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# $L^p$-HARDY–RELLICH AND UNCERTAINTY PRINCIPLE INEQUALITIES ON THE SPHERE

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ABSTRACT. In this paper, we study the Hardy–Rellich type inequalities and uncertainty principle on the geodesic sphere. Firstly, we derive $L^p$-Hardy inequalities via divergence theorem, which are in turn used to establish the $L^p$-Rellich inequalities. We also establish Heisenberg uncertainty principle on the sphere via the Hardy–Rellich type inequalities. The best constants appearing in the inequalities are shown to be sharp.

## 1. INTRODUCTION

This paper is concerned with Hardy, Rellich, and uncertainty principle inequalities. Motivated by [21], we derive Hardy type inequalities on the geodesic sphere. The Hardy inequalities obtained are then used to derive Rellich type and uncertainty principle inequalities with sharp constants. These inequalities play fundamental roles in tackling many problems from harmonic analysis, partial differential equations, and differential geometry as well as from physics and quantum mechanics.

Let $\mathbb{R}^N$, $N \geq 3$, be the $N$-dimensional Euclidean space. The classical Hardy inequality, for $f \in C_0^\infty(\mathbb{R}^N)$ and $1 < p < \infty$, is given as follows

$$\int_{\mathbb{R}^N} |\nabla f(x)|^p dx \geq \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} dx,$$


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where $((N-p)/p)^p$ is the best constant and never achieved. Improvement of the above inequality has become theme of many literatures in recent time because of its several applications, for instances, see [1, 2, 6, 7, 9, 8, 11, 13] and references therein.

In the Riemannian manifold setting, Carron [3] studied weighted $L^2$-Hardy inequalities under some geometric assumptions on the weight and obtained the following weighted $L^2$-Hardy inequality on compact or noncompact manifold $M$:

$$\int_M \rho^\alpha |\nabla f|^2 dV \geq \Big(\frac{C+\alpha-1}{2}\Big)^2 \int_M \frac{|f|^2}{\rho^{2-\alpha}} dV,$$

for all $f \in C_0^\infty(M)$, $\alpha \in \mathbb{R}$, $C > 1$, $C+\alpha-1 > 0$, and the positive weight function $\rho$ satisfies $|\nabla\rho|^p = 1$ and $\Delta\rho \geq \frac{C}{\rho}$. Recently, Kombe and Özaydin [14] (see also Kombe-Özaydin [15] and Kombe-Yener [16]) extended Caron's result to the general case $1 < p < \infty$. In these papers the authors proved several Hardy type, Rellich type, and even uncertainty principle inequalities on manifold satisfying certain geometric restriction. Yang, Su, and Kong [23] applied the above ideas to obtain the following Hardy inequality on Riemannian manifold with negative sectional curvature for $N \geq 3$, $1 < p < N-\alpha$, $\alpha \in \mathbb{R}$, and $f \in C_0^\infty(M)$:

$$\int_M \frac{|\nabla f|^p}{\rho^\alpha} dV \geq \Big(\frac{N-p-\alpha}{p}\Big)^p \int_M \frac{|f|^p}{\rho^{\alpha+p}} dV, \tag{1.1}$$

where $|\nabla\rho|^p = 1$, $\Delta\rho \geq \frac{N-1}{\rho}$, and the constant $\left(\frac{N-p-\alpha}{p}\right)^p$ is sharp. Using (1.1) they obtained Rellich type inequalities, for $1 < p < \infty$, $0 < \alpha < N-2$, and $f \in C_0^\infty(M)$,

$$\int_M \frac{|\Delta f|^p}{\rho^{2+\alpha-2p}} dV \geq \Big(\frac{[(p-1)(N-2)+\alpha](N-\alpha-2)}{p^2}\Big)^p \int_M \frac{|f|^p}{\rho^{\alpha+2}} dV$$

with sharp constant. They further showed that $L^p$-Hardy inequalities on Riemannian manifold can be globally refined by adding remainder terms like the Brezis and Vazques [2] improvement for $p \geq 2$.

