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AFFINE ACTIONS AND THE YANG–BAXTER EQUATION

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ABSTRACT. In this paper, the relations between the Yang–Baxter equation and affine actions are explored in detail. In particular, we classify the injective set-theoretic solutions of the Yang–Baxter equation in two ways: (i) by their associated affine actions of their structure groups on their derived structure groups, and (ii) by the C^*-dynamical systems obtained from their associated affine actions. On the way to our main results, several other useful results are also obtained.

The Yang–Baxter equation has been extensively studied in the literature since [14]. It plays important roles not only in statistical mechanics, but also in other areas, such as, quantum groups, link invariants, operator algebras, and the conformal field theory. In general, it is a rather challenging problem to find all solutions of the Yang–Baxter equation. Following a suggestion given in [6], many researchers have done a lot of work on studying a special but important class of solutions, which are now known as set-theoretic solutions. See, for example, [4, 5, 7, 8, 9, 10, 11, 13, 15] to name just a few, and the references therein.

The main aim of this paper is to explore the relations between the Yang–Baxter equation and affine actions on groups. The main ideas behind here are motivated by [7, 11, 13]. The rest of this paper is organized as follows. In section 1, we recall some necessary background on the Yang–Baxter equation which will be needed later. In section 2, we first introduce affine actions and some related notions; then associate to every solution of the Yang–Baxter equation a regular affine action of its structure group on its derived structure group (Proposition 2.5), and

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finally describe two constructions of solutions to the Yang–Baxter equation via their associated affine actions. Our main results of this paper are given in section 3. We classify injective solutions of the Yang–Baxter equation in terms of their associated affine actions (Theorem 3.3). We further obtain a connection with C^*-dynamical systems. It is shown that injective solutions can also be classified via their associated C^*-dynamical systems (Theorem 3.6). We end this paper with an appendix, which provides a commutation relation for semidirect product of solutions to the Yang–Baxter equation determined by cycle sets, which might be useful in the future studies.

1. The Yang–Baxter equation

In this section, we provide some background on the Yang–Baxter equation which will be useful later.

Let X be a (nonempty) set, and let $X^n := \overbrace{X \times \cdots \times X}^{n}$ for $n \geq 2$.

Definition 1.1. Let $R(x,y) = (\alpha_x(y), \beta_y(x))$ be a bijection on X^2. We call R a *set-theoretic solution of the Yang–Baxter equation* (abbreviated as *YBE*) if

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{1}$$

on X^3, where $R_{12} = R \times \mathrm{id}_X$ and $R_{23} = \mathrm{id}_X \times R$. The condition (1) is also known as the *braiding condition*. We often simply call R a *YBE solution on X*. Sometimes, we write it as R_X or a pair (R, X). A YBE solution R on X is said to be

- *involutive* if $R^2 = \mathrm{id}_{X^2}$;
- *nondegenerate* if, for all $x \in X$, α_x and β_x are bijections on X;
- *symmetric* if R is involutive and nondegenerate.

Some examples of YBE solutions are given in [15], where it is also shown that YBE solutions are intimately connected with higher-rank graphs.

Standing assumptions: All YBE solutions in the rest of this paper are always assumed to be set-theoretic and nondegenerate.

1.1. Two characterizations of YBE solutions.

The following lemma is well-known in the literature and also easy to prove.

Lemma 1.2. *Let $R(x,y) = (\alpha_x(y), \beta_y(x))$. Then R is a YBE solution on X if and only if the following properties hold true: for all $x, y, z \in X$,*

(i) $\alpha_x\alpha_y = \alpha_{\alpha_x(y)}\alpha_{\beta_y(x)}$,
(ii) $\beta_y\beta_x = \beta_{\beta_y(x)}\beta_{\alpha_x(y)}$, *and*
(iii) $\beta_{\alpha_{\beta_y(x)}(z)}(\alpha_x(y)) = \alpha_{\beta_{\alpha_y(z)}(x)}(\beta_z(y))$ (Compatibility Condition).

Furthermore, R is involutive if and only if

$$\alpha_{\alpha_x(y)}(\beta_y(x)) = x \quad \textit{and} \quad \beta_{\beta_y(x)}(\alpha_x(y)) = y \qquad \textit{for all } x, y \in X.$$

Let us associate to a given YBE solution an important object – its structure group.

Definition 1.3. Let $R(x,y)=(\alpha_x(y),\beta_y(x))$ be a YBE solution on X. The structure group of R, denoted as G_{R_X}, is the group generated by X with commutation relations determined by R:

$$G_{R_X}={}_{\rm gp}\langle X; xy=\alpha_x(y)\beta_y(x) \text{ for all } x,y\in X\rangle.$$

Sometimes we also write G_{R_X} as $G_{R,X}$ or G_X.

One can easily rephrase the characterization given in Lemma 1.2 in terms of actions of structure groups (see, e.g., [7, 9]).

Corollary 1.4. *A map $R(x,y)=(\alpha_x(y),\beta_y(x))$ is a YBE solution on X, if and only if*

(i) *α can be extended to a left action of G_{R_X} on X,*
(ii) *β can be extended to a right action of G_{R_X} on X, and*
(iii) *the compatibility condition in Lemma 1.2 (iii) holds.*

1.2. **Constructing YBE solutions from old to new.** There are several known constructions of YBE solutions from old to new. For our purpose, we only introduce two below.

▶ **Dual of R.** Let $R(x,y)=(\alpha_x(y),\beta_y(x))$ be a YBE solution on X. Define R° on X^2 by

$$R^\circ(x,y)=(\beta_x(y),\alpha_y(x)) \qquad \text{for all } x,y\in X.$$

We call R° the *dual of* R. It is also a YBE solution on X. Indeed, this can be seen by switching x and y in the first two identities, and x and z in the third one in Lemma 1.2. We give it such a name because we "dualize" the process $xy=\alpha_x(y)\beta_y(x)$ in G_{R_X} via $y\circ x=\beta_y(x)\circ\alpha_x(y)$ (by switching the factors on both sides).

Clearly, $R^{\circ\circ}=R$.

Let $\Phi: G_{R_X}\to G_{R^\circ_X}$ be defined via $\Phi(x):=x$ for $x\in X$ and $\Phi(xy):=y\circ x$ for all $x,y\in X$. Since $\Phi(xy)=\Phi(\alpha_x(y)\beta_y(x))$ for all $x,y\in X$, Φ can be extended to an anti-isomorphism from G_{R_X} to $G_{R^\circ_X}$.

▶ **Derived solution of R [7, 13].** Let $R(x,y)=(\alpha_x(y),\beta_y(x))$ be a YBE solution on X. Then

$$(x,y)\overset{R}{\mapsto}(\alpha_x(y),\beta_y(x))\overset{R}{\mapsto}\big(\alpha_{\alpha_x(y)}(\beta_y(x)),\beta_{\beta_y(x)}(\alpha_x(y))\big)$$

determines a YBE solution

$$(x,\alpha_x(y))\mapsto\big(\alpha_x(y),\alpha_{\alpha_x(y)}(\beta_y(x))\big),$$

namely,

$$R':(x,y)\mapsto\big(y,\alpha_y(\beta_{\alpha_x^{-1}(y)}(x))\big).$$

This solution R' is called the *derived solution of* R.

The *derived structure group* A_{R_X} *of* R is defined as

$$A_{R_X}=\big\langle X: x\bullet y=y\bullet\alpha_y(\beta_{\alpha_x^{-1}(y)}(x)) \text{ for all } x,y\in X\big\rangle.$$

As G_{R_X}, A_{R_X} is sometimes also written as $A_{R,X}$ or A_X.

Remark 1.5. It is often useful to think that A_{R_X} and G_{R_X} have the same generator set X with the relations

$$x \bullet y = x\alpha_{x^{-1}}(y) \qquad \text{for all } x, y \in X,$$

equivalently, $xy = x \bullet \alpha_x(y)$ for all $x, y \in X$.

Remark 1.6. (i) If $R(x, y) = (\alpha_x(y), x)$, then $R'(x, y) = (y, \alpha_y(x))$. Namely, $R' = R^\circ$.

(ii) As in [13], one can also define another derived YBE solution

$$'R(x, y) = \big(\beta_x(\alpha_{\beta_y^{-1}(x)}(y)), x\big) \qquad \text{for all } x, y \in X.$$

(iii) One can easily check the following: R is symmetric $\Leftrightarrow$ $'R = R'$ $\Leftrightarrow$ R° is symmetric $\Leftrightarrow$ A_{R_X} and $A_{R'_X}$ are abelian $\Leftrightarrow$ A_{R_X} and $A_{R^\circ_X}$ are abelian.

1.3. **A distinguished action of G_{R_X} on A_{R_X}.** Let $R(x, y) = (\alpha_x(y), \beta_y(x))$ be a YBE solution on X. By Corollary 1.4, both α and β^{-1} can be extended to actions of G_X on X. For our convenience, let

$$\phi_R(x, y) := \alpha_y(\beta_{\alpha_x^{-1}(y)}(x)),$$
$$\psi_R(x, y) := \beta_x(\alpha_{\beta_y^{-1}(x)}(y))$$

for all $x, y \in X$. Similar to [13, Theorem 2.3], one has the following.

