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PROJECTIONS AND ISOLATED POINTS OF PARTS OF THE SPECTRUM

PIETRO AIENA[1*] and SALVATORE TRIOLO[2]

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Abstract. In this paper, we relate the existence of certain projections, commuting with a bounded linear operator $T \in L(X)$ acting on Banach space X, with the generalized Kato decomposition of T. We also relate the existence of these projections with some properties of the quasi-nilpotent part $H_0(T)$ and the analytic core $K(T)$. Further results are given for the isolated points of some parts of the spectrum.

1. Introduction and preliminaries

This paper relates the existence of certain projections which commute with a bounded linear operator, defined on a complex Banach space, to the isolated points of certain parts of the spectrum $\sigma(T)$, as the approximative point spectrum $\sigma_{\rm ap}(T)$, or the surjective spectrum $\sigma_{\rm s}(T)$. The existence of such projections is also related to the properties of some spectral subspaces, as the quasi-nilpotent part $H_0(T)$ and the analytic core $K(T)$ of T. It is well-known, that 0 is an isolated point of $\sigma(T)$ exactly when $X = H_0(T) \oplus K(T)$ and $H_0(T)$ is nonempty. We generalize this result by considering isolated points of $\sigma_{\rm ap}(T)$ and $\sigma_{\rm s}(T)$, or more generally by considering isolated points of the Kato spectrum. In particular, we show that an operator $T \in L(X)$ admits the generalized Kato decomposition (GKD) if and only if there exists a commuting projection P such that $T+P$ is semiregular and TP is quasi nilpotent. By considering the particular case that $T+P$ is bounded below (respectively, onto) and TP is quasi nilpotent, we derive

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*Corresponding author .
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some known results. We also establish some results in the case when the quasi-nilpotent part $H_0(T)$ is closed. This article has as a partial motivation, that of commenting upon some results claimed in [11] (without a proof), which seem to be not true, or that was given without a correct proof An important tool in this paper is the so-called single valued extension property's introduced by Dunford in [6], [7]. This property has an important role in the local spectral theory and the Fredholm theory; see the recent monographs by Laursen and Neumann [15] and [1].

Definition 1.1. Let X be a complex Banach space, and let $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$, which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$.
An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C}\backslash\sigma(T)$. Moreover, the identity theorem for analytic function entails that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum. Note that the SVEP is inherited by the restriction to closed invariant subspaces; that is, if T has SVEP at λ_0 and M is a closed T-invariant subspace of X, then $T|M$ has SVEP at λ_0.

For a bounded linear operator T defined on a complex Banach space X, the *local resolvent set* of T at the point $x \in X$, denoted by $\rho_T(x)$, is defined as the union of all open subsets $\mathcal{U}$ of $\mathbf{C}$ such that there exists an analytic function $f : \mathcal{U} \to X$, which satisfies

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathcal{U} . \tag{1.1}$$

The local spectrum $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbf{C} \setminus \rho_T(x)$, and obviously, we have $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T.

For every subset $\mathcal{F}$ of $\mathbf{C}$, the *analytic spectral subspace* of T associated with $\mathcal{F}$ is the set

$$X_T(\mathcal{F}) := \{x \in X : \sigma_T(x) \subseteq \mathcal{F}\}.$$

It is easily seen from the definition that $X_T(\mathcal{F})$ is a linear subspace T-invariant of X. Furthermore, for every closed $\mathcal{F} \subseteq \mathbb{C}$, we have

$$(\lambda I - T)X_T(\mathcal{F}) = X_T(\mathcal{F}) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathcal{F}; \tag{1.2}$$

see [15, Proposition 1.2.16].

The *quasi-nilpotent part* of T is defined as follows:

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \| T^n(x) \|^{1/n} = 0\}.$$

It is easily seen that $N(T^n) \subseteq H_0(T)$ for every $n \in \mathbb{N}$, so $N^\infty(T) \subseteq H_0(T)$, where $N^\infty(T) = \bigcup_{n=1}^{\infty} N(T^n)$ denotes the hyper-kernel of T.

For a bounded operator $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exist a constant $c > 0$ and a sequence $(x_n)_{n=0,1,\ldots} \subset X$, such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\| x_n \| < c^n \| x \|$ for all $n \in \mathbb{N}$. Note that

$K(T) \subseteq R^\infty(T)) \subseteq R(T^n)$, where $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$ denotes the *hyper-range* of T and $T(K(T)) = K(T)$; see [1, Theorem 1.21]. It should be noted that $K(T) = X_T(\mathbb{C} \setminus \{0\})$; see [1, Theorem 2.19].

