Adv. Oper. Theory 4 (2019), no. 1, 251–264
https://doi.org/10.15352/aot.1804-1354
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

DOMINATED ORTHOGONALLY ADDITIVE OPERATORS IN LATTICE-NORMED SPACES

NARIMAN ABASOV[1] and MARAT PLIEV[2*]

Communicated by P. Aiena

Abstract. In this paper, we introduce a new class of operators in lattice-normed spaces. We say that an orthogonally additive operator T from a lattice-normed space (V, E) to a lattice-normed space (W, F) is dominated, if there exists a positive orthogonally additive operator S from E to F such that $⦀Tx⦀ \leq S ⦀x⦀$ for any element x of (V, E). We show that under some mild conditions, a dominated orthogonally additive operator has an exact dominant and obtain formulas for calculating the exact dominant of a dominated orthogonally additive operator. In the last part of the paper we consider laterally-to-order continuous operators. We prove that a dominated orthogonally additive operator is laterally-to-order continuous if and only if the same is its exact dominant.

1. Introduction and preliminaries

Nonlinear orthogonally additive functionals and operators in classical Banach spaces have been studied by a number of authors (see the incomplete list of references [4, 8, 7, 9, 10, 13, 14]). The theory of vector lattices gives a general framework for dealing with spaces of continuous and measurable functions. On the other hand, the theory of lattice-normed spaces is a useful instrument for analyzing spaces of continuous and measurable vector valued function. Order bounded orthogonally additive operators in vector lattices are introduced and studied by Mazón and Segura de León (see [15, 16, 20]). Recently, a new class

Date: Received: Apr. 27, 2018; Accepted: Aug. 2, 2018.
*Corresponding author .
2010 *Mathematics Subject Classification.* Primary 47H30; Secondary 47H99.
Key words and phrases. Lattice-normed space, vector lattice, orthogonally additive operator, dominated $\mathcal{P}$-operator, exact dominant, laterally-to-order continuous operator.

of orthogonally additive operators in vector lattices, where the condition of order boundness of an operator is replaced with a much weaker property, is investigated in [19]. A generalization of results of the article [19] to the setting of lattice-normed spaces is a challenging problem. The aim of these notes is to make the first attempt for solving this problem.

2. Preliminary information

In this section we state some basic facts concerning vector lattices and lattice-normed spaces. For the standard information we refer to [2, 12]. All vector lattices below are assumed to be Archimedean.

Let E be a vector lattice. A net $(x_\alpha)_{\alpha\in\Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net $(e_\alpha)_{\alpha\in\Lambda}$ in E_+ such that $e_\alpha \downarrow 0$ and $|x_\alpha - x| \leq e_\alpha$ for all $\alpha \in \Lambda$ satisfying $\alpha \geq \alpha_0$ for some $\alpha_0 \in \Lambda$. Two elements x, y of the vector lattice E are said to be *disjoint* (notation $x\bot y$), if $|x| \wedge |y| = 0$. The equality $x = \bigsqcup\limits_{i=1}^{n} x_i$ means that $x = \sum\limits_{i=1}^{n} x_i$ and $x_i \bot x_j$ for all $i \neq j$. If $n = 2$ we use the notation $x = x_1 \sqcup x_2$. An element y of E is called a *fragment* of an element $x \in E$ (in another terminology, a *component*), if $y \bot (x-y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. The set of all fragments of the element $x \in E$ is denoted by $\mathcal{F}_x$.

Lemma 2.1. [17, Prop.3.8] *Let E be a vector lattice. Then the binary relation $\sqsubseteq$ is a partial order on E.*

Definition 2.2. Let E be a vector lattice. The partial order $\sqsubseteq$ on E we call the *lateral order* on E. A subset $G \subseteq E$ is said to be *laterally bounded* in E if $G \subseteq \mathcal{F}_x$ for some $x \in E$.

Definition 2.3. Let E be a vector lattice, and let F be a real linear space. An operator $T : E \to F$ is called *orthogonally additive* if $T(x + y) = Tx + Ty$ for every disjoint elements $x, y \in E$.

It is clear that $T(0) = 0$. The set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Definition 2.4. Let E and F be vector lattices. An orthogonally additive operator $T : E \to F$ is said to be:

- *positive*, if $Tx \geq 0$ holds in F for all $x \in E$;
- *order bounded*, if it maps order bounded sets in E to order bounded sets in F;
- *laterally-to-order bounded*, if the set $T(\mathcal{F}_x)$ is order bounded in F for every $x \in E$.

Recall that the vector space of all order bounded orthogonally additive operators from E to F is called the space of *abstract Urysohn operators* and denoted by $\mathcal{U}(E, F)$. The order structure of $\mathcal{U}(E, F)$ has been the object of active research in last years [1, 11, 17, 18]. A laterally-to-order bounded orthogonally additive operator $T : E \to F$ is said to be a *Popov* operator (or $\mathcal{P}$-operator for brevity).

Since $\mathcal{F}_x$ is an order bounded set for every $x \in E$, every abstract Urysohn operator $T : E \to F$ is a $\mathcal{P}$-operator. The set of all $\mathcal{P}$-operators from E to F is denoted by $\mathcal{P}(E, F)$.

Consider some examples of $\mathcal{P}$-operators.

Example 2.5. We recall the vector space $\mathbb{R}^m$, $m \in \mathbb{N}$, as a vector lattice with the following coordinate-wise order: for any $x, y \in \mathbb{R}^m$ we set $x \leq y$ provided $e_i^*(x) \leq e_i^*(y)$ for all $i = 1, \dots, m$, where $(e_i^*)_{i=1}^m$ are the coordinate functionals on $\mathbb{R}^m$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$. Then $T \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if there are real functions $T_{i,j} : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$ satisfying $T_{i,j}(0) = 0$ such that

$$e_i^*\big(T(x_1, \dots, x_n)\big) = \sum_{j=1}^{n} T_{i,j}(x_j).$$

In this case we write $T = (T_{i,j})$.