Lemma 1.7. *ϕ is G_{R_X}-equivariant with respect to the action α:*

$$\phi_R(\alpha_g(x), \alpha_g(y)) = \alpha_g(\phi_R(x, y)) \qquad \textit{for all } x, y \in X \textit{ and } g \in G_{R_X}.$$

Proof. Notice that $\psi_{R^\circ}(x, y) = \phi_R(y, x)$ for all $x, y \in X$. Now first apply [13, Theorem 2.3] to R°, and then use the relation between R and R° to obtain the following:

$$\begin{aligned}
&\alpha_{g^{-1}}\psi_{R^\circ}(x, y) = \psi_{R^\circ}(\alpha_{g^{-1}}(x), \alpha_{g^{-1}}(y)) && \text{for all } x, y \in X, g \in G_{R^\circ_X}\\
\Rightarrow\ &\alpha_{g^{-1}}\phi_R(y, x) = \phi_R(\alpha_{g^{-1}}(y), \alpha_{g^{-1}}(x)) && \text{for all } x, y \in X, g \in G_{R^\circ_X}\\
\Rightarrow\ &\alpha_g\phi_R(y, x) = \phi_R(\alpha_g(y), \alpha_g(x)) && \text{for all } x, y \in X, g \in G_{R_X}.
\end{aligned}$$

We are done. $\square$

Let $\mathrm{Aut}_X(A_{R_X})$ be the group of all automorphisms of A_{R_X} preserving X.

Proposition 1.8 ([13]). *Keep the above notation. The action α of G_{R_X} on X induces an action of G_{R_X} on A_{R_X} preserving X. That is, there is a group homomorphism from G_{R_X} to* $\mathrm{Aut}_X(A_{R_X})$.

Proof. Notice that for all $g \in G_{R_X}$

$$\begin{aligned}
&x \bullet y = y \bullet \phi_R(x, y)\\
\Rightarrow\ &\alpha_g(x) \bullet \alpha_g(y) = \alpha_g(y) \bullet \phi_R(\alpha_g(x), \alpha_g(y)) && \text{(replacing } x, y \text{ by } \alpha_g(x), \alpha_g(y))\\
\Rightarrow\ &\alpha_g(x) \bullet \alpha_g(y) = \alpha_g(y) \bullet \alpha_g(\phi_R(x, y)) && \text{(by Lemma 1.7).}
\end{aligned}$$

This implies that α_g can be extended to an element in $\mathrm{Aut}_X(A_{R_X})$, as desired. $\square$

By Proposition 1.8, one has a generator preserving action $\alpha : G_{R_X} \curvearrowright A_{R_X}$.

2. Affine actions on groups

For a given YBE solution, we associate to it a regular affine action (Proposition 2.5). This plays a vital role in section 3. Conversely, in subsection 2.3, we use the two constructions of affine actions described in subsection 2.2 to construct new YBE solutions.

Let A be a group. Denote by $\mathrm{Aff}(A)$ the semidirect product

$$\mathrm{Aff}(A) = \mathrm{Aut}(A) \ltimes A,$$

where $(S,a)(T,b) = (ST, aS(b))$ for all $S,T \in \mathrm{Aut}(A)$ and $a,b \in A$. $\mathrm{Aff}(A)$ acts on A via $(S,a)b = aS(b)$.

Definition 2.1. Let G and A be groups. An affine action of G on A is a group homomorphism $\rho : G \to \mathrm{Aff}(A)$.

By definition, any affine action $\rho : G \to \mathrm{Aff}(A)$ has the following form:

$$\rho_g(a) = b(g)\pi_g(a) \qquad \text{for all } g \in G \text{ and } a \in A,$$

where $\pi : G \to \mathrm{Aut}(A)$ is a group homomorphism, called the *linear part* of ρ, and $b : G \to A$, called the *translational part* of ρ, is a 1-cocycle with respect to π in coefficient A:

$$b(g_1g_2) = b(g_1)\pi_{g_1}(b(g_2)) \qquad \text{for all } g_1, g_2 \in G.$$

We sometimes simply write $\rho = (\pi, b)$, and also write $b(g)$ as b_g for convenience.

Recall that a group action is called *regular* if it is transitive and free.

The following lemma should be known. But we include a proof below for completeness.

Lemma 2.2. *An affine action $\rho = (\pi, b)$ of a group G on a group A is regular if and only if b is bijective.*

Proof. ($\Rightarrow$): Since ρ is regular, for arbitrary x and y in A there is a unique $g \in G$ such that $\rho_g(x) = y$. Letting $x = e$ and $y \in A$ arbitrary shows that b is surjective.

Now suppose that $b(g_1) = b(g_2)$ for some $g_1, g_2 \in G$. Then $\rho_{g_1}(e) = b(g_1)\pi_{g_1}(e) = b(g_2)\pi_{g_2}(e) = \rho_{g_2}(e)$. So $g_1 = g_2$ as ρ is free. Thus b is injective.

($\Leftarrow$): Let $x, y \in A$. Since b is bijective, there is a unique $h_0 \in G$ such that $b(h_0) = x$, and further a unique $g \in G$ such that $b(gh_0) = y$. Then $\rho_g(x) = b(g)\pi_g(x) = b(g)\pi_g(b(h_0)) = b(gh_0) = y$. Thus ρ is transitive.

To show that ρ is free, suppose that there are $g_1, g_2 \in G$ such that $\rho_{g_1}(x) = \rho_{g_2}(x)$ for some $x \in A$. Then $b(g_1)\pi_{g_1}(x) = b(g_2)\pi_{g_2}(x)$. Since b is surjective, there is $g \in G$ such that $b(g) = x$. Hence $b(g_1)\pi_{g_1}(b(g)) = b(g_2)\pi_{g_2}(b(g))$; that is, $b(g_1g) = b(g_2g)$. But b is injective, $g_1g = g_2g$, and so $g_1 = g_2$. Therefore, ρ is free. ☐

Definition 2.3. Let ρ^i be an affine action of a group G on a group A_i $(i = 1, 2)$. A group homomorphism $\varphi : A_1 \to A_2$ is said to be G-equivariant relative to (ρ^1, ρ^2) if

$$\varphi \circ \rho^1_g = \rho^2_g \circ \varphi \qquad \text{for all } g \in G. \tag{2}$$

That is, for every $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} A_1 & \xrightarrow{\rho^1_g} & A_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A_2 & \xrightarrow[\rho^2_g]{} & A_2. \end{array}$$

If, furthermore, the above φ is bijective, then ρ^1 and ρ^2 are said to be conjugate.

Remark 2.4. (i) Let $\rho^i = (\pi^i, b^i)$ $(i = 1, 2)$. It is easy to see that (2) is equivalent to

$$\varphi \circ \pi^1_g = \pi^2_g \circ \varphi,$$
$$b^2_g = \varphi \circ b^1_g$$

for all $g \in G$. So, in particular, φ is also G-equivariant relative to (π^1, π^2).

(ii) If b^1 is surjective, then using the definition of 1-cocycles, it is easy to see that the second identity in (i) above determines the first one. In fact, from the second one has, for all $g, h \in G$,

$$\begin{aligned} b^2_{gh} = \varphi(b^1_{gh}) &\Rightarrow b^2_g \pi^2_g(b^2_h) = \varphi(b^1_g)\varphi(\pi^1_g(b^1_h)) \\ &\Rightarrow \pi^2_g(b^2_h) = \varphi(\pi^1_g(b^1_h)) && (\text{as } b^2_g = \varphi(b^1_g)) \\ &\Rightarrow \pi^2_g(\varphi(b^1_h)) = \varphi(\pi^1_g(b^1_h)) && (\text{as } b^2_h = \varphi(b^1_h)) \\ &\Rightarrow \pi^2_g \circ \varphi = \varphi \circ \pi^1_g && (\text{as } b^1(G) = A_1). \end{aligned}$$

2.1. **Affine actions associated to YBE solutions.** This subsection shows why we are interested in affine actions. We should mention that these actions are also considered in [13] in a different terminology.

Proposition 2.5 (and **Definition**). *Any YBE solution R on X induces a regular affine action ρ^X of G_X on A_X.*

The action ρ^X is called the affine action associated to R_X *and also denoted as ρ_{R_X} or even just ρ if the context is clear.*

Proof. The proof is completely similar to [13, Theorem 2.5]. We only sketch it here. By Proposition 1.8, there is an action $\alpha : G_X \curvearrowright A_X$.