It is easily seen from the definition of a quasi-nilpotent part that

$$x \in H_0(T) \Leftrightarrow Tx \in H_0(T). \tag{1.3}$$

The two subspaces $H_0(T)$ and $K(T)$ are in general not closed, and by [2],

$$H_0(\lambda I - T) \text{ closed} \Longrightarrow T \text{ has SVEP at } \lambda \tag{1.4}$$

and

$$H_0(\lambda I - T) \cap K(\lambda I - T) \text{ closed} \Rightarrow T \text{has SVEP at } \lambda. \tag{1.5}$$

Two classical quantities in operator theory are defined as follows. The *ascent* of an operator T is the smallest non-negative integer $p := p(T)$ such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T is the smallest non-negative integer $q := q(T)$ such that $R(T^q) = R(T^{q+1})$, and if such integer q does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$; see [1, Theorem 3.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ exactly when λ is a pole of the resolvent of T; see [10, Proposition 50.2]. Furthermore, if $\lambda \in \text{iso}\,\sigma(T)$, then the decomposition $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$ holds; see [1, Theorem 3.74]. If λ is a pole of the resolvent of T of order p, then $H_0(\lambda I - T) = N(\lambda I - T)^p$ and $K(\lambda I - T) = R(\lambda I - T)^p$; see [1, Theorem 3.74].
In what follows, we denote the dual of T by T^*. We have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.6}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda; \tag{1.7}$$

see [1, Theorem 3.8].

Recall that $T \in L(X)$ is said to be *bounded below*, if T is injective and $T(X)$ is closed. Let

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

denote the *approximate point spectrum*, and let $\sigma_{\mathrm{s}}(T)$ denote the *surjectivity spectrum* of T. It is well-known that

$$\sigma_{\mathrm{ap}}(T^*) = \sigma_{\mathrm{s}}(T) \quad \text{and} \quad \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*).$$

It is also well-known that if T is bounded below, then T^n is bounded below for every $n \in \mathbb{N}$. From the definition of the localized SVEP and from the identity $\sigma_{\mathrm{ap}}(T^*) = \sigma_{\mathrm{s}}(T)$, we easily obtain that

$$\sigma_{\mathrm{ap}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.8}$$

and dually

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{1.9}$$

Definition 1.2. An operator $T \in L(X)$ is said to be *left Drazin invertible*, if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$ is said to be *right Drazin invertible*, if $q := q(T) < \infty$ and $T^q(X)$ is closed. An operator $T \in L(X)$ is said to be *Drazin invertible*, if T is both left and right Drazin invertible, or equivalently $p(T) = q(T) < \infty$.

Evidently, if T is Drazin invertible, either T is invertible or 0 is a pole of the resolvent. The concept of pole may be sectioned as follows. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{ap}}(T)$, then λ is said to be a *left pole* of T, and dually, if $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$, then λ is said to be a *left pole* of T.

Every upper semi-Browder operator is left Drazin invertible. The following elementary lemma is well-known. Recall first that if $X = M \oplus N$, M and N are closed subspaces, then the pair (M, N) reduces T (i.e. both M and N are T-invariant) if and only if the projection P onto M along N commutes with T.

Lemma 1.3. *If M and N are closed T-invariant subspaces of X and $X = M \oplus N$, then T is bounded below if and only if the restrictions $T|M$ and $T|N$ are bounded below. Analogously, T is onto if and only if the restrictions $T|M$ and $T|N$ are onto. Consequently, $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T|M) \cup \sigma_{\mathrm{ap}}(T|N)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{s}}(T|M) \cup \sigma_{\mathrm{s}}(T|N)$.*

A bounded operator $T \in L(X)$ is said to be *semi-regular*, if $\ker T^n \subseteq T(X)$ for every $n \in \mathbb{N}$ and $T(X)$ is closed. Evidently, every bounded below operator, as well every onto operator, is semiregular. Moreover, if T is bounded below, then $H_0(T) = \{0\}$. The *Kato spectrum* is defined as

$$\sigma_{\mathrm{k}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

A well-known result shows that $\sigma_{\mathrm{k}} T)$ is a nonempty compact subset of $\mathbb{C}$ containing the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$; see [1, Theorem 1.75].

Lemma 1.4. *Let M and N be closed T-invariant subspaces of X, and let $X = M \oplus N$. Then T is semiregular if and only if both $T|M$ and $T|N$ are semiregular. Consequently, $\sigma_{\mathrm{k}}(T) = \sigma_{\mathrm{k}}(T|M) \cup \sigma_{\mathrm{k}}(T|N)$.*

Proof. Observe first that $\ker T|M = M \cap \ker T$. We also have $T(M) = M \cap T(X)$. To see this observe first that the inclusion $T(M) \subseteq M \cap T(X)$ is obvious. Conversely, if $y \in M \cap T(X)$, then $y \in M$ and $y = Tx$. Write $x = x_1 + x_2$, with $x_1 \in M$ and $x_2 \in N$. Then $y = Tx = Tx_1 + Tx_2$, and since $Tx_1 \in M$, we have $Tx_2 = y - Tx_1 \in M \cap N = \{0\}$; so $y = Tx_1 \in T(M)$.