Example 2.6. Let (B, Σ, ν) be a finite measure space. By $L_0(B, \Sigma, \nu)$ or $L_0(\nu)$ for brevity we denote the set of all measurable ν-a.e. finite functions. Let f and g be measurable functions on B. Then we write $f \leq g$, if $f(t) \leq g(t)$ holds for ν-almost all $t \in B$. Consider another finite measure space (A, Ξ, μ). We define the product measure space of (A, Ξ, μ) and (B, Σ, ν) as usual and denote by $(A \times B, \mu \times \nu)$ their completion. Consider the following properties of a function $K : A \times B \times \mathbb{R} \to \mathbb{R}$:

(1) $K(s, t, 0) = 0$ for $\mu \times \nu$-almost all $(s, t) \in A \times B$;
(2) $K(\cdot, \cdot, f(\cdot))$ is $\mu \times \nu$-measurable for all $f \in L_0(\nu)$.

Given $f \in L_0(B, \Sigma, \nu)$, the function $|K(s, \cdot, f(\cdot))|$ is ν-measurable for μ-almost all $s \in A$ and $h_f(s) := \int_B |K(s, t, f(t))|\, d\nu$ is a well defined μ-measurable function. Since the function h_f can be infinite on a set of positive measure, we define

$$\mathrm{Dom}_B(K) := \{f \in L_0(\nu) :\ h_f \in L_0(\mu)\}.$$

Define an operator $T : \mathrm{Dom}_B(K) \to L_0(\mu)$ by setting

$$(Tf)(s) := \int_B K(s, t, f(t))\, d\nu \ \ \mu\text{-a.e.} \tag{2.1}$$

Let E be an order ideal in $L_0(\nu)$, K be a function satisfying conditions (1), (2) above, and let $E \subset \mathrm{Dom}_B(K)$. Then by [19, Coroll. 5.4.], (2.1) defines an integral $\mathcal{P}$-operator from E to $L_0(\mu)$. This operator is called a *generalized Urysohn operator* with the kernel K. Some properties of generalized Urysohn operators are investigated in [19].

Consider the following order on $\mathcal{P}(E, F) : S \leq T$ whenever $T - S$ is a positive orthogonally additive operator in the above sense. Then $\mathcal{P}(E, F)$ becomes an ordered vector space. For Dedekind complete range lattices we have the following theorem.

Theorem 2.7. [19, Theorem 3.6] *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{P}(E, F)$ is a Dedekind complete vector lattice. Moreover, for all $S, T \in \mathcal{P}(E, F)$ and $x \in E$ the following relations hold:*

(1) $(T \vee S)(x) := \sup\{Ty + Sz :\ x = y \sqcup z\}$;

(2) $(T \wedge S)(x) := \inf\{Ty + Sz : \ x = y \sqcup z\}$;
(3) $(T)^+(x) := \sup\{Ty : \ y \sqsubseteq x\}$;
(4) $(T)^-(x) := -\inf\{Ty : \ y \sqsubseteq x\}$;
(5) $|Tx| \leq |T|(x)$.

Consider a vector space V and a real Archimedean vector lattice E. A map $\lvert\!\lvert \cdot \rvert\!\rvert : V \to E$ is said to be a *vector norm* if it satisfies the following axioms:

1) $\lvert\!\lvert v \rvert\!\rvert \geq 0$; $\lvert\!\lvert v \rvert\!\rvert = 0 \Leftrightarrow v = 0$ $(v \in V)$.
2) $\lvert\!\lvert v_1 + v_2 \rvert\!\rvert \leq \lvert\!\lvert v_1 \rvert\!\rvert + \lvert\!\lvert v_2 \rvert\!\rvert$ $(v_1, v_2 \in V)$.
3) $\lvert\!\lvert \lambda v \rvert\!\rvert = |\lambda| \, \lvert\!\lvert v \rvert\!\rvert$ $(\lambda \in \mathbb{R}, v \in V)$.

A vector norm is said to be *decomposable*, if

4) for all $e_1, e_2 \in E_+$ and $x \in V$ from $\lvert\!\lvert x \rvert\!\rvert = e_1 + e_2$ it follows that there exist $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $\lvert\!\lvert x_k \rvert\!\rvert = e_k$ $(k := 1, 2)$.

A triple $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ ((V, E) or V for short) is called a *lattice-normed space* if $\lvert\!\lvert \cdot \rvert\!\rvert$ is an E-valued vector norm in the vector space V. If the norm $\lvert\!\lvert \cdot \rvert\!\rvert$ is decomposable, then the space V itself is called *d*ecomposable. Two elements x, y of a lattice-normed space V are said to be *disjoint* (notation $x \perp y$), if $\lvert\!\lvert x \rvert\!\rvert \wedge \lvert\!\lvert y \rvert\!\rvert = 0$. We say that a net $(v_\alpha)_{\alpha \in \Lambda}$ *order converges* to an element $v \in V$ and write $v = (\mathrm{bo})-\lim\limits_{\alpha} v_\alpha$ if there exists a decreasing net $(e_\alpha)_{\alpha \in \Lambda}$ in E_+ such that $\inf_{\alpha \in \Lambda}(e_\alpha) = 0$ and $\lvert\!\lvert v - v_\alpha \rvert\!\rvert \leq e_\alpha$ for all $\alpha \in \Lambda$ satisfying $\alpha \geq \alpha_0$ for some $\alpha_0 \in \Lambda$. A net $(v_\alpha)_{\alpha \in \Lambda}$ is called *order fundamental* if the net $(v_\alpha - v_\beta)_{(\alpha, \beta) \in \Lambda \times \Lambda}$ order converges to zero. A lattice-normed space is called *order complete* if every order fundamental net order converges to an element of this space. Every decomposable order complete lattice-normed space is said to be a *Banach–Kantorovich space*.