Step 1: Extend the mapping

$$\rho : X \to G_X \ltimes_\alpha A_X, \ x \mapsto (x, x)$$

to a group homomorphism

$$\rho_G : G_X \to G_X \ltimes_\alpha A_X.$$

To do so, one needs to check that

$$\rho(x)\rho(y) = \rho(\alpha_x(y))\rho(\beta_y(x)) \qquad \text{for all } x, y \in X.$$

But

$$\rho(x)\rho(y) = (x, x)(y, y) = (xy, x \bullet \alpha_x(y)),$$

and similarly

$$\rho(\alpha_x(y))\rho(\beta_y(x)) = \big(\alpha_x(y)\beta_y(x), \alpha_x(y) \bullet \alpha_{\alpha_x(y)}(\beta_y(x))\big).$$

They are obviously equal.

<u>Step 2</u>: Let $p : G_X \ltimes_\alpha A_X \to A_X$ be the second projection to A_X, and let $b := p \circ \rho_G$. Then b is a 1-cocycle with respect to the action $G_X \overset{\alpha}{\curvearrowright} A_X$:

$$b(gh) = b(g) \bullet \alpha_g(b(h)) \qquad \text{for all } g, h \in G_X.$$

In fact, for all $g, h \in X$,

$$b(gh) = p\rho_G(gh) = p((g, g)(h, h)) = p(gh, g \bullet \alpha_g(h)) = g \bullet \alpha_g(h),$$

and

$$b(g) \bullet \alpha_g(b(h)) = p\rho_G(g) \bullet \alpha_g(p\rho_G(h)) = g \bullet \alpha_g(h).$$

<u>Step 3</u>: Check that b is bijective (see [13, Theorem 2.5]). $\square$

Remark 2.6. In what follows, we will frequently use that the simple fact that $b(x) = x$, for all $x \in X$, in the associated affine action $\rho^X = (\alpha, b)$ obtained from Proposition 2.5.

2.2. **Two constructions of affine actions.** In this subsection, we construct two new affine actions from given ones.

1° **Lifting.** This generalizes a construction given in [1, 2], which plays key roles in these works.

Let A and H be two groups, and let $\theta : H \to A$ be a homomorphism. Suppose that $\rho = (\pi, b)$ is a regular affine action of G on A and that σ is an action of G on H, such that θ is G-equivariant relative to (σ, π)

$$\theta \circ \sigma_g = \pi_g \circ \theta \qquad \text{for all } g \in G.$$

Introduce a new multiplication $\cdot$ on H via

$$x \cdot y := x\sigma_{b^{-1}\circ\theta(x)}(y) \qquad \text{for all } x, y \in H. \tag{3}$$

Then the *lifting of ρ from A to H* is defined as

$$\tilde{\rho}_x(y) = x \cdot y \qquad \text{for all } x, y \in H.$$

Conclusion 1. The lifting $\tilde{\rho}$ is an affine action of $(H, \cdot)$ on H. Furthermore, θ is $(H, \cdot)$-equivariant relative to $(\tilde{\rho}, \rho \circ b^{-1} \circ \theta)$.

Pictorially, one can summarize the above as follows: for all $g \in G$ and $h \in (H, \cdot)$,

$$\begin{tikzcd}
H \arrow[r, "\sigma_g"] \arrow[d, "\theta"'] & H \arrow[d, "\theta"] & \rightsquigarrow & H \arrow[r, dashed, "\tilde{\rho}_h"] \arrow[d, "\theta"'] & H \arrow[d, "\theta"] \\
A \arrow[r, "\pi_g"'] & A & \rightsquigarrow & A \arrow[r, "\rho_{b^{-1}\circ\theta(h)}"'] & A
\end{tikzcd}$$

Proof. One can show that $(H,\cdot)$ is indeed a group: $\cdot$ is closed and associative, the identity is (still) e, and the inverse of x in $(H,\cdot)$ is $\sigma_{b^{-1}\circ\theta(x)}(x)$. The verification is tedious and left to the reader.

Also, $\tilde{\rho}$ is an affine action of $(H,\cdot)$ on H. In fact,

$$\begin{aligned}
\tilde{\rho}_{x_1\cdot x_2}(y) &= (x_1\cdot x_2)\sigma_{b^{-1}\circ\theta(x_1\cdot x_2)}(y) \\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2)\sigma_{b^{-1}\circ\theta(x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2))}(y) && \text{(by (3))}\\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2)\sigma_{b^{-1}(\theta(x_1)\theta(\sigma_{b^{-1}\circ\theta(x_1)}(x_2)))}(y) && \text{(as } \theta \text{ is a homomorphism)}\\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2)\sigma_{b^{-1}(\theta(x_1)\pi_{b^{-1}\circ\theta(x_1)}\theta(x_2)))}(y) && \text{(as } \theta \text{ is } G\text{-equivariant)}\\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2)\sigma_{b^{-1}(\theta(x_1))b^{-1}\circ\theta(x_2)}(y) && \text{(as } b \text{ is a 1-cocycle w.r.t } \pi)\\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(x_2\sigma_{b^{-1}\circ\theta(x_2)}(y)) && \text{(as } \theta \text{ is an action)}\\
&= x_1\sigma_{b^{-1}\circ\theta(x_1)}(\rho_{x_2}(y)) && \text{(by (3))}\\
&= \tilde{\rho}_{x_1}(\tilde{\rho}_{x_2}(y)) && \text{(by (3)).}
\end{aligned}$$

Furthermore, $\rho\circ b^{-1}\circ\theta$ is an affine action of $(H,\cdot)$ on A. For this, since θ is G-equivariant for (σ,π) and b is 1-cocycle with respective to π, one has

$$\begin{aligned}
b^{-1}\circ\theta(h_1\cdot h_2) &= b^{-1}(\theta(h_1)\theta\circ\sigma_{b^{-1}(\theta(h_1))}(h_2))\\
&= b^{-1}(\theta(h_1)\pi_{b^{-1}(\theta(h_1))}(\theta(h_2)))\\
&= b^{-1}\circ\theta(h_1)b^{-1}\circ\theta(h_2).
\end{aligned}$$

Hence, for all $h_1, h_2\in H$ and $a\in A$, we get

$$\rho_{b^{-1}\circ\theta(h_1\cdot h_2)}(a) = b(b^{-1}\circ\theta(h_1)b^{-1}\circ\theta(h_2))\pi_{b^{-1}\circ\theta(h_1)b^{-1}\circ\theta(h_2)}(a)$$

and

$$\begin{aligned}
&\rho_{b^{-1}\circ\theta(h_1)}\rho_{b^{-1}\circ\theta(h_2)}(a)\\
&= \rho_{b^{-1}\circ\theta(h_1)}(b(b^{-1}\circ\theta(h_2)))\pi_{b^{-1}\circ\theta(h_2)}(a)\\
&= b(b^{-1}\circ\theta(h_1))\pi_{b^{-1}\circ\theta(h_1)}(b(b^{-1}\circ\theta(h_2)))\pi_{b^{-1}\circ\theta(h_2)}(a))\\
&= b(b^{-1}\circ\theta(h_1))\pi_{b^{-1}\circ\theta(h_1)}(b(b^{-1}\circ\theta(h_2)))\pi_{b^{-1}\circ\theta(h_1)b^{-1}\circ\theta(h_2)}(a).
\end{aligned}$$

This implies

$$\rho_{b^{-1}\circ\theta(h_1\cdot h_2)} = \rho_{b^{-1}\circ\theta(h_1)}\rho_{b^{-1}\circ\theta(h_2)}$$

as b is a 1-cocycle with respect to π.

Using the property that θ is G-equivariant relative to (σ,π) again, we have, for all $x, z\in H$,

$$\begin{aligned}
\theta(\tilde{\rho}_z(x)) = \theta(z\sigma_{b^{-1}\circ\theta(z)})(x) &= \theta(z)\theta(\sigma_{b^{-1}\circ\theta(z)}(x))\\
&= bb^{-1}(\theta(z))\pi_{b^{-1}\circ\theta(z)}(\theta(x))\\
&= \rho_{b^{-1}\circ\theta(z)}(\theta(x)).
\end{aligned}$$

Thus $\theta\circ\tilde{\rho}_z = \rho_{b^{-1}\circ\theta(z)}\circ\theta$, as desired. ☐

2° **Semidirect product.** Let ρ be an affine action of G on A, and let $\tilde{\rho} = (\tilde{\pi}, \tilde{b})$ be a regular affine action of $\tilde{G}$ on $\tilde{A}$. Suppose that $\theta : G \curvearrowright \tilde{G}$ is an action of G on $\tilde{G}$ such that

$$\theta_g(\tilde{b}^{-1}\tilde{\pi}_h\tilde{b}) = (\tilde{b}^{-1}\tilde{\pi}_{\theta_g(h)}\tilde{b})\theta_g \qquad \text{for all } g \in G, h \in \tilde{G}. \tag{4}$$

Then the *semidirect product of ρ and $\tilde{\rho}$ via θ* is defined as

$$\begin{aligned} \rho \ltimes_\theta \tilde{\rho} :& G \ltimes_\theta \tilde{G} \to \mathrm{Aff}(A \times \tilde{A}) \\ &(g, h) \mapsto (\rho_g, \tilde{\rho}_h \circ \tilde{b} \circ \theta_g \circ \tilde{b}^{-1}). \end{aligned}$$

Conclusion 2. The semidirect product $\rho \ltimes_\theta \tilde{\rho}$ is an affine action of $G \ltimes_\theta \tilde{G}$ on $A \times \tilde{A}$.