By induction we have $T^n(M) = M \cap T^n(X)$ for every $n \in \mathbb{N}$. Assume that T is semiregular. Then

$$\ker T|M = M \cap \ker T \subseteq M \cap T^n(X) = (T|M)^n(M)$$

for every $n \in \mathbb{N}$. Moreover, since $T(X) = T(M) \oplus T(N)$, $T(X)$ is closed if and only if $T(M)$ and $T(N)$ are closed. □

A bounded linear operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated as a GKD, if there exists a pair of T-invariant closed subspaces M and N such that $X = M \oplus N$, $T|M$ is semiregular and $T|N$ is quasi nilpotent. The operator $T \in L(X)$ is said to to be *of Kato type* if in the decomposition above $T|N$ is nilpotent. It is known that every semi-Fredholm operator is of Kato type.

For the proof of the following, see [1, Theorem 1.68] and [1, Corollary 1.69].

Lemma 1.5. *If $T \in L(X)$, then the following statements hold:*

(i) *T is quasi nilpotent $\Leftrightarrow H_0(T) = X$.*

(ii) *If (M, N) is a GKD for T, then $H_0(T) = H_0(T|M) \oplus H_0(T|N) = H_0(T|M) \oplus N$.*

Observe that if (M, N) is a GKD for T, then the pair of annihilator $(N^\perp, M^\perp)$ is a GKD for the dual T^*; see [1, Theorem 1.43]. In what follows, let $\operatorname{iso} K$ denote the set of all isolated points of a subset $K \subseteq \mathbb{C}$.

Lemma 1.6. *Suppose that $T \in L(X)$ admits a GKD $(M.N)$ and that $0 \in \sigma(T)$. Then*

(i) *T has SVEP at 0 if and only if $0 \in \operatorname{iso} \sigma_{\mathrm{ap}}(T)$.*

(ii) *T^* has SVEP at 0 if and only if $0 \in \operatorname{iso} \sigma_{\mathrm{s}}(T)$.*

Proof. (i) The implication $(\Leftarrow)$ has been observed above. To show the reverse implication assume that T has SVEP at 0 and that (M, N) is a GKD for T. Then the restriction $T|M$ has SVEP at 0 and $H_0(T) = N$; see [1, Theorem 3.14]. So $X = M \oplus H_0(T)$. Since $\ker T|M = \ker T \cap M \subseteq M \cap H_0(T) = \{0\}$, we then have that $T|M$ is injective. Since $T|M$ is semiregular, then $T(M)$ is closed; so $T|M$ is bounded below. By assumption $T|N$ is quasi nilpotent, so $\sigma_{\mathrm{ap}}(T|N) = \{0\}$. Therefore,

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T|M) \cup \sigma_{\mathrm{ap}}(T|N) = \sigma_{\mathrm{ap}}(T|M) \cup \{0\}.$$

But $0 \notin \sigma_{\mathrm{ap}}(T|M)$, and $\sigma_{\mathrm{ap}}(T|M)$ is closed, from which, we conclude that 0 is an isolated point of $\sigma_{\mathrm{ap}}(T)$.

(ii) We have only to show the implication $(\Leftarrow)$. The pair $(N^\perp, M^\perp)$ is a GKD for T^*, and hence if T^* has SVEP at 0, then $T^*|N^\perp$ has SVEP at 0, and as above it then follows that $T^*|N^\perp$ is injective. By [1, Lemma 3.12], we then conclude that $T|M$ is onto. Since $T|N$ is quasi nilpotent, then $\sigma_{\mathrm{s}}(T) = \{0\}$. Finally

$$\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{s}}(T|M) \cup \sigma_{\mathrm{s}}(T|N) = \sigma_{\mathrm{s}}(T|M) \cup \{0\}.$$

But $0 \notin \sigma_{\mathrm{s}}(T|M)$, and $\sigma_{\mathrm{s}}(T|M)$ is closed, from which we conclude that 0 is an isolated point of $\sigma_{\mathrm{s}}(T)$. □

2. Projections

The isolated points of the spectrum $\sigma(T)$ have been characterized by Koliha [12] and [13] as follows:

Theorem 2.1. *Let $T \in L(X)$. Then there exists a projection $0 \neq P \in L(X)$ commuting with T such that $T + P$ is invertible and TP is quasi nilpotent if and only if $0 \in \operatorname{iso} \sigma(T)$.*

An operator T for which either T is invertible or $0 \in \operatorname{iso}\sigma(T)$ is said some time to be *generalized Drazin invertible*. The result of Theorem 2.1 has been established in the more abstract framework of Banach algebras. The methods used in this paper will be those of local spectral theory involving the spectral subspaces $H_0(T)$ and $K(T)$ and may be considered a refinement of result of Theorem 2.1.