Hardy–Rellich type and uncertainty principle inequalities have also been established for various settings such as Poincaré model [15], Carnot groups [13], Heisenberg groups [11, 22], and Lie groups [19] to mention but a few. In 2016, Xiao [21] studied $L^2$-Hardy inequality on the unit sphere and used his result to derive $L^2$-Rellich inequality. In 2017, Sun and Pan [20] applied the same approach to extend Xiao's result to the general $p$. The approach is used in this paper is similar in some sense to Xiao [21] and Sun and Pan [20].

The classical uncertainty principle of quantum mechanics known as Heisenberg–Pauli–Weyl inequality is presented as follows:

$$\Big(\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx\Big)\Big(\int_{\mathbb{R}^N} |\nabla f(x)|^2 dx\Big) \geq \Big(\frac{N}{2}\Big)^2 \Big(\int_{\mathbb{R}^N} |f(x)|^2 dx\Big)^2$$

for all $f \in L^2(\mathbb{R}^N)$ with the best constant $(N/2)^2$. Uncertainty principle inequalities have also undergone some sort of improvement and restriction to model manifolds. For examples, see [15] when the manifold is the hyperbolic space,

[11] on Heisenberg groups, and [19] on Lie groups. In the present paper the uncertainty principle obtained is a consequence of the $L^p$-weighted type inequality. The uncertainty principle on the sphere has varieties of applications in harmonic analysis, signal analysis, and information theory. There are some other works treating uncertainty principle on the sphere using different techniques [6, 12, 18].

The rest of the paper is planned as follows. In Section 2 we recall some basic facts about the sphere and then present the main results of this paper. Section 3 is devoted to the proof of our main results on weighted Hardy and Rellich type inequalities and the sharpness of their constants. We prove the weighted Hardy inequality by adding remainder terms. We solve an improved version of Rellich inequality which connects first and second order derivatives. In section 4 we establish an Heisenberg uncertainty principle inequality on the sphere as a natural consequence of the weighted Hardy inequality.

## 2. Notations and main results

2.1. **Sphere.** The sphere is an example of a compact Riemannian manifold. Consider the $N$-dimensional sphere of radius $r > 0$

$$\mathbb{S}^N = \{x = (x_1, x_2, \dots, x_{N+1}) \in \mathbb{R}^{N+1} : |x|^2 = \sum_{j=i}^{N+1} x_j^2 = r^2\}.$$

It is clear that $\mathbb{S}^N$ is a smooth orientable submanifold of $\mathbb{R}^{N+1}$, endowed with canonical Riemannian structure defined by restriction of standard Euclidean metric and with constant sectional curvature $1/r^2$. For detail discussion on the sphere see any standard book on Riemannian geometry like [4, 5, 10].

In this paper we shall deal with the unit $N$-sphere $\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} : |x| = 1\}$. Let $(\theta_1, \theta_2, \dots, \theta_N)$ be angular variables on $\mathbb{S}^N$; we set $\theta = \theta_N$, where $x_{N+1} = |x| \cos\theta_N$. The associated weight function is given as $\Theta(\theta, \xi) = (\sin\theta)^{N-1}$, $\xi \in \mathbb{S}^{N-1}$, and by polar coordinate transformation

$$\int_{\mathbb{S}^N} f dV = \int_{\mathbb{S}^{N-1}} \int_0^\pi f (\sin\theta)^{N-1} d\theta d\sigma, \qquad f \in L^1(\mathbb{S}^N)$$

and

$$Vol(\mathbb{S}^N) = Vol(\mathbb{S}^{N-1}) \int_0^\pi (\sin\theta)^{N-1} d\theta,$$

