and *precompact* operators. Although it seems that terms *completely continuous* and *precompact* are more accurate; we found that *compact* is more convenient for our purpose.

Let $\mathcal{A}$ be a unital W^*-algebra, and let $T : H_{\mathcal{A}} \to H_{\mathcal{A}}$ be a “compact” operator. Then T is compact [2, Theorem 4.5].

2. Main results

In what follows, we assume that the C^*-algebra is unital.

Definition 2.1. Let M be a Hilbert C^*-module. A topological space (M, τ) is called “continuous” if every “compact” operator is compact; that is, it maps the norm unit ball from M to a totally bounded set in (M, τ). A Hilbert C^*-module M is “continuous” if there is a topology τ on M such that (M, τ) is “continuous”.

An infinite C^*-algebra $\mathcal{A}$, as Hilbert C^*-module over itself with norm topology, is not “continuous”, because the identity operator $\Theta_{e,e}(\Theta_{e,e}(x) = e\langle e, x\rangle = x)$ is "compact", but the image of the unit ball is not totally bounded in Banach space sense.

The standard Hilbert W^*-module $H_{\mathcal{A}}$ is "continuous" (see [2, Theorem 4.5]).

Lemma 2.2. *Let $T \in B^a(H_{\mathcal{A}})$ be a "compact" operator. Then there is a "compact" operator $T^\# \in B^a(H_{\mathcal{A}}^\#)$ such that the diagram*

$$\begin{array}{ccc} H_{\mathcal{A}}^\# & \xrightarrow{T^\#} & H_{\mathcal{A}}^\# \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i} \\ H_{\mathcal{A}} & \xrightarrow[T]{} & H_{\mathcal{A}} \end{array}$$

is commutative.

Proof. Let $T = \Theta_{x,y}$ be a "compact" operator for any $x, y \in H_{\mathcal{A}}$. We define the operator $\Theta_{x\otimes 1+N, y\otimes 1+N}$ on $H_{\mathcal{A}} \otimes \mathcal{A}^{**}/N$ by

$$\Theta_{x\otimes 1+N,y\otimes 1+N}(z \otimes b + N) = (x \otimes 1 + N)[y \otimes 1 + N, z \otimes b + N], \tag{2.1}$$

and its continuous extension to $H_{\mathcal{A}}^\#$ by

$$\Theta_{x\otimes 1+N,y\otimes 1+N}(z) = \lim_n \Theta_{x\otimes 1+N,y\otimes 1+N}(z_n),$$

for $z \in H_{\mathcal{A}}^\#$, where $H_{\mathcal{A}} \otimes \mathcal{A}^{**}/N \ni z_n \to z \in H_{\mathcal{A}}^\#$. This operator is "compact" on $H_{\mathcal{A}}^\#$, and it holds

$$\begin{aligned} \Theta_{x\otimes 1+N,y\otimes 1+N}(i(z)) &= \Theta_{x\otimes 1+N,y\otimes 1+N}(z \otimes 1 + N) \\ &= (x \otimes 1 + N)[y \otimes 1 + N, z \otimes 1 + N] = (x \otimes 1 + N) \langle y, z \rangle \\ &= x \langle y, z \rangle \otimes 1 + N = i(x \langle y, z \rangle) = i(\Theta_{x,y}(z)) \end{aligned}$$

for every $z \in H_{\mathcal{A}}$. Thus, for $z \in H_{\mathcal{A}}$, we have $\Theta_{x\otimes 1+N,y\otimes 1+N}(i(z)) = i(\Theta_{x,y}(z))$.

Next, we get

$$\begin{aligned} ||\Theta_{x\otimes 1+N,y\otimes 1+N}|| &= \sup_{\substack{z\in H_{\mathcal{A}}^\# \\ ||z||\leqslant 1}} ||\Theta_{x\otimes 1+N,y\otimes 1+N}(z)|| \\ &= \lim_{n\to\infty} \sup_{\substack{z_n\in H_{\mathcal{A}}, b_n\in\mathcal{A}^{**} \\ ||z_n\otimes b_n+N||\leqslant 1}} ||(x \otimes 1 + N)[y \otimes 1 + N, z_n \otimes b_n + N]|| \\ &= \lim_{n\to\infty} \sup_{\substack{z_n\in H_{\mathcal{A}}, b_n\in\mathcal{A}^{**} \\ ||z_n\otimes b_n+N||\leqslant 1}} ||(x \otimes 1 + N) \langle y, z_n \rangle b_n|| \\ &= \lim_{n\to\infty} \sup_{\substack{z_n\in H_{\mathcal{A}}, b_n\in\mathcal{A}^{**} \\ ||z_n\otimes b_n+N||\leqslant 1}} ||x \langle y, z_n \rangle \otimes b_n + N||. \end{aligned}$$