An important question in local spectral theory and the Fredholm theory is to find conditions which ensure that $H_0(T)$ or $K(T)$ is closed. It should be noted that if $0 \in \operatorname{iso}\sigma(T)$, then $H_0(T)$ is closed, since it coincides with the range $P(X)$ of the spectral projection P associated with the set $\{0\}$, while $K(T)$ is closed since coincides with the kernel $\ker P$. Precisely, we have

$$0 \in \operatorname{iso}\sigma(T) \;\Leftrightarrow\; H_0(T) \text{ and } K(T) \text{ are closed and } X = H_0(T) \oplus K(T), \quad (2.1)$$

where $\oplus$ is the topological direct sum; see Mbekhta [16]. In this case $T|K(T)$ is invertible and $T|H_0(T)$ is quasi nilpotent. Mbekhta's result has been improved by Schmoeger [18], in the following way:

$$0 \in \operatorname{iso}\sigma(T) \;\Leftrightarrow\; X = H_0(T) \oplus K(T), K(T) \text{ is closed}, H_0(T) \neq \{0\}, \quad (2.2)$$

where here $\oplus$ denotes the algebraic direct sum.

The operators which admit a GKD may be characterized by means of commuting projection in the following way.

Theorem 2.2. *If $T \in L(X)$, the following statements are equivalent:*

(i) *T admits a GKD;*

(ii) *there exists a commuting projection P such that $T + P$ is semiregular and TP is quasi nilpotent.*

In this case $K(T) = \ker P$. If $0 \in \sigma(T)$, then $0 \in iso\,\sigma_{\mathrm{k}}(T)$.

Proof. (i) $\Rightarrow$ (ii) Suppose that T has a GKD (M, N). Then $T|M$ is semiregular and $T|N$ is quasi nilpotent. Let P be the projection of X onto N along M. Then $M = \ker P$ and $N = P(X)$. The pair (M, N) reduces T, so $PT = TP$. Since also $T + P$ and TP are reduced by $(M.N)$, by Lemma 1.4, the restriction $(T + P)|M = T|M$ is also semiregular. Furthermore,

$$(T + P)|N = (T + P)|P(X) = T|N + I|N$$

is invertible, hence $T + P = (T + P)|M \oplus (T + P)|N$ is semiregular. We have, by part (ii) of Lemma 1.5 and since $T|N$ is quasi nilpotent,

$$H_0(TP) = H_0((TP)|M) \oplus H_0((TP)|N) = H_0(0) \oplus H_0(T|N) = M \oplus N = X;$$

so, by part (i) of Lemma 1.5, TP is quasi nilpotent.

(ii) $\Rightarrow$ (i) Suppose that P is a commuting projection for which $T + P$ is semiregular and TP is quasi nilpotent. Then $X = \ker P \oplus P(X)$, and the pair $(\ker P, P(X))$ reduces T and hence reduces also $T + P$ and TP. By Lemma 1.4, the restriction $(T + P)|\ker P = T|\ker P$ is semiregular.

We show now that the restriction $T|P(X)$ is quasi nilpotent. Let $x \in P(X)$ be arbitrary chosen. Then

$$\|(T|P(X))^n\|^{\frac{1}{n}} = \|T^n P^n x\|^{\frac{1}{n}} = \|(TP)^n x\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty,$$

so $x \in H_0(T|P(X))$ and hence $P(X) \subseteq H_0(T|P(X))$. The reverse inclusion $H_0(T|P(X)) \subseteq P(X)$ is obvious, so $H_0(T|P(X)) = P(X)$, and this implies, by Lemma 1.5, that $T|P(X)$ is quasi nilpotent. Therefore, the pair $(\ker P, P(X))$ is a GKD for T, and hence $K(T) = K(T|\ker P)$; see [1, Theorem 1.41]. On the other hand, since the restriction $T|\ker P$ is semiregular, by [1, Theorem 1.24], it then follows that $K(T)$ is closed.

Suppose that 0 belongs to $\sigma(T)$. We know that $T|P(X)$ is quasi nilpotent, and the Kato spectrum is nonempty; so $\sigma_{\rm k}(T|P(X)) = \{0\}$. By Lemma 1.4 we then have

$$\sigma_{\rm k}(T) = \sigma_{\rm k}(T|\ker P) \cup \sigma_{\rm k}(T|P(X)) = \sigma_{\rm k}(T|\ker P) \cup \{0\}.$$

Since $0 \notin \sigma_{\rm k}(T|\ker P)$ and $\sigma_{\rm k}(T|\ker P)$ is closed, it then follows that 0 is an isolated point of $\sigma_{\rm k}(T)$. □

Corollary 2.3. *If $T \in L(X)$, the following statements are equivalent:*

(i) *T is of Kato type;*

(ii) *there exists a commuting projection P such that $T + P$ is semiregular and TP is nilpotent.*

Proof. (i) ⇒ (ii) Let (M, N) be a GKD for T such that $T|N$ is nilpotent. If P is the projection of X onto N along M, we have $(TP)|M = (TP)|\ker P = 0$. Let $\nu \in \mathbb{N}$ be such that $((T|N)^\nu = (TP)|P(X)))^\nu = 0$. Then

$$(TP)^\nu = ((TP)|M)^n u \oplus (TP|N)^\nu = 0,$$

so TP is nilpotent.
(ii) ⇒ (i) Suppose that P is a commuting projection such that $T + P$ is semiregular and TP is nilpotent. As in the proof of Theorem 2.2, the pair $(\ker P, P(X)$ is a GKD for T. Furthermore, $T|P(X) = (TP)|P(X)$ is nilpotent. □

The proof of the following result may be found in [1, Theorem 2.31].