where $dV$ denotes the standard volume element on $\mathbb{S}^N$, and $d\sigma$ and $Vol(\mathbb{S}^{N-1}) = (2\pi)^{N/2}/\Gamma(N/2)$, respectively, denote the canonical measure and the volume of the unit $(N-1)$-sphere. A function $f = f(\theta)$ which depends only on $\theta$ is called radial. In this case using the radial part of the Laplace–Beltrami operator $\Delta_{\mathbb{S}^N}$ we have

$$\Delta_{\mathbb{S}^N} f(\theta) = (\sin\theta)^{1-N} \frac{d}{d\theta}\Big((\sin\theta)^{N-1} \frac{d}{d\theta} f\Big)$$

while the gradient of a function $f$ on $\mathbb{S}^N$ is $|\nabla_{\mathbb{S}^N} f(\theta)| = |\frac{d}{d\theta} f(\theta)|$.

The geodesic distance between $x$ and an arbitrary point $q \in \mathbb{S}^N$ is denoted as $d(x, q)$. Note that the points on the sphere are all the same distance from the origin, which is a fixed point and all geodesics of the sphere are closed curves.

Consider two points $x$ and $y$ on a sphere of radius $r > 0$ centered at the origin of $\mathbb{R}^N$, the distance between the two points is given by $d(x, y) = r\arccos((x \cdot y)/r)$, while $x \cdot y = r^2 \cos\theta$, where $\theta$ is the angle between vectors $x$ and $y$. Hence, the minimal geodesic joining two points on the unit sphere can be taken to be $\theta$. Throughout we denote $\Delta = \Delta_{\mathbb{S}^N}$ and $\nabla = \nabla_{\mathbb{S}^N}$.

2.2. **Main results.** Our first result is the following Hardy type inequalities

**Theorem 2.1.** *Let $N \geq 3$, $0 \leq \alpha < N-p$, and $1 < p < \infty$; then there exists a positive constant $A = A(N, \alpha, p)$ such that, for all $f \in C^\infty(\mathbb{S}^N)$,*

$$\int_{\mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} f|^p}{(\sin\theta)^\alpha} dV + A(N,\alpha,p) \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p-2}} dV \geq \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p}} dV, \tag{2.1}$$

*where*

$$A(N,\alpha,p) = \min\Big\{1, \frac{p}{2}\Big\}\Big(\frac{N-p-\alpha}{p}\Big)^p + \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}.$$

*Moreover, the constant $\left(\frac{N-p-\alpha}{p}\right)^p$ appearing in (2.1) is sharp in the sense that*

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} f|^p}{(\sin\theta)^\alpha} dV + A(N,\alpha,p)\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p-2}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p}} dV} \leq \Big(\frac{N-p-\alpha}{p}\Big)^p.$$

We now present a weighted $L^P$-Hardy type inequality of form (2.1) but with additional remainder term.

**Theorem 2.2.** *Let $N \geq 3$, $0 \leq \alpha < N-p$, and $2 \leq p < \infty$. Let $g$ be a positive smooth function on $\mathbb{S}^N$ such that $\langle \nabla g, \nabla h\rangle_{\mathbb{S}^N} \geq 0$, where $h$ is defined by $h = g^{-1/p}(\sin\theta)^{\frac{N-p-\alpha}{p}} f$. Then there exists a positive constant $A = A(N,\alpha,p)$ such that, for all $f \in C^\infty(\mathbb{S}^N)$, the following inequality holds*

$$\int_{\mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} f|^p}{(\sin\theta)^\alpha} dV + A(N,\alpha,p)\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p-2}} dV \geq \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p}} dV + \frac{c(p)}{p^p}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^\alpha}\frac{|\nabla g|^p}{g^p} dV, \tag{2.2}$$

*where $c(p)$ is a constant depending only on $p$.*

There is a corresponding inequality with different additional remainder theorem for $1 < p < 2$. The second result is the following Rellich type inequality.