Since

$$\lim_{n\to\infty} \sup_{\substack{z_n\in H_{\mathcal{A}}, b_n\in\mathcal{A}^{**} \\ ||z_n\otimes b_n+N||\leqslant 1}} ||x \langle y, z_n \rangle \otimes b_n + N|| = \lim_{n\to\infty} \sup_{\substack{z_n\in H_{\mathcal{A}}, ||z_n||\leqslant 1 \\ b_n\in\mathcal{A}^{**}, ||b_n||\leqslant 1}} ||x \langle y, z_n \rangle|| \cdot ||b_n||,$$

we have

$$\begin{aligned}||\Theta_{x\otimes 1+N,y\otimes 1+N}|| &= \lim_{n\to\infty}\sup_{\substack{z_n\in H_{\mathcal{A}},||z_n||\leqslant 1\\ b_n\in\mathcal{A}^{**},||b_n||\leqslant 1}}||x\langle y,z_n\rangle||\cdot||b_n||\\ &= \lim_{n\to\infty}\sup_{z_n\in H_{\mathcal{A}},||z_n||\leqslant 1}||x\langle y,z_n\rangle||\\ &= \lim_{n\to\infty}\sup_{z_n\in H_{\mathcal{A}},||z_n||\leqslant 1}||\Theta_{x,y}(z_n)|| = ||\Theta_{x,y}||.\end{aligned}$$

For the sum of two operators of the form (2.1), it holds

$$\begin{aligned}&||\left(\Theta_{x\otimes 1+N,y\otimes 1+N}+\Theta_{x_1\otimes 1+N,y_1\otimes 1+N}\right)(z\otimes b+N)||\\ =\ & ||\Theta_{x\otimes 1+N,y\otimes 1+N}(z\otimes b+N)+\Theta_{x_1\otimes 1+N,y_1\otimes 1+N}(z\otimes b+N)||\\ =\ & ||(x\otimes 1+N)[y\otimes 1+N,z\otimes b+N]+(x_1\otimes 1+N)[y_1\otimes 1+N,z\otimes b+N]||\\ =\ & ||(x\otimes 1+N)\langle y,z\rangle b+(x_1\otimes 1+N)\langle y_1,z\rangle b||\\ =\ & ||(x\langle y,z\rangle\otimes b+N)+(x_1\langle y_1,z\rangle\otimes b+N)||\\ =\ & ||(x\langle y,z\rangle+x_1\langle y_1,z\rangle)\otimes b+N)||\\ =\ & ||x\langle y,z\rangle+x_1\langle y_1,z\rangle||\cdot||b|| = ||\left(\Theta_{x,y}+\Theta_{x_1,y_1}\right)(z)||\cdot||b||.\end{aligned}$$

When we take the supremum over $z\otimes b+N\in H_{\mathcal{A}}\otimes\mathcal{A}^{**}/N$, we get

$$||\Theta_{x\otimes 1+N,y\otimes 1+N}+\Theta_{x_1\otimes 1+N,y_1\otimes 1+N}|| = ||\Theta_{x,y}+\Theta_{x_1,y_1}||$$

on $H_{\mathcal{A}}\otimes\mathcal{A}^{**}/N$, and, from its continuity, it follows that

$$||\Theta_{x\otimes 1+N,y\otimes 1+N}+\Theta_{x_1\otimes 1+N,y_1\otimes 1+N}|| = ||\Theta_{x,y}+\Theta_{x_1,y_1}|| \tag{2.2}$$

on $H_{\mathcal{A}}^{\#}$.

If $T_n=\sum_{i=1}^{n}\Theta_{x_i,y_i}$, then $T_n^{\#}=\sum_{i=1}^{n}\Theta_{x_i\oplus 1+N,y_i\oplus 1+N}$, and from (2.2), we have

$$||T_n^{\#}|| = ||T_n||,$$

and

$$||T_n^{\#}-T_m^{\#}|| = ||T_n-T_m||.$$

The operator T is "compact"; so there exists a sequence of operators $T_n=\sum_{i=1}^{n}\Theta_{x_i,y_i}$, which converges in norm to the operator T. Hence, the sequence T_n is a Cauchy sequence; so $T_n^{\#}$ is a Cauchy sequence, too. From completeness of space $B^a(H_{\mathcal{A}}^{\#})$, we have that $T_n^{\#}$ converges to a "compact" operator $T^{\#}$. Since

$$\Theta_{x\otimes 1+N,y\otimes 1+N}(i(z)) = i\left(\Theta_{x,y}(z)\right)\text{ for all } z\in H_{\mathcal{A}}$$

and $T_n^{\#}=\sum\limits_{i=1}^{n}\Theta_{x_i\oplus 1+N,y_i\oplus+N}$ converges to $T^{\#}$, we have that $T^{\#}(i(z))=i(T^{\#}(z))$, so the diagram is commutative. □