Theorem 2.4. *Suppose that $H_0(T)$ is closed or that $H_0(T) \cap K(T)$ is closed. Then $H_0(T) \cap K(T) = \{0\}$.*

In the case of semiregular operators we can say the following.

Corollary 2.5. *If $T \in L(X)$ is semiregular and either $H_0(T)$ or $H_0(T) \cap K(T)$ is closed, then $H_0(T) = \{0\}$. In this case T is bounded below.*

Proof. If T is semiregular, then $T(H_0(T)) = H_0(T)$; see [1, Corollary 1.71]. Suppose first that $H_0(T)$ is closed. Then $H_0(T) \subseteq K(T)$, and hence, by Theorem 2.4, $H_0(T) = H_0(T) \cap K(T) = \{0\}$. Consider the other case that $H_0(T) \cap K(T)$ is closed. Since T is semiregular, then $\ker T \subseteq T^n(X)$ for every $n \in \mathbb{N}$ and this is equivalent to say that $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$; see [1, Corollary 1.6]. Moreover, by [1, Theorem 1.24], $K(T) = T^{\infty}(X)$ is closed. The semiregularity of T also implies, by [1, Theorem 1.70], that

$$H_0(T) \subseteq \overline{H_0(T)} = \overline{\mathcal{N}^{\infty}(T)} \subseteq \overline{T^{\infty}(X)} = \overline{K(T)} = K(T),$$

and hence $H_0(T) \cap K(T) = H_0(T)$ is closed. From the first part of the proof, we then obtain $H_0(T) = \{0\}$. Finally, from the inclusion $\ker T \subseteq H_0(T) = \{0\}$, we see that T is injective, and since $T(X)$ is closed, we conclude that T is bounded below. □

The implication

$$H_0(T) closed \Rightarrow H_0(T) = \{0\}$$

for a semiregular operator has been first noted in [16]. The condition $H_0(T)$ closed is satisfied by several classes of operators; for instance, for every multiplier T on a commutative semisimple Banach algebra, we have $H_0(T) = \ker T$, in particular for every convolution operator defined on a group algebra $L_1(G)$, where G is a locally compact Abelian group. Also every $H(p)$ operator (i.e., operators for which $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ for some $p \in \mathbb{N}$) has closed quasi-nilpotent part, in particular every *generalized scalar operator*; see [17] for details. Every spectral operator (in the sense of Dunford [8]) has closed quasi-nilpotent part; see [16, Lemma 2.13].

Theorem 2.6. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *there exists a pair of proper subspaces (M, N) which reduces T such that $T = T|M \oplus T|N$, $T|M$ is bounded below, and $T|N$ is quasi nilpotent;*

(ii) *there exists a commuting projection $P \neq 0$ such that $T+P$ is bounded below and TP is quasi nilpotent;*

(iii) *$H_0(T)$ is complemented with a T-invariant closed subspace M such that $T(M)$ is closed;*

In this case both subspaces $H_0(T)$ and $K(T)$ are closed. Precisely, for every projection P which satisfies (ii), we have $H_0(T) = P(X)$. Moreover,

(iv) *$H_0(T) \cap K(T) = \{0\}$.*

(v) *$0 \in iso\, \sigma_{\mathrm{ap}}(T)$.*

Proof. (i) ⇒ (ii) Clearly, (M, N) is a GKD for T, since every bounded below operator is semiregular. As in the proof of Theorem 2.2, if P is the projection of T onto N along M, then $M = \ker P$, $N = P(X)$, and TP is quasi nilpotent. Moreover, $T+P = (T+P)|M \oplus (T+P)|N$, where $(T+P)|M = T|M$ is bounded below, by Lemma 1.3, while $(T+P)|N = T|N + I_N$, I_N the identity on N is invertible, and hence also bounded below. Therefore, again by Lemma 1.3, $T+P$ is bounded below.
(ii) ⇒ (i) Take $M := \ker P$ and $N := P(X)$. As in the proof of Theorem 2.2, the restriction $T|M = (T+P)|M$ is bounded below, while $T|N = T|P(X)$ is quasi nilpotent.
(iii) ⇔ (ii) see [4].

To conclude the proof observe that $K(T)$ is closed by Theorem 2.2. To show that $H_0(T)$ is closed, observe that, by Lemma 1.5, we have $H_0(T) = H_0(T|M) \oplus N$.

Since $T|M$ is bounded below, we have $H_0(T|M) = \{0\}$, so $H_0(T) = \{0\} \oplus N = N$ is closed.

(iv) $H_0(T) \cap K(T)$ is closed, so, by Theorem 2.4, $H_0(T) \cap K(T) = \{0\}$.