Consider some traditional examples of lattice-normed spaces.

Example 2.8. We start with simple extreme cases, namely vector lattices and normed spaces. If $V = E$, then the absolute value of an element can be taken as its lattice norm: $\lvert\!\lvert v \rvert\!\rvert := |v| = v \vee (-v)$; $v \in E$. The decomposability of this norm follows from the Riesz decomposition property holding in every vector lattice (see [2, Theor. 1.13]). If $E = \mathbb{R}$, then V is a normed space.

Example 2.9. Let K be a compact topological space, and let X be a normed space. Let $V := C(K, X)$ be the space of all continuous vector-valued functions from K to X and $E := C(K, \mathbb{R})$. Given $f \in V$, we define its lattice norm by the relation $\lvert\!\lvert f \rvert\!\rvert : t \mapsto \|f(t)\|_X$ $(t \in K)$. Then $\lvert\!\lvert \cdot \rvert\!\rvert$ is a decomposable vector norm (see [12, Lemma 2.3.2]).

Example 2.10. Let E be a Banach function space over a finite measure space (Ω, Σ, μ), and let X be a Banach space. The notation $L_0(\Omega, \Sigma, \mu, X)$ or $L_0(\mu, X)$ for brevity denotes the space of (equivalence classes of) all strongly μ-measurable vector functions acting from Ω to X. As usual, vector-functions are equivalent, if they have equal values at almost all points of the set Ω. The *Köthe–Bochner* space $E(X)$ over (Ω, Σ, μ) is defined as

$$E(X) := \{f \in L_0(\mu, X) : \ \|f(\cdot)\|_X \in E\}.$$

For a given $f \in E(X)$ the map $f \mapsto \|f(\cdot)\|_X$ is the E-valued vector norm which is denoted by $\lvert\!\lvert f \rvert\!\rvert$ and $E(X)$ is a lattice-normed space with a decomposable vector norm (see [12, Lemma 2.3.7]). Since E is a Banach space, the lattice-normed space $E(X)$ is a Banach space with respect to the norm

$$\|f\|_{E(X)} := \|\|f(\cdot)\|_X\|_E, \quad f \in E(X).$$

It is worth noting that lattice-normed were invented by Kantorovich in the first part of 20 century. Recently, some classes of operators in lattice-normed spaces were investigated in [3, 6].

3. Main results

In this section we introduce a new class of dominated orthogonally additive operators in lattice-normed spaces and calculate the exact dominant of a dominated operator. It was still noted by Mazón and Segura de León that in the context of the theory of orthogonally additive operators laterally-to-order continuous operators play the same role as order continuous operators in the setting of the theory of regular linear operators. In the last part of the section we investigate laterally-to-order continuous orthogonally additive operators. We show that a dominated orthogonally additive operator is laterally-to-order continuous if and only if its exact dominant is.

Definition 3.1. Let (V, E) and (W, F) be lattice-normed spaces. A map $T : V \to W$ is said to be an *orthogonally additive* if $T(u + v) = Tu + Tv$ for any $u, v \in V$ with $u \perp v$. An orthogonally additive map $T : V \to W$ is said to be a *dominated Popov operator* (or *dominated $\mathcal{P}$-operator* for brevity) if there exists a positive orthogonally additive operator $S : E \to F$ such that $\lvert Tv \rvert \leq S \lvert v \rvert$ for any $v \in V$. In this case we say that S is a *dominant* for T. The set of all dominants of an operator T is denoted by $\mathfrak{D}(T)$. If there is the least element in $\mathfrak{D}(T)$ with respect to the order induced by $\mathcal{P}_+(E, F)$, then it is called the *least* or the *exact dominant* of T, and it is denoted by $\lvert T \rvert$. The set of all dominated $\mathcal{P}$-operators from V to W is denoted by $\mathcal{DP}(V, W)$.

Consider some examples of dominated $\mathcal{P}$-operators.

Example 3.2. Suppose $(V, E) = (W, F) = (\mathbb{R}, \mathbb{R})$. The notation $\mathbb{R}^{\mathbb{R}}$ denotes the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with point-wise order. Assign by definition

$$\mathbb{R}_0^{\mathbb{R}} := \{f \in \mathbb{R}^{\mathbb{R}} : f(0) = 0\}.$$

Then $\mathcal{DP}(\mathbb{R}, \mathbb{R}) = \mathbb{R}_0^{\mathbb{R}}$.

Example 3.3. Let X and Y be normed spaces. Consider the lattice-normed spaces $(X, \mathbb{R})$ and $(Y, \mathbb{R})$. A given map $T : X \to Y$ is an element of $\mathcal{DP}(X, Y)$ if and only if there exists a function $f : \mathbb{R} \to \mathbb{R}_+$ such that $f(0) = 0$, and the inequality $\|Tx\| \leq f(\|x\|)$ holds for every $x \in X$.

Example 3.4. Let E and F be vector lattices with F Dedekind complete. Consider the lattice-normed spaces (E, E) and (F, F) where the lattice valued norms coincide with the modules. We may show that the vector space $\mathcal{DP}(E, F)$ coincides with $\mathcal{P}(E, F)$. Indeed, any $\mathcal{P}$-operator $T : E \to F$ is dominated by positive one, namely by its module $|T|$, and we have that $\mathcal{DP}(E,F) \subset \mathcal{P}(E,F)$. On the other hand, since any positive orthogonally additive operator is laterally-to-order bounded we deduce that $\mathcal{DP}(E,F) \subset \mathcal{D}_{\mathcal{P}}(E,F)$.

Now we need some auxiliary information.