Lemma 2.3. *Let $T^{\#}\in B^a(H_{\mathcal{A}}^{\#})$ be a "compact" operator, and let j be the mapping $j\colon H_{\mathcal{A}}^{\#}\to H_{\mathcal{A}^{**}}$, $j(x\otimes a+N)=(x_1a,x_2a,\dots)$ for $x=(x_1,x_2,\dots)\in H_{\mathcal{A}}, a\in$*

$\mathcal{A}^{**}$. Then there exists a "compact" operator $T_{**} \in B^a(H_{\mathcal{A}^{**}})$, such that the diagram

$$\begin{array}{ccc} H_{\mathcal{A}^{**}} & \xrightarrow{T_{**}} & H_{\mathcal{A}^{**}} \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j} \\ H_{\mathcal{A}}^{\#} & \xrightarrow[T^{\#}]{} & H_{\mathcal{A}}^{\#} \end{array}$$

is commutative.

Proof. First we prove that the mapping $j\colon H_{\mathcal{A}}^{\#} \to H_{\mathcal{A}^{**}}$? ?defined by ???$j(x \otimes a + N) = (x_1 a, x_2 a, \dots)$, where $x = (x_1, x_2, \dots) \in H_{\mathcal{A}}, a \in \mathcal{A}^{**}$, is an isometric isomorphism. From

$$\begin{aligned} [x \otimes a - y \otimes b, x \otimes a - y \otimes b] &= a^* \langle x, x\rangle a - a^* \langle x, y\rangle b - b^* \langle y, x\rangle a + b^* \langle y, y\rangle b \\ &= a^* \langle x, x\rangle a - a^* \langle x, y\rangle b - b^* \langle y, x\rangle a + b^* \langle y, y\rangle b \\ &= \sum_{n=1}^{\infty} (x_i a)^* x_i a - \sum_{n=1}^{\infty} (x_i a)^* y_i b - \sum_{n=1}^{\infty} (y_i b)^* x_i a + \sum_{n=1}^{\infty} (y_i b)^* y_i b \\ &= \sum_{n=1}^{\infty} (x_i a - y_i b)^* (x_i a - y_i b), \end{aligned}$$

it follows that j is well defined and injective.

The mapping j maps $H_{\mathcal{A}}^{\#}$ onto $H_{\mathcal{A}^{**}}$, because if $x = (x_1, x_2, \dots) \in H_{\mathcal{A}^{**}}$, then $j(\sum_{i=1}^{+\infty} e_i \otimes x_i) = x$.

From the equality of the inner products

$$\begin{aligned} [x \otimes a + N, y \otimes b + N] &= a^* \langle x, y\rangle b = a^* \left(\sum_{n=1}^{\infty} x_i^* y_i \right) b = \sum_{n=1}^{\infty} (x_i a)^* y_i b \\ &= \langle j(x \otimes a + N), j(y \otimes b + N)\rangle , \end{aligned}$$

we get that j is an isomorphism.

For "compact" operator $T^{\#} \in B^a(H_{\mathcal{A}}^{\#})$, there exists a "compact" operator $T_{**} \in B^a(H_{\mathcal{A}^{**}}), T_{**} = j \circ T^{\#} \circ j^{-1}$ (for $T^{\#} = \Theta_{x\otimes a, y\otimes b}$ we have $T_{**} = \Theta_{xa,yb}$). □

Theorem 2.4. *The standard Hilbert C^*-module $H_{\mathcal{A}}$ and the Hilbert W^*-module $H_{\mathcal{A}}^{\#}$ are "continuous".*

Proof. Let T be a "compact" on $H_{\mathcal{A}}$. Then there are "compact" operators $T^{\#} \in B^a(H_{\mathcal{A}}^{\#})$ and $T_{**} \in B^a(H_{\mathcal{A}^{**}})$, such that diagram

$$\begin{array}{ccc} H_{\mathcal{A}^{**}} & \xrightarrow{T_{**}} & H_{\mathcal{A}^{**}} \\ {\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle j} \\ H_{\mathcal{A}}^{\#} & \xrightarrow{T^{\#}} & H_{\mathcal{A}}^{\#} \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i} \\ H_{\mathcal{A}} & \xrightarrow{T} & H_{\mathcal{A}} \end{array}$$

is commutative.