(v) Since $H_0(T)$ is closed, then T has SVEP at 0. By Lemma 1.6, then $0 \in \text{iso}\, \sigma_{\text{ap}}(T)$. □

The next result is dual, in a sense, to that of Theorem 2.6.

Theorem 2.7. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *there exists a pair of closed subspaces (M, N) which reduces T such that $T = T|M \oplus T|N$, $T|M$ is onto, and $T|N$ quasi nilpotent;*

(ii) *there exists a commuting projection $P \neq 0$ such that $T + P$ is onto and TP is quasi nilpotent;*

(iii) *$K(T)$ is complemented by a T-invariant subspace N contained in $H_0(T)$.*

Furthermore, if one of the equivalent conditions (i)–(iii) is satisfied we have

(iv) *$K(T)$ is closed and $X = H_0(T) + K(T)$.*

(v) *$0 \in \text{iso}\, \sigma_{\text{s}}(T)$.*

Proof. The equivalence (i) ⇔ (iii) has been proved in [4]. (i) ⇒ (ii) Since every surjective operator is semiregular, then the pair (M, N) is a GKD for T. If P is the projection of X onto N along M, then $(T + P)|M = T|M$ is onto, while $(T + P)|N = T|N + I_N$, is invertible, and hence onto. By Lemma 1.3 then $T + P$ is onto, and, as in the proof of Theorem 2.2, TP is quasi nilpotent.

(ii) ⇒ (i) Also here take $M. = \ker P$ and $N := P(X)$. Then, by arguing as in the proof of Theorem 2.2, we obtain that $T|M = (T + P)|M$ is onto, while $T|N = T|P(X)$ is quasi nilpotent.

(iv) Suppose that the equivalent conditions (i)–(iii) are satisfied. Then $K(T)$ is closed by Theorem 2.2. Now,

$$H_0(T) = H_0(T|M) \oplus H_0(T|N) = H_0(T|M) \oplus N \supseteq N,$$

while $K(T) = K(T|M) = M$, since $T|M$ is onto. Therefore, $X = M \oplus N \subseteq K(T) + H_0(T)$ and hence $X == K(T) + H_0(T)$.

(v) The condition $X == K(T) + H_0(T)$ entails that T^* has SVEP at 0; see [1, Theorem 2.33]. So, by Lemma 1.6, $0 \in \text{iso}\, \sigma_{\text{s}}(T)$. □

Actually, in [9], it has been proved that if $0 \in \sigma(T)$, then

$$X = H_0(T) + K(T) \Leftrightarrow 0 \in \text{iso}\, \sigma_{\text{s}}(T).$$

Theorem 2.8. *Suppose that either $H_0(T)$ or $H_0(T) \cap K(T)$ is closed. Then each one of the condition (i)–(iii) of Theorem 2.7 is equivalent to the condition $0 \in \text{iso}\, \sigma(T)$, or is equivalent to the condition $0 \in \text{iso}\, \sigma_{\text{s}}(T)$.*

Proof. If $0 \in \text{iso}\, \sigma(T)$, then (ii) of Theorem 2.7 trivially holds, by Theorem 2.1.

Conversely, if (ii) of Theorem 2.7 holds, we have $X = H_0(T)+K(T)$. Moreover, since $0 \in \text{iso}\, \sigma_{\text{s}}(T)$, then $H_0(T) \neq \{0\}$, otherwise if were $H_0(T) = \{0\}$, we would have $X = K(T)$, and hence $0 \notin \sigma_{\text{s}}(T)$. Our assumption that $H_0(T)$ is closed, or $H_0(T) \cap K(T)$ is closed, entails, by Theorem 2.4, that $H_0(T) \cap K(T) = \{0\}$, so

X is the algebraic sum of $H_0(T)$ and $K(T)$. Since $K(T)$ is closed, by Theorem 2.7 and $H_0(T) \neq \{0\}$, then Schmoeger's result (2.2) entails that $0 \in \operatorname{iso}\sigma(T)$. Clearly, $0 \in \operatorname{iso}\sigma(T)$ implies $0 \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$, since every isolated point of the spectrum belongs to $\sigma_{\mathrm{s}}(T)$. The argument above shows that if $0 \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$, then $0 \in \operatorname{iso}\sigma(T)$; so the proof is complete. □

Example 2.9. The assumption that $H_0(T)$ is closed is essential in Theorem 2.8. To see this, let R denote the *right shift* on the Hilbert space $\ell_2(\mathbb{N})$, defined as

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}).$$

The Hilbert space adjoint of R is the *left shift* L defined as

$$L(x_1, x_2, \dots) := (x_2, , x_3, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}).$$

The operator L is onto, precisely we have $\sigma_{\mathrm{ap}}(R) = \sigma_{\mathrm{s}}(L)) = \Gamma$, where Γ denotes the unit circle of $\mathbb{C}$ and $\sigma(L) = D(0,1)$, the unit closed disc of $\mathbb{C}$.