Proof. First notice that (4) guarantees that the mapping

$$(g, h) \mapsto (\pi_g, \pi_h\tilde{b} \circ \theta_g \circ \tilde{b}^{-1})$$

is a group homomorphism from $G \ltimes_\theta \tilde{G}$ to $\mathrm{Aut}(A \times \tilde{A})$. The tedious verification is left to the reader.

We now show the following identity:

$$\theta_g(\tilde{b}^{-1}\tilde{\rho}_h\tilde{b}) = (\tilde{b}^{-1}\tilde{\rho}_{\theta_g(h)}\tilde{b})\theta_g \qquad \text{for all } g \in G \text{ and } h \in \tilde{G}. \tag{5}$$

In fact, one has

$$\begin{aligned} &\theta_g(h_1h_2) = \theta_g(h_1)\theta_g(h_2) && \text{for all } g \in G, h_1, h_2 \in \tilde{G} \\ \Rightarrow\ &\tilde{b}(\theta_g(h_1h_2)) = \tilde{b}(\theta_g(h_1)\theta_g(h_2)) \\ \Rightarrow\ &\tilde{b}(\theta_g(\tilde{b}^{-1}(\tilde{b}(h_1)\tilde{\pi}_{h_1}(\tilde{b}(h_2))))) = \tilde{b}(\theta_g(h_1))\tilde{\pi}_{\theta_g(h_1)}(\tilde{b}\theta_g(h_2)) && (\text{as } \tilde{b} \text{ is a 1-cocycle}) \\ \Rightarrow\ &\tilde{b}\theta_g\tilde{b}^{-1}\tilde{\rho}_{h_1}(\tilde{b}(h_2)) = \tilde{\rho}_{\theta_g(h_1)}(\tilde{b}(\theta_g(h_2))) && (\text{by the definition of } \tilde{\rho}) \\ \Rightarrow\ &\theta_g(\tilde{b}^{-1}\tilde{\rho}_{h_1}\tilde{b}) = (\tilde{b}^{-1}\tilde{\rho}_{\theta_g(h_1)}\tilde{b})\theta_g. \end{aligned}$$

Set $\Gamma := \rho \ltimes_\theta \tilde{\rho}$. In order to show that Γ is an affine action, it suffices to check that

$$\Gamma_{(g,h)(g',h')} = \Gamma_{(g,h)}\Gamma_{(g',h')} \qquad \text{for all } g, g' \in G, h, h' \in \tilde{G}.$$

For this, let $y \in G$, and let $t \in \tilde{G}$. We have

$$\begin{aligned} \Gamma_{(g,h)(g',h')}(y, t) &= \Gamma_{(gg', h\theta_g(h'))}(y, t) \\ &= \big(\rho_{gg'}(y), \tilde{\rho}_{h\theta_g(h')}\,\tilde{b}\,\theta_{gg'}\tilde{b}^{-1}(t)\big) \\ &= \big(\rho_{gg'}(y), \tilde{\rho}_{h\theta_g(h')}\,\tilde{b}\,\theta_g\theta_{g'}\tilde{b}^{-1}(t)\big) \\ &= \big(\rho_{gg'}(y), \tilde{\rho}_h\tilde{b}\tilde{b}^{-1}\tilde{\rho}_{\theta_g(h')}\,\tilde{b}\,\theta_g\theta_{g'}\tilde{b}^{-1}(t)\big) \\ &= \big(\rho_{gg'}(y), \tilde{\rho}_h\tilde{b}\theta_g\tilde{b}^{-1}\tilde{\rho}_{h'}\,\tilde{b}\,\theta_{g'}\tilde{b}^{-1}(t)\big), && (\text{by (5)}) \end{aligned}$$

and

$$\begin{aligned} \Gamma_{(g,h)}\Gamma_{(g',h')}(y, t) &= \Gamma_{(g,h)}\big(\rho_{g'}(y), \tilde{\rho}_{h'}\tilde{b}\theta_{g'}\tilde{b}^{-1}(t)\big) \\ &= \big(\rho_g\rho_{g'}(y), \tilde{\rho}_h\tilde{b}\theta_g\tilde{b}^{-1}(\tilde{\rho}_{h'}\tilde{b}\theta_{g'}\tilde{b}^{-1}(t))\big). \end{aligned}$$

We are done. ∎

When θ is the trivial action, then the condition (4) is redundant and the corresponding affine action is just the direct product of ρ and $\tilde{\rho}$.

An application of the above semidirect product construction is given in the appendix.

2.3. Constructing YBE solutions.

Let us first recall the following result.

Theorem 2.7. [11] *Let G be a group. Then following two groups of data are equivalent:*

(1) *There is a pair of left-right actions (α, β) of the group G on G, which is compatible (i.e., $gh = \alpha_g(h)\beta_h(g)$ for all g, h in G).*
(2) *There is a regular affine action $\rho = (\pi, b)$ of G on some group A.*

Proof. This is proved in [11]. Since the idea of the proof will be useful later, we sketch it below.

(i)⇒(ii): Let $A := G$ as sets but the multiplication $\odot$ on A is given by

$$g \odot h = g\alpha_{g^{-1}}(h) \qquad \text{for all } g, h \in G,$$

namely,

$$gh = g \odot \alpha_g(h) \qquad \text{for all } g, h \in G.$$

This implies that the identity mapping "id" is a (bijective) 1-cocycle with respect to α.

(ii)⇒(i): Set

$$\alpha_g(h) := b^{-1} \circ \pi_g \circ b(h) \quad \text{and} \quad \beta_h(g) := \alpha_g(h)^{-1}gh$$

for all $g, h \in G$. □

Remark 2.8. (i) Let G and A be groups. Given a regular affine action ρ of G on A, by Theorem 2.7 and [11, Corollary 3], one obtains a YBE solution on G given by $R(g, h) = (\alpha_g(h), \beta_h(g))$ for all $g, h \in G$.

(ii) Let R be a YBE solution on X, and let ρ^X be its associated regular affine action of G_X on A_X (see Proposition 2.5). From (i) above, there is a YBE solution $\bar{R}$ on G_X. From its construction, one can see that this is nothing but the universal extension of R mentioned in [11, Theorem 9].

Remark 2.9. This remark shows that there is a natural generalization of the relation $\beta_h(g) = \alpha^{-1}_{\alpha_g(h)}(g)$ holding for symmetric YBE solutions (see Lemma 1.2).

Let us return to the proof of (i)⇒(ii) in Theorem 2.7. The property of $b :=$ id being a 1-cocycle with respect to α gives

$$g \odot h = g\alpha_{g^{-1}}(h) = g\alpha_g^{-1}(h) \qquad \text{for all } g, h \in G,$$

which implies

$$gh = g \odot \alpha_g(h) \qquad \text{for all } g, h \in G.$$

In particular,

$$\bar{g} = \alpha_g(g^{-1}) \qquad \text{for all } g \in G.$$

To distinguish, we write $\bar{g}$ as the inverse of g in A, while g^{-1} as the inverse of g in G as usual.

If (α, β) is a compatible pair, then we claim

$$\beta_h(g) = \alpha_{\alpha_g(h)^{-1}} \circ \mathrm{Ad}_{\overline{\alpha_g(h)}}(g),$$

where $\mathrm{Ad}_{\overline{\alpha_g(h)}}$ acts on A.

Indeed, since (α, β) is a compatible pair, one has

$$\begin{aligned}
& g \odot h = g\alpha_{g^{-1}}(h) = h\beta_{\alpha_{g^{-1}}(h)}(g) \\
\Rightarrow\ & g \odot \alpha_g(h) = \alpha_g(h)\beta_h(g) \\
\Rightarrow\ & \beta_h(g) = \alpha_g(h)^{-1}(g \odot \alpha_g(h)) \\
&\qquad = \alpha_g(h)^{-1} \odot \alpha_{\alpha_g(h)^{-1}}\big(g \odot \alpha_g(h)\big) \\
&\qquad = \alpha_{\alpha_g(h)^{-1}}\left[\alpha_{\alpha_g(h)}(\alpha_g(h)^{-1}) \odot g \odot \alpha_g(h)\right] \ \ (\text{as } \alpha_g \in \mathrm{Aut}(A)) \\
&\qquad = \alpha_{\alpha_g(h)^{-1}}\left[\overline{\alpha_g(h)} \odot g \odot \alpha_g(h)\right] \\
&\qquad = \alpha_{\alpha_g(h)^{-1}} \circ \mathrm{Ad}_{\overline{\alpha_g(h)}}(g).
\end{aligned}$$

This proves our claim.

In particular, if the YBE solution R on G determined by (α, β) is symmetric, then A is abelian [11]. So in this case Ad_g $(g \in A)$ is nothing but the identity mapping on A.