Definition 3.5. Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space. The characteristic function of a measurable set $D \in \Sigma$ is denoted by 1_D. A function $f : \Omega \to X$ is called *simple*, if there exist $x_1, \dots, x_n \in X$ and $\Omega_1, \dots, \Omega_n \in \Sigma$ such that $f = \sum\limits_{i=1}^{n} x_i 1_{\Omega_i}$. A function $f : \Omega \to X$ is called *strongly μ-measurable*, if there exits a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} \|f(\cdot) - f_n(\cdot)\|_X = 0$ μ-almost everywhere.

Definition 3.6. Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space. A strongly μ-measurable function $f : \Omega \to X$ is said to be *Bochner* integrable, if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \int\limits_{\Omega} \|f(\cdot) - f_n(\cdot)\|_X \, d\mu = 0$. In this case the *Bochner integral* $\int\limits_A f(t)\, d\mu$ of f is defined for each $A \in \Sigma$ by

$$\int\limits_A f(t)\, d\mu = \lim_{n\to\infty} \int\limits_A f_n(t)\, d\mu,$$

where $\int\limits_A f_n(t)\, d\mu$ of a simple function $f_n = \sum\limits_{i=1}^{k(n)} x_i^n 1_{\Omega_i^n}$ is defined by

$$\int\limits_A f_n(t)\, d\mu = \sum_{i=1}^{k(n)} x_i^n \mu(\Omega_i^n).$$

The following proposition gives a concise characterization of Bochner integrable functions and explains the denotation $L_1(X)$ for the set of all Bochner integrable functions.

Proposition 3.7. [5, Page 45] *A strongly μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int\limits_{\Omega} \|f(\cdot)\|_X \, d\mu < \infty$.*

Definition 3.8. Let (Ω, Σ, μ) be a finite measure space, E be a Banach function space in $L_0(\mu)$ and X be a Banach space. Let $K : \Omega \times \mathbb{R} \to X$ be a function satisfying the following conditions:

(C_0) $K(t, 0) = 0$ for μ-almost all $t \in \Omega$;
(C_1) $K(t, \cdot)$ is continuous with respect to the norm of X for μ-almost all $t \in \Omega$.

Since for every simple function $p = \sum\limits_{k=1}^{n} c_k 1_{\Omega_k}$, $c_k \in \mathbb{R}$, $\Omega_k \in \Sigma$ the function $K(t, p(t))$ is Bochner μ-measurable by (C_1) and any $f \in L_0(\mu)$ is the pointwise limit of a sequence of simple functions, it follows that the function $K(t, f(t))$ is Bochner μ-measurable. Assign

$$\mathrm{Dom}(K) := \{f \in L_0(\mu) : \ K(\cdot, f(\cdot)) \in L_1(\mu, X)\}.$$

If $E \subset \mathrm{Dom}(K)$, we can define an orthogonally additive operator $T : E \to X$ by

$$Tf = \int_A K(t, f(t))\, d\mu(t). \tag{3.1}$$

where the integral in (3.1) is the Bochner integral.

Proposition 3.9. *Let $E \subset Dom(K)$. Then equation (3.1) defines a dominated $\mathcal{P}$-operator $T : E \to X$.*

Proof. Since $E \subset \mathrm{Dom}(K)$ and by Proposition 3.7, one has $\|K(\cdot, f(\cdot))\| \in L_1(\mu)$ for any $f \in E$. Hence, it is defined a positive orthogonally additive operator $S : E \to \mathbb{R}$ by

$$Sf = \int_A \|K(t, f(t))\|\, d\mu(t).$$

Then for any $f \in E$ the following inequality holds:

$$\begin{aligned} |Tf| = \|Tf\| &= \Big\| \int_A K(t, f(t))\, d\mu(t) \Big\| \\ &\leq \int_A \|K(t, f(t)\|)\, d\mu(t) = S(f). \end{aligned}$$

Thus S is a dominant for T and the proof is finished. □

Definition 3.10. A subset D of a vector lattice E is said to be a *lateral ideal in E, if the following conditions hold:*

(1) *if $x, y \in D$ and $x \perp y$, then $x + y \in D$;*
(2) *if $x \in D$, then $y \in D$ for any $y \in \mathcal{F}_x$.*

Consider some examples.

Example 3.11. Let E be a vector lattice. Every order ideal in E is a lateral ideal.

Example 3.12. Let E be a vector lattice. Then $E_+ = \{e \in E : 0 \leq e\}$ is a lateral ideal.

Example 3.13. Let E be a vector lattice, and let $e \in E$. Then $\mathcal{F}_e$ is a lateral ideal.

Example 3.14. Let E, F be a vector lattices, and let $T \in \mathcal{P}_+(E, F)$. Then $\mathcal{N}_T := \{e \in E : T(e) = 0\}$ is a lateral ideal.

Let (V, E) be a decomposable lattice-normed space. Assign

$$\widetilde{E}_+ := \{e \in E_+ : e = \bigsqcup_{i=1}^{n} |v_i| \,;\, v_i \in V; \, n \in \mathbb{N}\}.$$

Example 3.15. The set $\widetilde{E}_+$ is a lateral ideal. We need to check (1) and (2) of Definition 3.10. First, it is clear that the sum of two disjoint elements of $\widetilde{E}_+$ belongs to $\widetilde{E}_+$. Second, take $e \in \widetilde{E}_+$ and $e_0 \sqsubseteq e$. Then, by definition, there exists a family $v_1, \dots, v_n$ of mutually disjoint elements of V such that $e = \bigsqcup_{i=1}^{n} |v_i|$ and

$e = e_0 \sqcup (e - e_0)$. Taking into account the decomposability of V, we can find two families $(u_1, \dots, u_n)$ and $(w_1, \dots, w_n)$ of mutually disjoint elements of V such that

$$v_i = u_i + w_i;\ u_i \perp w_i;\ i \in \{1, \dots, n\};$$
$$e_0 = \bigsqcup_{i=1}^{n} |u_i|\ ;\ (e - e_0) = \bigsqcup_{i=1}^{n} |w_i|\ .$$

Definition 3.16. Let E, F be vector lattices, and let D be a lateral ideal in E. A map $T : D \to F$ is said to be

(1) *orthogonally additive*, if $T(e + f) = Te + Tf$ for any disjoint elements $e, f \in D$;
(2) *positive*, if $Te \geq 0$ for any $e \in D$.