Let $H_{\mathcal{A}^{**}}$ be the respective Hilbert $\mathcal{A}^{**}$-module, where $\mathcal{A}$ is a unital C^*-algebra. We define a locally convex topology τ_{**} on $H_{\mathcal{A}^{**}}$ (of the form (1.1)) by the family of seminorms

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j)|^2}, \quad x = (\xi_1, \xi_2, \dots), \tag{2.3}$$

where φ is a bounded linear functional on $\mathcal{A}$ (since any normal state on $\mathcal{A}^{**}$ is a bounded linear functional on $\mathcal{A}$), and $y = (\eta_1, \eta_2, \dots)$ is a sequence of elements in $\mathcal{A}^{**}$, such that

$$\sup_{j \geqslant 1} \varphi(\eta_j^* \eta_j) = 1. \tag{2.4}$$

Define a topology $\tau^{\#}$ as the topology induced on the module $H_{\mathcal{A}}^{\#}$ by τ_{**}; that is, $A \in \tau^{\#}$ if $i(A) \in \tau_{**}$, and define a topology τ as the topology induced on the module $H_{\mathcal{A}}$ by τ_{**} , $\tau = (i^{-1} \circ j^{-1})(\tau_{**})$.

Since T_{**} is "compact", then it is also compact [2, Theorem 4.5]; so $T_{**}(B)$ is totally bounded. From $j(B_1) \subset B$ and $(j \circ i)(B_2) \subset B$, where B_1 and B_2 are the unit balls in $H_{\mathcal{A}}^{\#}$ and $H_{\mathcal{A}}$, respectively, it follows that the sets $T_{**}(j(B_1)) = j(T^{\#}(B_1))$ and $T_{**}((j \circ i)(B_2)) = (j \circ i)(T(B))$ are totally bounded in $(H_{\mathcal{A}^{**}}, \tau^{**})$. Hence $T^{\#}(B_1)$ is totally bounded in $(H_{\mathcal{A}}^{\#}, j^{-1}(\tau_{**})) = (H_{\mathcal{A}}^{\#}, \tau^{\#})$, and $T(B)$ is totally bounded in $(H_{\mathcal{A}}, i^{-1} \circ j^{-1}(\tau_{**})) = (H_{\mathcal{A}}, \tau)$. Therefore, the Hilbert modules $H_{\mathcal{A}}$ and $H_{\mathcal{A}}^{\#}$ are "continuous". □

Acknowledgement. The author was supported in part by the Ministry of education and science, Republic of Serbia, Grant 174034.

References

1. M. Frank, *Self-duality and C^*-reflexivity of Hilbert C^*-modules*, Z. Anal. Anwend. **9** (1990), no. 2, 165–176.
2. D. J. Kečkić and Z. Lazović, *Compact and "compact" operators on standard Hilbert modules over W^*-algebras*, Ann. Funct. Anal. **9** (2018), no. 2, 258–270.
3. E. C. Lance, *Hilbert C^*-modules: A Toolkit for Operator algebraists*, London Math. Soc. Lecture Note Ser. **210**, Cambridge Univ. Press, Cambridge, 1995.
4. V. M. Manuilov, *Diagonalization of compact operators in Hilbert modules over finite W^*-algebras*, Ann. Global Anal. Geom. **13** (1995), no. 3, 207–226.

5. V. M. Manuilov, *Diagonalization of compact operators on Hilbert modules over C^*-algebras of real rank zero* (in Russian), Mat. Zametki **62** no. 6 (1997), 865–870; English translation in Math. Notes **62** (1997), 726–730.
6. V. M. Manuilov and E. V. Troitsky, *Hilbert C^*-Modules*, Transl. Math. Monogr. **226** Amer. Math. Soc., Providence, 2005.
7. W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.
8. W. L. Paschke, *Inner product modules arising from compact automorphism groups of von Neumann algebras*, Trans. Amer. Math. Soc. **224** (1976), 87–102.

Faculty of Mathematics, University of Belgrade, Studentski trg 16-18, 11000 Belgrade, Serbia.
E-mail address: zlatkol@matf.bg.ac.rs