Making use of Theorem 2.7, Remark 2.8, and the constructions of affine actions in subsection 2.2, we get two constructions of YBE solutions on groups.

Lifting revisited. Let R_X be a YBE solution. In the lifting construction on affine actions, let $G = G_X$, $A = A_X$, and ρ be the affine action associated to R_X. Then (3) becomes

$$x \cdot y = x\sigma_{\theta(x)}(y) \qquad \text{for all } x, y \in H.$$

In this case, $\tilde{\rho}_x(y) = x \cdot y$ is a regular affine action, and so it yields a YBE solution on $(H, \cdot)$.

Semidirect product revisited. Let R_X and R_Y be two YBE solutions. Let $G = G_X$, $A = A_X$, $\tilde{G} = G_Y$, and $\tilde{A} = A_Y$ in the semidirect product construction on affine actions. Suppose that θ is an action of G_X on G_Y satisfying (4). In this case,

$$\rho^X \ltimes_\theta \rho^Y(g, h) = (\rho^X_g, \rho^Y_h \circ \theta_g),$$

and $\rho^X \ltimes_\theta \rho^Y$ is also regular. It follows that $\rho^X \ltimes_\theta \rho^Y$ determines a YBE solution, say $\bar{R}$, on $G_X \ltimes_\theta G_Y$. Notice that if θ is the trivial action, then $\bar{R}$ is nothing but the trivial extension of R_X and R_Y in the sense of [7] (also, see [15, 2.2 2°]).

3. Classifying solutions of the Yang–Baxter equation via their associated affine actions

In this section, we state and prove our main results in this paper. We classify all injective YBE solutions in terms of their associated regular affine actions (Theorem 3.3). Furthermore, a connection with C^*-dynamical systems is obtained: All injective YBE solutions can also be classified via their associated C^*-dynamical systems (Theorem 3.6).

Let R_X be a YBE solution. Denote by ι_G and ι_A the natural mappings from X into G_X and A_X, respectively.

Definition 3.1. If ι_G is injective, then R_X is said to be injective.

It is known from [13] that ι_G is injective if and only if so is ι_A. Also, every symmetric YBE solution is injective.

Let R_X and R_Y be two YBE solutions. Recall that a mapping $h : X \to Y$ is a *YB-homomorphism between R_X and R_Y*, if $R_Y(h \times h) = (h \times h)R_X$. This amounts to saying that

$$\alpha^Y_{h(x)}(h(y)) = h(\alpha^X_x(y)) \quad \text{and} \quad \beta^Y_{h(x)}(h(y)) = h(\beta^X_x(y)) \tag{6}$$

for all $x, y \in X$. In this case, we also say that R_X is *homomorphic to R_Y via h*. Of course, if h is bijective, then R_X and R_Y are called *isomorphic*.

If R_X and R_Y are symmetric, then only one of the two identities in (6) suffices.

Proposition 3.2. *Let R_X and R_Y be two arbitrary YBE solutions. If R_X is homomorphic to R_Y via h, then h induces group homomorphisms $h_G : G_X \to G_Y$ and $h_A : A_X \to A_Y$ such that h_A is G_X-equivariant relative to $(\rho^X, \rho^Y \circ h_G)$.*

If h is furthermore bijective, then ρ^X and $\rho^Y \circ h_G$ are conjugate.

Proof. For convenience of notation, let $R_X(x_1, x_2) = (\alpha^X_{x_1}(x_2), \beta^X_{x_2}(x_1))$ for all $x_1, x_2 \in X$, and let $R_Y(y_1, y_2) = (\alpha^Y_{y_1}(y_2), \beta^Y_{y_2}(y_1))$ for all $y_1, y_2 \in Y$.

Notice that since $h : X \to Y$ is a YB-homomorphism between R_X and R_Y, it is easy to check that h can be extended to a group homomorphism, say h_G, from G_X to G_Y. Indeed, it follows from (6) and the definition of G_Y that

$$h(\alpha^X_x(y))h(\beta^X_y(x)) = \alpha^Y_{h(x)}(h(y))\beta^Y_{h(y)}(h(x)) = h(x)h(y)$$

for all $x, y \in X$. Obviously, $\rho^Y \circ h_G$ is an affine action of G_X on A_Y.

Similarly, one can extend h to a group homomorphism, say h_A, from A_X to A_Y. In fact, repeatedly using (6) yields

$$\begin{aligned} &h(\alpha^X_x(y)) \bullet h(\alpha^X_{\alpha^X_x(y)}(\beta^X_y(x))) \\ &= \alpha^Y_{h(x)}(h(y)) \bullet \alpha^Y_{h(\alpha^X_x(y))}(h(\beta^X_y(x))) \\ &= \alpha^Y_{h(x)}(h(y)) \bullet \alpha^Y_{\alpha^Y_{h(x)}(h(y))}(\beta^Y_{h(y)}(h(x))) \end{aligned}$$

for all $x, y \in X$. But the definition of A_Y gives

$$h(x) \bullet \alpha^Y_{h(x)}(h(y)) = \alpha^Y_{h(x)}(h(y)) \bullet \alpha^Y_{\alpha^Y_{h(x)}(h(y))}(\beta^Y_{h(y)}(h(x))),$$

and so

$$\begin{aligned} h(x) \bullet h(\alpha^X_x(y)) &= h(x) \bullet \alpha^Y_{h(x)}(h(y)) \\ &= h(\alpha^X_x(y)) \bullet h(\alpha^X_{\alpha^X_x(y)}(\beta^X_y(x))) \qquad \text{for all } x, y \in X. \end{aligned}$$

In what follows, we show that h_A is G_X-equivariant relative to ρ^X and $\rho^Y \circ h_G$. By Remark 2.4 it is equivalent to show

$$\alpha^Y_{h_G(g)}(h_A(a)) = h_A(\alpha^X_g(a)) \qquad \text{for all } g \in G_X, a \in A_X, \tag{7}$$
$$h_A(b^X(g)) = b^Y(h_G(g)) \qquad \text{for all } g \in G_X. \tag{8}$$

Applying (6) and Proposition 1.8, one has that

$$\alpha^Y_{h_G(g)}(h_A(x)) = h_A(\alpha^X_g(x)) \qquad \text{for all } g \in G_X, x \in X.$$

Now from this identity and Proposition 1.8, one can easily verify (7).

For (8), first notice that it is true when $g \in X$, as both sides are equal to $h(g)$. Then the general case follows from (7) and the definition of 1-cocycles.

The last assertion of the proposition is clear. □

The following theorem generalizes the case of symmetric YBE solutions (see, e.g., [7]).

Theorem 3.3. *Let R_X and R_Y be two injective YBE solutions. Then they are isomorphic, if and only if there is a group isomorphism $\phi : G_X \to G_Y$ such that $\phi(X) = Y$, and ρ^X and $\rho^Y \circ \phi$ are conjugate.*

Proof. "Only if" part: Let $h : X \to Y$ be a YB-isomorphism between R_X and R_Y. Keep the same notation used in the proof of Proposition 3.2. Then $\phi := h_G$ has all desired properties, and furthermore ρ^X and $\rho^Y \circ \phi$ are conjugate via h_A.

"If" part: As before, write $\rho^X = (\alpha^X, b^X)$ and $\rho^Y = (\alpha^Y, b^Y)$. Let $h : A_X \to A_Y$ be a G_X-equivariant mapping relative to $(\rho^X, \rho^Y \circ \phi)$. Then by Remark 2.4 we have

$$h \circ \alpha^X_g = \alpha^Y_{\phi(g)} \circ h, \tag{9}$$
$$b^Y \circ \phi(g) = h \circ b^X(g) \tag{10}$$

for all $g \in G_X$.

On the other hand, it follows from the proof of Theorem 2.7 and Remark 2.8 that ρ^X and ρ^Y induce YBE solutions $\bar{R}_X$ and $\bar{R}_Y$ on G_X and G_Y, respectively. Actually,

$$\bar{R}_X(g_1, g_2) = (\tilde{\alpha}^X_{g_1}(g_2), \tilde{\beta}^X_{g_2}(g_1)) \qquad \text{for all } g_1, g_2 \in G_X,$$
$$\bar{R}_Y(g_1', g_2') = (\tilde{\alpha}^Y_{g_1'}(g_2'), \tilde{\beta}^Y_{g_2'}(g_1')) \qquad \text{for all } g_1', g_2' \in G_Y,$$

where

$$\tilde{\alpha}^X_{g_1} := (b^X)^{-1} \alpha^X_{g_1} b^X, \quad \tilde{\beta}^X_{g_2}(g_1) := \tilde{\alpha}^X_{g_1}(g_2)^{-1} g_1 g_2,$$

and similarly for $\tilde{\alpha}^Y$, $\tilde{\beta}^Y$.

From (10) one has

$$\phi = (b^Y)^{-1} \circ h \circ b^X.$$

We claim that ϕ is actually a YB-isomorphism between $\bar{R}_X$ and $\bar{R}_Y$. To this end, we must show that the two identities in (6) hold true.