For a pair of orthogonally additive maps T and G from D to F assign

$$T \leq G \Leftrightarrow (G - T) \text{ is a positive map.}$$

Recall that an ordered set $(V, \leq)$ is said to be *directed downward* if for any $u, v \in V$ there exists $w \in V$ such that $w \leq u$ and $w \leq v$. Now we need the following auxiliary lemma.

Lemma 3.17. *Let E, F be vector lattices with F Dedekind complete, D be a lateral ideal in E, and $(T_\xi)_{\xi \in \Xi}$ be a downward directed set of positive orthogonally additive maps from D to F. For any $e \in D$, put*

$$Re = \inf\{T_\xi e : \xi \in \Xi\}. \tag{3.2}$$

Then (3.2) defines a positive orthogonally additive map from D to F.

Proof. Take an element e of D, and assign $\mathcal{R}_e := \{T_\xi e : \xi \in \Xi\}$. Since $(T_\xi)_{\xi \in \Xi}$ is the downward directed set, we have that $\mathcal{R}_e$ is the downward directed set of positive elements of the vector lattice F for any $e \in D$. From Dedekind completeness of F it follows that the map R is well defined and positive. We show that R is an orthogonally additive map. Take any pair e, f of disjoint elements of D. Then for any $\xi \in \Xi$ we may write

$$T_\xi(e + f) = T_\xi e + T_\xi f \Rightarrow\ R(e + f) \leq T_\xi e + T_\xi f.$$

Passing to the infimum in the right-hand side of the above inequality over all $\xi \in \Xi$, we deduce that

$$R(e + f) \leq Re + Rf.$$

On the other side, we have that

$$T_\xi(e + f) = T_\xi e + T_\xi f \Rightarrow\ T_\xi(e + f) \geq Re + Rf$$

and passing to the infimum in the left-hand side of the above inequality over all $\xi \in \Xi$, we get

$$R(e + f) \geq Re + Rf.$$

Therefore $R(e + f) = Re + Rf$ and the proof is finished. □

Definition 3.18. Let E, F be vector lattices with F Dedekind complete, and let D be a lateral ideal in E. With any positive orthogonally additive map $T : D \to F$, we can associate a map $\widetilde{T}_D : E \to F$ defined by

$$\widetilde{T}_D e = \sup\{Te_0 : \, e_0 \sqsubseteq e; \, e_0 \in D\}.$$

The map $\widetilde{T}_D$ is said to be the *minimal extension* (with respect to D) of T.

Lemma 3.19. [19, Theorem 4.4.] *Let E, F be vector lattices with F Dedekind complete, D be a lateral ideal in E, and $T : D \to F$ be a positive orthogonally additive map. Then $\widetilde{T}_D \in \mathcal{P}_+(E, F)$ and $\widetilde{T}_D e = Te$ for any $e \in D$.*

The following theorem is the first main result of the article.

Theorem 3.20. *Let (V, E) and (W, F) be lattice-normed spaces with V decomposable and F Dedekind complete. Then every dominated $\mathcal{P}$-operator $T : V \to W$ has an exact dominant $\lvert T \rvert$. Moreover the exact dominant of a dominated $\mathcal{P}$-operator $T : V \to W$ can be calculated by the following formulas:*

(1) $\lvert T \rvert (e) = \sup\left\{ \sum\limits_{i=1}^{n} \lvert Tu_i \rvert : \, \bigsqcup\limits_{i=1}^{n} \lvert u_i \rvert = e, \, n \in \mathbb{N} \right\} \quad (e \in \widetilde{E}_+)$;

(2) $\lvert T \rvert (e) = \sup\left\{ \lvert T \rvert (e_0) : \, e_0 \in \widetilde{E}_+, \, e_0 \sqsubseteq e \right\} \quad (e \in E)$.

Proof. First we show that $\mathfrak{D}(T)$ is the downward directed set. Indeed, assume that $S_1, S_2 \in \mathfrak{D}(T)$. Take $x \in V$ with $e = \lvert x \rvert$, and assume that $e = f \sqcup h$. Then by the decomposability of V, there exist $y, z \in V$ such that $x = y + z$, $\lvert y \rvert = f$, $\lvert z \rvert = h$, and we may write

$$\begin{aligned} \lvert Tx \rvert &= \lvert T(y+z) \rvert \leq \lvert Ty \rvert + \lvert Tz \rvert \\ &\leq S_1(f) + S_2(h). \end{aligned}$$

Passing to the infimum in the right-hand side of the above inequality over all $h, f \in E_+$ with $e = f \sqcup h$, we deduce that

$$\lvert Tx \rvert \leq (S_1 \wedge S_2) \lvert x \rvert .$$

For any positive operator $S \in \mathfrak{D}(T)$ by $\overline{S}$, we denote the restriction S to positive cone E_+. Clearly, $\overline{S}$ is a positive orthogonally additive map from E_+ to F. It is not difficult to check that $\{\overline{S} : \, S \in \mathfrak{D}(T)\}$ is the downward directed set of positive orthogonally additive maps from E_+ to F. Now applying Lemma 3.17 we have that $R : E_+ \to F$ defined by

$$Re = \inf\{\overline{S}e : \, S \in \mathfrak{D}(T)\}$$

is a positive orthogonally additive map and

$$\lvert Tx \rvert \leq R \lvert x \rvert \leq S \lvert x \rvert$$

for any $x \in V$ and $S \in \mathfrak{D}(T)$. Then by Lemma 3.19 $\widetilde{R}$ is a positive orthogonally additive operator from E to F and $\widetilde{R}e = Re$ for any $e \in E_+$. Thus $\widetilde{R} = \lvert T \rvert$. Now we prove the second part of the theorem. Take $e \in \widetilde{E}_+$, and denote by