Define $T := L \oplus Q$, where Q is any quasi-nilpotent operator on $\ell_2(\mathbb{N})$. Note that the quasi-nilpotent part $H_0(T)$ is not closed, otherwise T would have SVEP at 0, by the implication (1.4), and hence, by [1, Theorem 2.9], also L has SVEP at 0, and it is well known that this is not true; see[1, p. 71]. Now, $0 \in \sigma_{\mathrm{s}}(T)$, by Theorem 2.7, while

$$\sigma(T) = \sigma(L) \cup \sigma(Q) = \mathbf{D}(0,1),$$

thus $0 \notin \operatorname{iso}\sigma(T)$.

Remark 2.10. The class of operators, which satisfies the condition (iii) of Theorem 2.6, has been introduced in [11]. These operators have been called the *left generalized Drazin invertible operators.* In the same article, every $T \in L(X)$, which satisfies the condition (i) of Theorem 2.7, was called a *right generalized Drazin invertible operator.* We are convinced that [11] contains mistakes.

(i) On pages 1637–8 of [11] the authors asserted that if T is left generalized Drazin invertible, then T is generalized Drazin invertible. This is not true. For instance, if $T = L \oplus Q$ is as in Remark 2.9, then, by using the definition of [11], T is right generalized Drazin invertible, but not generalized Drazin invertible, since 0 is not isolated in $\sigma(T)$. The adjoint T^* is left generalized Drazin invertible, but not generalized Drazin invertible.

(ii) The authors said (without proof) that T is left generalized Drazin invertible if and only if $0 \in \operatorname{iso}\sigma_{\mathrm{ap}}(T)$ [11, Theorem 3.16], and that T is right generalized Drazin invertible [11, Theorem 3.17] if and only if $0 \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$. As observed before, the condition $0 \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$ is only equivalent to say that $0 \in \operatorname{iso}\sigma(T)$ and $X = H_0(T)+K(T)$, so the claimed equivalences without additionally assumptions seem not be justified.

(iii) On page 1642 of [11] the authors said that the left (respectively, right) generalized Drazin spectrum is contained in the left (respectively, right) Drazin spectrum, in other worlds if T is left (respectively, right) Drazin invertible, then T is left (respectively, right) generalized Drazin invertible. Also this implication appears not justified for Banach space operators. If T is left Drazin invertible, then $H_0(T)$ is closed, since $H_0(T) = \{0\}$ if $0 \notin \sigma_{\mathrm{ap}}(T)$, while if $0 \in \sigma_{\mathrm{ap}}(T)$, then 0 is a left pole of T and hence $H_0(T) = \ker T^p$, where p is the ascent of T; see [3, Theorem 2.4]. But we do not see why $H_0(T)$ is complemented by a T-invariant subspace M for which $T(M)$ is closed. The same can be said for right Drazin invertible operators. In this case, $K(T) = X$ if T is surjective, while if $0 \in \sigma_{\rm ap}(T)$, then $K(T) = T^q(X)$, q the descent of T [3, Theorem 2.4]. But we do not see why $K(T)$ is complemented by a T-invariant subspace N for which $N \subseteq H_0(T)$.

However, in presence of SVEP we have the following.

Corollary 2.11. *If $T \in L(X)$ has SVEP, then each one of the conditions (i)–(iii) of Theorem 2.7 is equivalent to the condition $0 \in iso\,\sigma_{\rm s}(T)$, and analogously, if T^* has SVEP, then the each one of the conditions (i)–(iii) of Theorem 2.6 is equivalent to the condition $0 \in iso\,\sigma_{\rm ap}(T)$.*

Proof. If T has SVEP, then $\sigma(T) = \sigma_{\rm s}(T)$, while if T^* has SVEP, then $\sigma(T) = \sigma_{\rm ap}(T)$; see [1, Corollary 2.45]. The statements then immediately follow from Theorem 2.1. □

For Hilbert space operators we also have the following theorem.

Theorem 2.12. *Every left (respectively right), Drazin invertible operator defined on a Hilbert space is left (respectively right), generalized Drazin invertible.*

Proof. From [3], any left Drazin invertible operator defined on a Hilbert space admits the decomposition $T = T_1 \oplus T_2$, where T_1 is bounded below and T_2 is nilpotent. Analogously, any right Drazin invertible operator defined on a Hilbert space admits the decomposition $T = T_1 \oplus T_2$, where T_1 is onto and T_2 is nilpotent. Therefore, Theorems 2.6 and 2.7 apply. □

We consider now the case where TP is nilpotent.

Theorem 2.13. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *there exists a pair of closed subspaces (M, N), which reduces T such that $T = T|M \oplus T|N$, $T|M$ is bounded below (respectively, onto), and $T|N$ is nilpotent;*

(ii) *there exists a commuting projection $P \neq 0$ such that $T + P$ is bounded below (respectively, onto) and TP is nilpotent.*

In this case 0 is a left pole of T (respectively, 0 is a right pole of T).