► Firstly, we check

$$\phi \circ \tilde{\alpha}^X_g = \tilde{\alpha}^Y_{\phi(g)} \circ \phi \qquad \text{for all } g \in G_X.$$

But this follows from (9), the definitions of $\tilde{\alpha}^X$ and $\tilde{\alpha}^Y$:

$$\begin{aligned}(9) &\Rightarrow hb^X((b^X)^{-1}\alpha_g^X b^X) = b^Y((b^Y)^{-1}\alpha^Y_{\phi(g)}b^Y)((b^Y)^{-1}hb^X)\\ &\Rightarrow [(b^Y)^{-1}hb^X][(b^X)^{-1}\alpha_g^X b^X] = [(b^Y)^{-1}\alpha^Y_{\phi(g)}b^Y][(b^Y)^{-1}hb^X]\\ &\Rightarrow \phi\circ\tilde{\alpha}_g^X = \tilde{\alpha}^Y_{\phi(g)}\circ\phi.\end{aligned}$$

► Secondly, we verify that

$$\phi\circ\tilde{\beta}_g^X = \tilde{\beta}^Y_{\phi(g)}\circ\phi \qquad \text{for all } g\in G_X.$$

Since b^X is a 1-cocycle with respect to α^X in coefficient A_X, one has that, for all $g_1, g_2 \in G_X$,

$$\begin{aligned}& b^X_{g_1g_2} = b^X_{g_1}\bullet\alpha^X_{g_1}(b^X_{g_2})\\ &\Rightarrow g_1(b^X)^{-1}(\alpha^X_{g_1^{-1}}(b^X_{g_2})) = (b^X)^{-1}(b^X_{g_1}\bullet b^X_{g_2}) = g_1\odot g_2\\ &\Rightarrow g_1\tilde{\alpha}^X_{g_1^{-1}}(g_2) = g_1\odot g_2\\ &\Rightarrow g_1^{-1}\tilde{\alpha}^X_{g_1}(g_2) = g_1^{-1}\odot g_2\\ &\Rightarrow \tilde{\alpha}^X_{g_1}(g_2)^{-1} = (g_1^{-1}\odot g_2)^{-1}g_1^{-1}.\end{aligned} \tag{11}$$

Similarly,

$$\tilde{\alpha}^Y_{g_1'}(g_2')^{-1} = (g_1'^{-1}\odot g_2')^{-1}g_1'^{-1} \qquad \text{for all } g_1', g_2'\in G_Y. \tag{11'}$$

Now define a new multiplication $\odot$ on G_X by

$$g_1\odot g_2 = (b^X)^{-1}(b^X_{g_1}\bullet b^X_{g_2}) \qquad \text{for all } g_1, g_2\in G_X,$$

and similarly on G_Y. Then it is easy to check that $(G_X,\odot)$ and $(G_Y,\odot)$ are groups. In what follows, we claim that ϕ is also a group homomorphism from $(G_X,\odot)$ to $(G_Y,\odot)$. As a matter of fact, for all g_1, $g_2\in G_X$, one has

$$\begin{aligned}\phi(g_1\odot g_2) &= \phi\circ(b^X)^{-1}(b^X_{g_1}\bullet b^X_{g_2})\\ &= (b^Y)^{-1}\circ h(b^X_{g_1}\bullet b^X_{g_2}) && \text{(by (10))}\\ &= (b^Y)^{-1}(h(b^X_{g_1})\bullet h(b^X_{g_2})) && \text{(as } h: A_X\to A_Y \text{ is a homomorphism)}\\ &= (b^Y)^{-1}(b^Y_{\phi(g_1)}\bullet b^Y_{\phi(g_2)}) && \text{(by (10))}\\ &= \phi(g_1)\odot\phi(g_2) && \text{(by the definition of } \odot \text{ in } G_Y).\end{aligned}$$

We now have

$$\begin{aligned}
&\phi \circ \tilde{\beta}^X_{g_2}(g_1)\\
&= \phi(\tilde{\alpha}^X_{g_1}(g_2)^{-1} g_1 g_2) && \text{(by the definition of } \tilde{\beta}^X)\\
&= \phi((g_1^{-1} \odot g_2)^{-1} g_1^{-1} g_1 g_2) && \text{(by (11))}\\
&= \phi(g_1^{-1} \odot g_2)^{-1} \phi(g_2) && \text{(as } \phi \text{ is a homomorphism from } G_X \text{ to } G_Y)\\
&= (\phi(g_1^{-1}) \odot \phi(g_2))^{-1} \phi(g_2) && \text{(by the above claim)}\\
&= (\phi(g_1)^{-1} \odot \phi(g_2))^{-1} \phi(g_2) && \text{(as } \phi \text{ is a homomorphism from } G_X \text{ to } G_Y)\\
&= (\phi(g_1)^{-1} \odot \phi(g_2))^{-1}\\
&\quad \phi(g_1)^{-1} \phi(g_1) \phi(g_2)\\
&= \tilde{\alpha}^X_{\phi(g_1)}(\phi(g_2))^{-1} \phi(g_1) \phi(g_2) && \text{(by (11'))}\\
&= \tilde{\beta}^Y_{\phi(g_2)}(\phi(g_1)) && \text{(by the definition of } \tilde{\beta}^Y)
\end{aligned}$$

for all $g_1, g_2 \in G_X$.

Therefore, ϕ is a YB-isomorphism between $\bar{R}_X$ and $\bar{R}_Y$.

Recall from Remark 2.8 that $\bar{R}_X$ is an extension of R_X from X to G_X and similarly for $\bar{R}_Y$. Since R_X and R_Y are injective and $\phi(X) = Y$, the restriction $\phi|_X$ yields a YB-isomorphism between R_X and R_Y. $\square$

We are now ready to provide a characterization when the extensions $\bar{R}_X$ and $\bar{R}_Y$ are isomorphic.

Theorem 3.4. *Let R_X and R_Y be two arbitrary YBE solutions. Then the extensions $\bar{R}_X$ and $\bar{R}_Y$ on G_X and G_Y are YB-isomorphic, if and only if there is a group isomorphism $\phi : G_X \to G_Y$ such that ρ^X and $\rho^Y \circ \phi$ are conjugate.*

Proof. ($\Leftarrow$): It directly follows from the proof of "If" part of Theorem 3.3.

($\Rightarrow$): Let $h : G_X \to G_Y$ be a YB-isomorphism between $\bar{R}_X$ and $\bar{R}_Y$. Now consider $h|_{\iota_G(X)}$. Then completely similar to the proof of Proposition 3.2, $h|_{\iota_G(X)}$ can be extended to an isomorphism ϕ from G_X to G_Y, such that ρ^X and $\rho^Y \circ \phi$ are conjugate. $\square$

In the rest of this section, we provide a connection with C^*-dynamical systems. For any group G, by $C^*(G)$ we mean the group C^*-algebra of G. Since all groups here are assumed to be discrete, $C^*(G)$ is unital. Furthermore, G can be canonically embedded to $C^*(G)$ as its unitary generators. For the background on C^*-dynamical systems which is needed below, refer to [3].

Proposition 3.5.

(i) *A YBE solution R_X determines an action π^X of G_X on $M_2(C^*(A_X))$ such that*

$$\pi^X_g(\mathrm{diag}(x, y)) = \mathrm{diag}(\gamma_g(x), \zeta_g(y)) \qquad \textit{for all } g \in G_X, x, y \in C^*(A_X),$$

where γ and ζ are representations of G_X on $C^*(A_X)$.

(ii) *If h is a YB-homomorphism between R_X and R_Y, then there are group homomorphisms $h_G : G_X \to G_Y$ and $h_A : A_X \to A_Y$ such that the inflation $h_A^{(2)}$ is G_X-equivariant relative to $(\pi^X, \pi^Y \circ h_G)$.*

Proof. (i) Let π^X be defined as

$$\begin{aligned}\pi_g^X \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} &= \begin{pmatrix} \alpha_g^X(a_1) & \alpha_g^X(a_2)(b_g^X)^* \\ b_g^X \alpha_g^X(a_3) & b_g^X \alpha_g^X(a_4)(b_g^X)^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & b_g^X \end{pmatrix} (\alpha_g^X)^{(2)} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (b_g^X)^* \end{pmatrix}\end{aligned}$$

for all $g \in G_X$ and $a_1, a_2, a_3, a_4 \in A_X$. Then one can use the properties of α and b to easily check that π^X is an action of G_X on the matrix C^*-algebra $M_2(C^*(A_X))$. Also $\gamma_g(\cdot) = \alpha_g^X(\cdot)$ and $\zeta_g(\cdot) = b_g^X \alpha_g^X(\cdot)(b_g^X)^*$ are two representations of G_X on $C^*(A_X)$.