Ge the right-hand side of the formula (1). For any decomposition $e = \bigsqcup_{i=1}^{n} |v_i|$, $v_1, \dots, v_n \in V$, we may write

$$\sum_{i=1}^{n} |Tv_i| \leq |T| \Big(\bigsqcup_{i=1}^{n} |v_i| \Big) = |T| (e).$$

Since a vector lattice F is Dedekind complete, we have that $G : \widetilde{E}_+ \to F$ is a well defined positive map. We claim that G is an orthogonally additive map. Indeed, take two disjoint elements $e, f \in \widetilde{E}_+$. Assume that $e = \bigsqcup_{i=1}^{n} |u_i|$ and $f = \bigsqcup_{j=1}^{m} |v_j|$, $e \perp f$. Then we may write

$$\sum_{i=1}^{n} |Tu_i| + \sum_{k=1}^{m} |Tv_k| \leq G(e+f).$$

Passing to the supremum in the left-hand side of the above inequality over all disjoint decompositions of e and f, we obtain that

$$Ge + Gf \leq G(e+f).$$

Now we prove the reverse inequality. Take a decomposition $e + f = \bigsqcup_{k=1}^{n} |w_k|$. By the decomposability of the vector norm in V, there exist finite families of mutually disjoint elements $u_1, \dots, u_n \in V$ and $v_1, \dots, v_n \in V$, such that

$$e = \bigsqcup_{i=1}^{n} |u_i| \, ; f = \bigsqcup_{i=1}^{n} |v_i| \, ;$$
$$w_i = u_i + v_i; \quad 1 \leq i \leq n.$$

Then we have

$$\sum_{k=1}^{n} |Tw_k| \leq \sum_{k=1}^{n} |Tv_k| + \sum_{k=1}^{n} |Tu_k| \leq Ge + Gf.$$

Passing to the supremum in the left-hand side of the above inequality over all disjoint decomposition of the element $e + f$, we obtain the reverse inequality

$$G(e+f) \leq G(e) + G(f).$$

Thus G is a positive orthogonally additive map from $\widetilde{E}_+$ to F and $|Tx| \leq G\,|x|$ for any $x \in V$. Now applying Lemma 3.19, we have that $\widetilde{G} \in \mathcal{P}_+(E, F)$. We show that $\widetilde{G}$ is the exact dominant of T. Indeed, take $S \in \mathfrak{D}(T)$ and $e \in \widetilde{E}_+$. Then for any decomposition $e = \bigsqcup_{i=1}^{n} |u_i|$, we may write

$$\sum_{i=1}^{n} |Tu_i| \leq S \Big(\bigsqcup_{i=1}^{n} |u_i| \Big) = Se.$$

Passing to the supremum in the left-hand side of the above inequality over all disjoint decomposition of the element e, we obtain

$$\widetilde{G}e = Ge \leq Se, \quad e \in \widetilde{E}_+.$$

Finally, for any $e \in E$, we have

$$\widetilde{G}e_0 = Ge_0 \leq Se_0 \leq Se, \quad e_0 \sqsubseteq e, \quad e_0 \in \widetilde{E}_+.$$

Passing to the supremum in the left-hand side of the above inequality over all fragments $e_0 \in \widetilde{E}_+$ of an element e, we deduce that the inequality $\widetilde{G}e \leq Se$ holds for any element $e \in E$, and the proof is completed. □

Corollary 3.21. *Let (V, E) and (W, F) be the same as Theorem 3.20. Then an orthogonally additive operator $T : V \to W$ is dominated if and only if the set*

$$\mathcal{G}e = \Big\{ \sum_{i=1}^{n} |\!|Tu_i|\!| : \bigsqcup_{i=1}^{n} |\!|u_i|\!| = e,\, n \in \mathbb{N} \Big\}$$

is order bounded for any $e \in \widetilde{E}_+$.

Proof. Necessity is obvious. Assume that $\mathcal{G}e$ is an order bounded set for any $e \in \widetilde{E}_+$. Define a map $G : \widetilde{E}_+ \to F$ in the same way as in the proof of Theorem 3.20. Taking into account the order boundness of the set $\mathcal{G}e$, $e \in \widetilde{E}_+$, and Dedekind completeness of the vector lattice F, we deduce that G is a positive orthogonally additive map from $\widetilde{E}_+$ to F, $\widetilde{G} \in \mathcal{P}_+(E, F)$ and $|\!|T|\!| = \widetilde{G}$. □

Definition 3.22. Let (V, E) be a lattice-normed space. A net $(v_\alpha)_{\alpha\in\Lambda} \subset V$ (sequence $(v_n)_{n\in\mathbb{N}} \subset V$) is said to be *laterally* convergent to $v \in V$, if $v = (bo) - \lim_\alpha v_\alpha$ ($v = (bo) - \lim_n v_n$) and $\big|\!\big|v_\beta - v_\gamma\big|\!\big| \perp \big|\!\big|v_\gamma\big|\!\big|$ for all $\beta, \gamma \in \Lambda$, $\beta \geq \gamma$ ($|\!|v_n - v_m|\!| \perp |\!|v_m|\!|$ for all $n, m \in \mathbb{N}$, $n \geq m$).

Definition 3.23. Let (V, E) and (W, F) be lattice-normed spaces. An orthogonally additive operator $T : V \to W$ is said to be:

(1) *laterally-to-order* continuous provided T sends laterally convergent nets in W to order convergent nets in W;
(2) *σ-laterally-to-order* continuous provided T sends laterally convergent sequences in W to order convergent sequences in W.