Proof. (i) ⇔ (ii) If $T = T|M \oplus T|N$, where $T|M$ is bounded below and $T|N$ is nilpotent, then, as in the proof of Theorem 2.6, $T + P$ is bounded below and, evidently, $TP = T|P(X) = T|N$ is nilpotent. Conversely, if T satisfies (ii) $M := \ker P$, $N := P(X)$, then $T|M$ is bounded below. Let ν be the order of nilpotency of TP; that is, $(TP)^\nu = 0$. If $x \in P(X)$, then

$$(T|P(X))^\nu x = T^\nu P^\nu x = (TP)^\nu x = 0,$$

so $T|N$ is nilpotent of order ν.

To show that 0 is a left pole of T, observe first that $p(T|M) = 0$, since $T|M$ is injective, while $p(T|N) = \nu$. From the decomposition $X = M \oplus N$, we have $p(T) = p(T|M) + p(T|N) = p(T|N) = \nu$. Furthermore,

$$\begin{aligned} T^{\nu+1}(X) &= (T|\ker T)^{\nu+1}(\ker P) \oplus (T|P(X))^{\nu+1}(P(X) \\ &= (T|\ker T)^{\nu+1}(\ker P) \oplus \{0\}. \end{aligned}$$

But $(T|\ker T)$ is bounded below entails that also $(T|\ker T)^{\nu+1}$ is bounded below, so $(T|\ker T)^{\nu+1}(\ker P)$ is closed and hence $T^{\nu+1}(X)$ is closed. Since by Theorem 2.6, $0 \in \sigma_{\rm ap}(T)$; it then follows that 0 is a left pole of the resolvent of T.

The case when $T+P$ is onto is similar to $T|\ker P$ is onto, so the descent $q(T|\ker T) = 0$ while $q(T|P(X)) = \nu$, and hence

$$q(T) = q(T|\ker T) + q(T|P(X)) = q(T|P(X)) = \nu,$$

Since, by Theorem 2.7, $0 \in \sigma_{\rm s}(T)$; it then follows that 0 is a right pole of the resolvent of T.

□

Corollary 2.14. *If $T \in L(X)$, the following statements are equivalent:*

(i) *there exists a commuting projection $P \neq 0$ such that $T+P$ is invertible and TP is nilpotent;*

(ii) 0 *is a pole of the resolvent of T.*

Proof. The implication (i) ⇒ (ii) is clear from Theorem 2.13. Suppose that 0 is a pole of order p of the resolvent of T, and let P denote the spectral projection associated with $\{0\}$. Then $PT = TP$ and $H_0(T) = P(X) = \ker T^p$, while $K(T) = \ker P = T^p(X)$; see [1, Theorem 3.74]. From the spectral decomposition theorem, we also know that $\sigma(T|P(X)) = \{0\}$; that is, $T|P(X)$ is quasi nilpotent, and $T+P|\ker P = T|\ker P$ is invertible. Now, $(T+P)|P(X) = T|P(X)+I_{P(X)}$, $I_{P(X)}$ the identity on $P(X)$, so also $(T+P)|P(X)$ is invertible, and hence $T+P = (T+P)|\ker P \oplus (T+P)|P(X)$ is invertible. Finally, $(TP)^p x = T^p P x = 0$ for every $x \in X$; so TP is nilpotent of order p. □

Recall that $\lambda \in \mathbb{C}$ is said to be a *Riesz point* of T, if λ is pole of the resolvent of T and $\lambda I - T$ is Fredholm (i.e. $\alpha(\lambda I - T) := \dim \ker(\lambda I - T)$ and $\beta(\lambda I - T) := \operatorname{codim}(\lambda I - T)(X)$ are both finite).

Corollary 2.15. *If $T \in L(X)$, the following statements are equivalent:*

(i) *there exists a commuting finite rank projection P such that $T+P$ is invertible and TP is nilpotent;*

(ii) 0 *is a Riesz point of T.*

Proof. The equivalence is obvious, if $0 \notin \sigma(T)$ (in this case $P = 0$). Assume that $0 \in \sigma(T)$. (i) ⇒ (ii). By Theorem 2.14, then 0 is a pole of the resolvent and hence has ascent and descent both finite. Furthermore, since $T|\ker P$ is invertible, we have

$$\alpha(T) = \alpha(T|\ker P) + \alpha(T|P(X) = \alpha(T|P(X) < \infty.$$

This implies that $\beta(T) = \alpha(T) < \infty$, by [1, Theorem 3.4].

(ii) ⇒ (i) If 0 is a Riesz point, then 0 is a pole of T, so there exists a commuting projection $P \neq 0$ such that $T+P$ is invertible and TP is nilpotent. Moreover, if p is the order of the pole, then $P(X) = \ker T^p$ is finite-dimensional, since $\ker T$ is finite-dimensional. □

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[1]Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy.
E-mail address: paiena@unipa.it

[2]DEIM, Università di Palermo Viale delle Scienze, I-90128 Palermo, Italy.
E-mail address: salvatore.triolo@unipa.it