(ii) Since h is a YB-isomorphism between R_X and R_Y, as in the proof of Proposition 3.2, it induces group homomorphisms $h_G : G_X \to G_Y$ and $h_A : A_X \to A_Y$ satisfying (7) and (8). Then we extend h_A to a C^*-homomorphism, still denoted by h_A, from $C^*(A_X)$ to $C^*(A_Y)$. Furthermore, its inflation $h_A^{(2)} : M_2(C^*(A_X)) \to M_2(C^*(A_Y))$ gives a G_X-equivariant mapping relative to π^X and $\pi^Y \circ h_G$. In fact, a simple calculation gives

$$h_A^{(2)} \circ \pi_g^X \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} h_A(\alpha_g^X(a_1)) & h_A(\alpha_g^X(a_2))h_A((b_g^X)^*) \\ h_A(b_g^X)h_A(\alpha_g^X(a_3)) & h_A(b_g)h_A(\alpha_g^X(a_4))h_A((b_g^X)^*) \end{pmatrix}$$

and

$$\pi_{h_G(g)}^Y \circ h_A^{(2)} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha_{h_G(g)}^Y(h_A(a_1)) & \alpha_{h_G(g)}^Y(h_A(a_2))(b_{h_G(g)}^Y)^* \\ b_{h_G(g)}^Y \alpha_{h_G(g)}^Y(h_A(a_3)) & b_{h_G(g)}^Y \alpha_{h_G(g)}(a_4)(b_{h_G(g)}^Y)^* \end{pmatrix}.$$

Then apply (7) and (8) to obtain the right hand sides equal. □

By Proposition 3.5, from the associated regular affine action ρ^X of a given YBE solution R_X, one obtains a C^*-dynamical system $(G_X, M_2(C^*(A_X)), \pi^X)$.

Theorem 3.6. *Two injective YBE solutions R_X and R_Y are isomorphic, if and only if there is a group isomorphism $\phi : G_X \to G_Y$ mapping X onto Y such that $(G_X, M_2(C^*(A_X)), \pi^X)$ and $(G_X, M_2(C^*(A_Y)), \pi^Y \circ \phi)$ are conjugate.*

Proof. ($\Rightarrow$): Keep the same notation as in the proof of Proposition 3.5. If $h : X \to Y$ is a YB-isomorphism between R_X and R_Y, then $\phi := h_G$, and π^X and $\pi^Y \circ \phi$ are equivalent via $h_A^{(2)}$.

($\Leftarrow$): Let $\mathfrak{h} : M_2(C^*(A_X)) \to M_2(C^*(A_Y))$ be an intertwining homomorphism between π^X and $\pi^Y \circ \phi$. Let us write $\mathfrak{h} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$. Then $\mathfrak{h}$ acts as a 2×2 matrix multiplication of the applications of h_{ij}. Then $\mathfrak{h} \circ \pi_g^X \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \pi_{\phi(g)}^Y \circ \mathfrak{h} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ yields

$$h_{11} \alpha_g^X(a) = \alpha_{\phi(g)}^Y h_{11}(a) \qquad \text{for all } a \in A_X.$$

Also $\mathfrak{h}\circ\pi_g^X\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}=\pi_{\phi(g)}^Y\circ\mathfrak{h}\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$ yields $(h_{11}(b_g^X))^*=(b_{\phi(g)}^Y)^*$, which implies

$$h_{11}(b_g^X)=b_{\phi(g)}^Y.$$

The above two identities give (9) and (10). Then applying the proof of "If" part of Theorem 3.3 ends the proof. □

Appendix A. A Commutation Relation for Semidirect Products

In this appendix, we prove a commutation relation for semidirect products of YBE solutions derived from cycle sets, which might be useful in the future studies. We further describe a connection between the structure group of the semidirect product of two YBE solutions and the semidirect product of their structure groups.

Definition A.1. A nonempty set X with a binary operation $\cdot$ is called a cycle set, if

$$(x\cdot y)\cdot(x\cdot z)=(y\cdot x)\cdot(y\cdot z)\qquad \text{for all } x,y,z\in X.$$

A cycle set X is said to be nondegenerate if $x\mapsto x\cdot x$ is bijective.

The main motivation to study cycle sets is the following theorem due to Rump ([12]): There is a one-to-one correspondence between the set of symmetric YBE solutions and the set of nondegenerate cycle sets. In fact, let $(X,\cdot)$ be a nondegenerate cycle set. If we let $\ell_x(y)=x\cdot y$, then

$$R(x,y)=\left(\ell_{\ell_y^{-1}(x)}(y),\ell_y^{-1}(x)\right)$$

is a symmetric YBE solution on X. Conversely, given a symmetric YBE solution $R(x,y)=(\alpha_x(y),\beta_y(x))$ on X, let $x\cdot y=\beta_x^{-1}(y)$. Then $(X,\cdot)$ is a nondegenerate cycle set.

The following small lemma turns out very handy.

Lemma A.2. *Keep the above notation. Then*

$$G_{R_X}=_{\mathrm{gp}}\langle X;(y\cdot x)y=(x\cdot y)x \textit{ for all } x,y\in X\rangle.$$

Proof. By Lemma 1.2, $\alpha_x(y)=\beta^{-1}_{\beta_y(x)}(y)$ for all $x,y\in X$. Hence

$$\begin{aligned} R(x,y)=(\beta^{-1}_{\beta_y(x)}(y),\beta_y(x)) &\Leftrightarrow R(\beta_y^{-1}(x),y)=(\beta_x^{-1}(y),x)\\ &\Leftrightarrow R(y\cdot x,y)=(x\cdot y,x)\end{aligned}$$

for all $x,y\in X$. □

Let us now recall Rump's semidirect product of cycle sets below.

Definition A.3. Let X and S be two finite cycle sets, and let π be an action of X on S. That is, $\pi: X\times S\to S,\ (x,s)\mapsto\pi_x(s)$, satisfies

(1) $\pi_x(s\cdot t)=\pi_x(s)\cdot\pi_x(t)$ for every $x\in X$ and for all $s,t\in S$;
(2) $\pi_{y\cdot x}\pi_y(s)=\pi_{x\cdot y}\pi_x(s)$ for all $x,y\in X$ and $s\in S$;
(3) $\pi_x\in\mathrm{Sym}(S)$ for every $x\in X$.

Set

$$\gamma_{x,y}(s,t) = \pi_{x\cdot y}(s) \cdot \pi_{y\cdot x}(t).$$

Now define

$$(x,s)\cdot(y,t) := (x\cdot y, \gamma_{x,y}(s,t)). \tag{12}$$

Then this gives a cycle structure on $X \times S$, which is denoted by $X \ltimes_\pi S$, called the *semidirect product of X and S by π*. The symmetric YBE solution determined by $X \ltimes_\pi S$ is written as $R_{X\ltimes_\pi S}$.

Remark A.4. Notice that for Definition A.3 (i) one has

$$\begin{aligned}\pi_x(s\cdot t) = \pi_x(s)\cdot\pi_x(t) &\Leftrightarrow \pi_x(\beta_s^{-1}(t)) = \beta^{-1}_{\pi_x(s)}(\pi_x(t))\\ &\Leftrightarrow \beta_{\pi_x(s)}(\pi_x(t)) = \pi_x(\beta_s(t))\\ &\Leftrightarrow \pi_x(\alpha_s(t)) = \alpha_{\pi_x(s)}(\pi_x(t))\end{aligned}$$

for all $s,t \in S$ and $x \in X$. In particular, this shows that, for every $x \in X$, π_x is a YB-isomorphism between R_S and itself.

In what follows, let us write

$$R_X(x,y) = (\alpha_x(y), \beta_y(x)) \quad \textit{and} \quad R_S(s,t) = (\tilde\alpha_s(t), \tilde\beta_t(s)).$$

Corollary A.5. *Let X and S be cycles sets, and let π be an action of X on S. Then the YBE solution $R_{X\ltimes_\pi S}$ is explicitly given by the following formula*

$$R_{X\ltimes_\pi S}\big((x,s),(y,t)\big) = \Big(\big(\alpha_x(y), \tilde\alpha_s(\pi_x(t))\big), \big(\beta_y(x), \pi^{-1}_{\alpha_x(y)}(\tilde\beta_{\pi_x(t)}(s))\big)\Big)$$

for all $x,y \in X$ and $s,t\in S$.

Proof. First observe that

$$x\cdot y = \beta_x^{-1}(y) \Rightarrow x\cdot\beta_x(y) = y \Rightarrow y\cdot\beta_y(x) = x$$

and

$$x\cdot y = \beta_x^{-1}(y) \Rightarrow \beta_y(x)\cdot y = \beta^{-1}_{\beta_y(x)}(y) = \alpha_x(y).$$

The above identities will be frequently used in what follows.