The vector space of all laterally to-order continuous (σ-laterally-to-order continuous) dominated $\mathcal{P}$-operators from V to W is denoted by $\mathcal{DP}_c(E, F)$ ($\mathcal{DP}_{\sigma c}(E, F)$).

Consider some examples.

Example 3.24. Let $(V, E) = (W, F) = (\mathbb{R}, \mathbb{R})$. Let $C(\mathbb{R})$ denote the space of all continuous function on $\mathbb{R}$. Assign by definition

$$C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : f(0) = 0\}.$$

Then $\mathcal{DP}_c(\mathbb{R}, \mathbb{R}) = \mathcal{DP}_{\sigma c}(\mathbb{R}, \mathbb{R}) = C_0(\mathbb{R})$.

Proposition 3.25. *Let (A, Ξ, μ) and (B, Σ, ν) be finite measure spaces, and let $T : L_0(\nu) \to L_0(\mu)$ be a generalized Urysohn operator with the kernel K as in Example 2.6. Then T is a σ-laterally-to-order continuous operator.*

Proof. Take a sequence $(f_n) \subset L_0(\nu)$, which laterally converges to f. Recall that if $f : B \to \mathbb{R}$ is a ν-measurable function, then the *support* of f is the set $\mathrm{supp}(f) = \{t \in B : f(t) \neq 0\}$. Clearly, $\mathrm{supp}(f)$ is a ν-measurable set. By definition $(f - f_n) \perp f_n$ for any $n \in \mathbb{N}$. In means that

$$\nu\{t \in B : t \in \mathrm{supp}(f - f_n) \cap \mathrm{supp}(f_n)\} = 0.$$

Consequently $f_n = f1_{D_n}$ and $1_{D_n} \xrightarrow{\text{a.e.}} 1_D$, where $D_n \in \Sigma$, $D_n = \mathrm{supp}(f_n)$, $n \in \mathbb{N}$, and $D = \mathrm{supp}(f)$. Now we have

$$\begin{aligned}(Tf_n)(s) &= \int_B K(s,t,f_n(t))\,d\nu = \int_B K(s,t,f1_{D_n}(t))\,d\nu \\ &= \int_{D_n} K(s,t,f(t))\,d\nu \overset{n\to\infty}{\longrightarrow} \int_D K(s,t,f(t))\,d\nu = \int_B K(s,t,f(t))\,d\nu\end{aligned}$$

for μ-almost all $s \in A$. □

Definition 3.26. Let E, F be vector lattices, and let D be a lateral ideal in E. A map $T : D \to F$ is said to be a *laterally-to-order continuous* provided T sends laterally convergent nets in D to order convergent nets in F.

Lemma 3.27. [19, Theorem 4.5.] *Let E, F be Dedekind complete vector lattices, D be a lateral ideal in E, and $T : D \to F$ be a laterally-to-order continuous positive orthogonally additive map. Then the minimal extension $\widetilde{T}_D$ of T is a laterally-to-order continuous positive orthogonally additive operator from E to F.*

The next theorem is the second main result of the article.

Theorem 3.28. *Let (V, E) be a decomposable lattice-normed space, and let (W, F) be a Banach–Kantorovich space. Then for a dominated $\mathcal{P}$-operator $T : V \to W$ the following statements hold:*

(1) *T is a laterally-to-order continuous if and only if its exact dominant $|\!|T|\!| : E \to F$ is;*
(2) *T is a σ-laterally-to-order continuous if and only if its exact dominant $|\!|T|\!| : E \to F$ is.*

Proof. We shall prove only statement (1) because the proof of (2) is similar. Take a laterally convergent net $(v_\alpha)_{\alpha\in\Lambda} \subset V$ with $v = (bo) - \lim_\alpha v_\alpha$. Since $(v - v_\alpha)$ and v_α are disjoint elements for any $\alpha \in \Lambda$, we may write

$$\begin{aligned}|\!|Tv - Tv_\alpha|\!| &= \big|\!\big|T(v - v_\alpha + v_\alpha) - Tv_\alpha\big|\!\big| \\ &= \big|\!\big|T(v - v_\alpha)\big|\!\big| \le |\!|T|\!|\,(|\!|v - v_\alpha|\!|) \to 0.\end{aligned}$$

Hence, T is a laterally-to-order continuous operator. Let us prove the converse assertion. Suppose $T \in \mathcal{DP}_c(V, W)$. Take an element $e \in \widetilde{E}_+$ and a laterally convergent net $(e_\alpha)_{\alpha\in\Lambda} \subset \widetilde{E}_+$; so that $e = (o) - \lim_\alpha e_\alpha$. Assign

$$g = \sup_\alpha \sup\Big\{\sum_{i=1}^n |\!|Tv_i|\!| \,:\, v_1, \dots, v_n \in V;\, \bigsqcup_{i=1}^n |\!|v_i|\!| = e_\alpha,\, n \in \mathbb{N}\}.$$

Then we have $g = \sup_\alpha |\!|T|\!|\,(e_\alpha) \le |\!|T|\!|\,(e)$. We show that $|\!|T|\!|\,(e) \le \sup_\alpha |\!|T|\!|\,(e_\alpha)$. Consider a finite family of mutually disjoint elements $v_1, \dots, v_n$ of V with the property $\bigsqcup\limits_{i=1}^{n} \lvert v_i \rvert = e$. Given $\alpha \in \Lambda$, we associate with each $i \in \{1, \dots, n\}$ a representation $v_i = u_{i,\alpha} + w_{i,\alpha}$, $u_{i,\alpha} \perp w_{i,\alpha}$ for every $i \in \{1, \dots, n\}, \alpha \in \Lambda$; so that

$$\lvert v_i \rvert = \lvert u_{i,\alpha} \rvert + \lvert w_{i,\alpha} \rvert ; \bigsqcup_{i=1}^{n} \lvert u_{i,\alpha} \rvert = e_\alpha; \bigsqcup_{i=1}^{n} \lvert w_{i,\alpha} \rvert = e - e_\alpha.$$

Since (e_α) laterally converges to e, we have $\lvert v_i - u_{i,\alpha} \rvert = \lvert w_{i,\alpha} \rvert$, and therefore $u_{i,\alpha}$ laterally converges to v_i for every $i \in \{1, \dots, n\}$. Then we have

$$\sum_{i=1}^{n} \lvert T v_i \rvert = (o) - \lim_{\alpha} \Big(\sum_{i=1}^{n} \lvert T u_{i,\alpha} \rvert \Big).$$

On the other hand, for any $\beta \in \Lambda$, we have

$$\begin{aligned} \sum_{i=1}^{n} \lvert T u_{i,\beta} \rvert &\leq \Big\{ \sum_{i=1}^{n} \lvert T u_i \rvert : u_1, \dots, u_n \subset V; \bigsqcup_{i=1}^{n} \lvert u_i \rvert = e_\beta; \, n \in \mathbb{N} \Big\} \\ &= \lvert T \rvert (e_\beta) \leq \sup_{\alpha} \lvert T \rvert (e_\alpha) = g. \end{aligned}$$

Passing to the (o)-limit over β in the latter inequalities, we obtain that $\sum\limits_{i=1}^{n} \lvert T u_i \rvert \leq g$. Finally, taking the supremum over all mutually disjoint $\{u_1, \dots, u_n\}$ we deduce that $\lvert T \rvert (e) \leq g$. Thus, we have proved that the operator $\lvert T \rvert$ is laterally-to-order continuous on the lateral ideal $\widetilde{E}_+$. By Lemma 3.27, we obtain that $\lvert T \rvert$ is a laterally-to-order continuous operator from E to F. □

Acknowledgments. The authors are thankful to the anonymous referee for the careful reading of the text and valuable remarks. M. Pliev was supported by the Russian Foundation for Basic Research (the grant number 17-51-12064).

References

1. N. Abasov and M. Pliev, *On extensions of some nonlinear maps in vector lattices*, J. Math. Anal. Appl. **455** (2017), 516–527.
2. C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Reprint of the 1985 original. Springer, Dordrecht, 2006.
3. A. Aydin, E. Yu Emelyanov, N. Erkursun Ozcan, and M. A. A. Marabeh, *Compact-like operators in lattice-normed spaces*, Indag. Math. **29** (2018), no. 2, 633–656.
4. J. Batt *Nonlinear integral operators on $C(S, E)$*, Studia Math. **48** (1973), 145–177.
5. J. Diestel and J. J. Uhl, *Vector measures*, With a foreword by B. J. Pettis. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
6. X. Fang and M. Pliev, *Narrow orthgonally additive operators in lattice-normed spaces*, Siberian Math. J. **58** (2017), no. 1, 134–141.
7. W. A. Feldman, *Lattice preserving maps on lattices of continuous functions*, J. Math. Anal. Appl. **404** (2013), 310–316.
8. W. A. Feldman, *A characterization of non-linear maps satisfying orthogonality properties*, Positivity **21** (2017), no. 1, 85–97.
9. W. A. Feldman and S. Pramod, *A characterization of positively decomposable non-linear maps between Banach lattices*, Positivity **12** (2008), 495–502.
10. N. Friedman and M. Katz *A representaion theorem for additive functionals*, Arch. Rational Mech. Anal. **21** (1966), no. 1, 49–57.

11. H. I. Gumenchuk, *On the sum of narrow and finite-rank orthogonally additive operators*, Ukrainian Math. J. **67** (2016), no. 12, 1831–1837.
12. A. G. Kusraev, *Dominated operators*, Translated from the 1999 Russian original by the author. Translation edited and with a foreword by S. Kutateladze. Mathematics and its Applications, 519. Kluwer Academic Publishers, Dordrecht, 2000.
13. M. Marcus and V. Mizel *Representaion theorem for nonlinear disjointly additive functionals and operators on Sobolev spaces*, Trans. Amer. Math. Soc. **226** (1977), 1–45.
14. M. Marcus and V. Mizel *Extension theorem of Hahn-Banach type for nonlinear disjontly additive functionals and operators in Lebesgue spaces*, J. Funct. Anal. **24** (1977), 303–335.
15. J. M. Mazón and S. Segura de León, *Order bounded orthogonally additive operators*, Rev. Roumane Math. Pures Appl. **35** (1990), no. 4, 329–353.
16. J. M. Mazón and S. Segura de León, *Uryson operators*, Rev. Roumane Math. Pures Appl. **35** (1990), no. 5, 431–449.
17. V. Orlov, M. Pliev, and D. Rode *Domination problem for AM-compact abstract Uryson operators*, Arch. Math. (Basel) **107** (2016), no. 5, 543–552.
18. M. Pliev *Domination problem for narrow orthogonally additive operators*, Positivity **21** (2017), no. 1, 23–33.
19. M. Pliev and K. Ramdane *Order unbounded orthogonally additive operators in vector lattices*, Mediterr. J. Math. **15** (2018), no. 2, Art. 55, 20 pp.
20. S. Segura de León, *Bukhvalov type characterization of Urysohn operators*, Studia Math. **99** (1991), no. 3, 199–220.

[1]Department of Mathematics, MAI – Moscow Aviation Institute (National Research University), Moscow, 121552 Russia.
E-mail address: abasovn@mail.ru

[2]Laboratory of Functional Analysis, Southern Mathematical Institute of the Russian Academy of Sciences, Vladikavkaz 362027, Russia.
E-mail address: plimarat@yandex.ru