Suppose that

$$R_{X\ltimes_\pi S}((x,s),(y,t)) = \big(\alpha'_{(x,s)}(y,t), \beta'_{(y,t)}(x,s)\big) \qquad \text{for all } x,y\in X, s,t\in S.$$

Let $\beta'_{(y,t)}(x,s) = (z,p)$. Then

$$(x,s) = \ell_{(y,t)}(z,p) = (y,t)\cdot(z,p) = (y\cdot z, \pi_{y\cdot z}(t)\cdot\pi_{z\cdot y}(p)).$$

So $z = \beta_y(x)$ and

$$\begin{aligned}s = \pi_{y\cdot z}(t)\cdot\pi_{z\cdot y}(p) &\Rightarrow \pi_{z\cdot y}(p) = \tilde\beta_{\pi_{y\cdot z}(t)}(s) \Rightarrow p = \pi^{-1}_{z\cdot y}(\tilde\beta_{\pi_{y\cdot z}(t)}(s))\\ &\Rightarrow p = \pi^{-1}_{\beta_y(x)\cdot y}(\tilde\beta_{\pi_{y\cdot\beta_y(x)}(t)}(s)) \Rightarrow p = \pi^{-1}_{\alpha_x(y)}(\tilde\beta_{\pi_x(t)}(s)).\end{aligned}$$

Thus

$$\beta'_{(y,t)}(x,s) = \big(\beta_y(x), \pi^{-1}_{\alpha_x(y)}(\tilde\beta_{\pi_x(t)}(s))\big).$$

Now

$$\alpha'_{(x,s)}(y,t) = \beta'^{-1}_{\beta'_{(y,t)}(x,s)}(y,t) =: (u,v).$$

Then

$$\begin{aligned}(u,v) &= \big(\beta_y(x), \pi^{-1}_{\alpha_x(y)}(\tilde{\beta}_{\pi_x(t)}(s))\big)\cdot(y,t)\\ &= \big(\beta_y(x)\cdot y, \pi_{\beta_y(x)\cdot y}(\pi^{-1}_{\alpha_x(y)}(\tilde{\beta}_{\pi_x(t)}(s))\cdot \pi_{y\cdot\beta_y(x)}(t))\big)\\ &= \big(\beta_y(x)\cdot y, \pi_{\alpha_x(y)}(\pi^{-1}_{\alpha_x(y)}(\tilde{\beta}_{\pi_x(t)}(s))\cdot \pi_x(t))\big)\\ &= \big(\beta_y(x)\cdot y, \tilde{\beta}_{\pi_x(t)}(s)\cdot \pi_x(t)\big)\\ &= \big(\beta_y(x)\cdot y, \tilde{\alpha}_s(\pi_x(t))\big).\end{aligned}$$

Therefore

$$\alpha'_{(x,s)}(y,t) = \big(\beta_y(x)\cdot y, \tilde{\alpha}_s(\pi_x(t))\big).$$

This ends the proof. ☐

Lemma A.6. *Let X and S be cycle sets, and let π be an action of X on S. Then π can be extended to an action of G_{R_X} on G_{R_S}.*

Proof. Notice that, in Definition A.3, (i) says that π_x is a cycle morphism on S for every $x \in X$, and (ii) says that $\pi_{y\cdot x}\pi_y = \pi_{x\cdot y}\pi_x$ for all $x, y \in X$. Thus Lemma A.2 and the latter imply that the action π can be extended to an action $\pi : G_{R_X} \curvearrowright S$.

Applying Lemma A.2 to G_{R_S}, one has

$$(s\cdot t)s = (t\cdot s)t \qquad \text{for all } s,t\in S.$$

Since $\pi_x(s), \pi_x(t) \in S$ by Definition A.3 (iii), replacing s and t by $\pi_x(s)$ and $\pi_x(t)$, respectively, in the identity obtained above gives

$$(\pi_x(s)\cdot\pi_x(t))\pi_x(s) = (\pi_x(t)\cdot\pi_x(s))\pi_x(t) \qquad \text{for all } x\in X, s,t\in S.$$

This implies

$$\pi_x(s\cdot t)\pi_x(s) = \pi_x(t\cdot s)\pi_x(t) \qquad \text{for all } x\in X, s,t\in S$$

as π_x is a cycle morphism on S. Therefore, by Lemma A.2, π can be extended to an action $\pi : G_{R_X} \curvearrowright G_{R_S}$. ☐

Under the conditions of Lemma A.6, one can form the semidirect product $G_{R_X} \ltimes_\pi G_{R_S}$, where

$$\pi : G_{R_X}\times G_{R_S} \to G_{R_S},\ (x,s)\mapsto \pi_x(s).$$

It is also worth mentioning that the identity (4) automatically holds true for the action π of G_{R_X} on G_{R_S} obtained in Lemma A.6. In fact, let $\theta = \pi$, and so it suffices to show that $\pi_x(\tilde{b}^{-1}\alpha_s\tilde{b}) = (\tilde{b}^{-1}\alpha_{\pi_x(s)}\tilde{b})\pi_x$ for all $x \in X$ and $s \in S$. But the restrictions b and $\tilde{b}$, respectively, onto X and S are the identity mappings. Thus this amounts to $\pi_x(\alpha_s(t)) = \alpha_{\pi_x(s)}(\pi_x(t))$. But this holds true by Remark A.4.

Therefore, one obtains a regular affine action $\rho^X \ltimes_\pi \rho^S$ of $G_{R_X} \ltimes_\pi G_{R_S}$ on $A_{R_X} \times A_{R_S}$ (see subsection 2.2), and so a YBE solution $\hat{R}$ on $G_{R_X} \ltimes_\pi G_{R_S}$. (see subsection 2.3). It is natural to write $\hat{R}|_{(X\times S)^2}$ as $R_X \ltimes_\pi R_S$, called the *semidirect product of R_X and R_S by π*. Then we obtain the following commutation relation:

Proposition A.7 (**Commutation Relation for SemiDirect Products**). *Let X and S be cycle sets, and let π be an action of X on S. Then*

$$R_X \ltimes_\pi R_S = R_{X\ltimes_\pi S}.$$

Proof. Assume that

$$\hat{R}((x,s),(y,t)) = \big(\hat{\alpha}_{(x,s)}(y,t), \hat{\beta}_{(y,t)}(x,s)\big) \qquad \text{for all } x,y \in X, s,t \in S.$$

It follows from subsection 2.2 2° that

$$\hat{\alpha}_{(x,s)}(y,t) = (\alpha_x(y), \tilde{\alpha}_s(\pi_x(t))).$$

Then an easy calculation yields

$$\begin{aligned}\hat{\beta}_{(y,t)}((x,s)) &= \hat{\alpha}_{(x,s)}(y,t)^{-1}(xy, s\pi_x(t))\\ &= (\alpha_x(y)^{-1}xy, \pi_{\alpha_x(y)^{-1}}(\tilde{\alpha}_s(\pi_x(t))^{-1}s\pi_x(t)))\\ &= (\beta_y(x), \pi^{-1}_{\alpha_x(y)}(\tilde{\beta}_{\pi_x(t)}(s))).\end{aligned}$$

Therefore comparing the formula of $R_{X\ltimes_\pi S}$ given in Corollary A.5 yields the desired commutation relation. □

For the structure groups G_{R_X}, G_{R_S} and $G_{R_X\ltimes_\pi R_S}$, we have the following.

Proposition A.8. *Keep the above notation. Then there is a group homomorphism*

$$\Pi : G_{R_X\ltimes_\pi R_S} \to G_{R_X} \ltimes_\pi G_{R_S}.$$

Proof. By Proposition A.7, $G_{R_X\ltimes_\pi R_S} = G_{R_{X\ltimes_\pi S}}$. Applying Lemma A.2 to $G_{R_{X\ltimes_\pi S}}$, we have the following relations

$$((x,s)\cdot(y,t))(x,s) = ((y,t)\cdot(x,s))(y,t) \qquad \text{for all } x,y \in X,\ \ s,t \in S.$$

From (12), this is equivalent to

$$(x\cdot y, \gamma_{x,y}(s,t))(x,s) = (y\cdot x, \gamma_{y,x}(t,s))(y,t).$$

Let $\Pi : X \ltimes_\pi S \to G_{R_X} \ltimes_\pi G_{R_S}$ be defined via

$$\Pi(x,s) = (x,s) \qquad \text{for all } x \in X, s \in S.$$

Simple calculations show that

$$\begin{aligned}\Pi(x\cdot y, \gamma_{x,y}(s,t))\,\Pi(x,s) &= \big(x\cdot y, \pi_{x\cdot y}(s)\cdot\pi_{y\cdot x}(t)\big)\,(x,s)\\ &= \big((x\cdot y)x, (\pi_{x\cdot y}(s)\cdot\pi_{y\cdot x}(t))\pi_{x\cdot y}(s)\big)\end{aligned}$$

and

$$\begin{aligned}\Pi(y\cdot x, \gamma_{y,x}(t,s))\,\Pi(y,t) &= (y\cdot x, \pi_{y\cdot x}(t)\cdot\pi_{x\cdot y}(s))(y,t)\\ &= \big((y\cdot x)y, (\pi_{y\cdot x}(t)\cdot\pi_{x\cdot y}(s))\pi_{y\cdot x}(t)\big).\end{aligned}$$

Therefore, by Lemma A.2, Π can be extended a group homomorphism, still denoted by Π, from $G_{R_X\ltimes_\pi R_S}$ to $G_{R_X} \ltimes_\pi G_{R_S}$. □

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