**Theorem 2.3.** *Let $N \geq 3$, $0 \leq \alpha < N-p$, and $1 < p < \infty$. There holds, for $f \in C^\infty(\mathbb{S}^N)$ and $q \in \mathbb{S}^N$,*

$$\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{\alpha+2-2p}} dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}} dV \geq \mathcal{C}\int_{\mathbb{S}^N}\Big(\frac{|f|^p}{\theta^p} + \frac{|f|^p}{(\pi-\theta)^p}\Big) dV, \tag{2.3}$$

*where $\mathcal{D} = \mathcal{B} + \mathcal{C}\sup_{\theta\in(0,\pi)}|V(\theta)|$,*

$$\mathcal{B} = \mathcal{B}(N, \alpha, p) = \Big(\frac{4(p-1)A(N,\alpha,p) + \alpha(N-\alpha-1)}{p^2}\Big)^p,$$

$$\mathcal{C} = \mathcal{C}(N, \alpha, p) = \Big(\frac{(N-\alpha-2)[(p-1)(N-2)+\alpha]}{p^2}\Big)^p,$$

*and*

$$|V(\theta)| = \Big|\frac{(\sin\theta)^{\alpha+2-2p}}{\theta^p} + \frac{(\sin\theta)^{\alpha+2-2p}}{(\pi-\theta)^p} = \frac{1}{(\sin\theta)^{2p}}\Big|.$$

*Furthermore the constant* $\mathcal{C} = \mathcal{C}(N,\alpha,p)$ *is sharp in the sense that*

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{\alpha-2-2p}} dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha-2-2p}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{d(x,q)^p} dV} \le \mathcal{C}(N,\alpha,p)$$

*and*

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{\alpha-2-2p}} dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha-2-2p}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{9\pi-d(x,q)0^p} dV} \le \mathcal{C}(N,\alpha,p).$$

The third result is the uncertainty principle on the sphere.

**Theorem 2.4.** *Let* $N \ge 3$, $1 < p < \infty$, *and* $\xi \in \mathbb{S}^N$; *there holds the following inequality, for all functions* $f \in C^\infty(\mathbb{S}^N)$,

$$\begin{aligned}\Big(\int_{\mathbb{S}^N} |d(x,\xi)^q||f|^p dV\Big)^{p/q}\Big(\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^p dV + C\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV\Big)& \\ \ge \Big(\frac{N-p}{p}\Big)^p\Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p&,\end{aligned} \tag{2.4}$$

*where* $p$ *and* $q$ *are Hölders conjugate; that is,* $1/p + 1/q = 1$.

## 3. Hardy and Rellich inequalities

3.1. **Proof of Hardy inequalities.** Before we prove Theorem 2.1 we quickly recall the following vectorial inequalities (see [17, 23]).

**Lemma 3.1.** *There exists some constant* $c(p) > 0$ *such that, for all* $u, v \in \mathbb{R}^N$, *there holds the following conditions*

(1) $|u+v|^p - |u|^p - p|u|^{p-2}\langle u, v\rangle \ge c(p)\frac{|v|^2}{(|u|+|v|)^{2-p}}$, $\qquad p < 2$,
(2) $|u+v|^p - |u|^p - p|u|^{p-2}\langle u, v\rangle \ge c(p)|v|^p$, $\qquad p \ge 2$.

*For our purpose we pull the above two conditions together to obtain the condition below.*

(3) $|u+v|^p - |u|^p - p|u|^{p-2}\langle u, v\rangle \ge c(p)|\mathcal{G}(u,v)|$, *where* $|\mathcal{G}(u,v)| > 0$.

*Proof of Theorem 2.1.* Let $f = (\sin\theta)^{-\frac{N-p-\alpha}{p}}\varphi$; we have by a straightforward computation and condition (3) above that

$$\begin{aligned}
&|\nabla f|^p \\
&= \Big| -\frac{N-p-\alpha}{p}(\sin\theta)^{\frac{\alpha-N}{p}}\cos\theta\ \varphi + (\sin\theta)^{-\frac{N-p-\alpha}{p}}\nabla\varphi\Big|^p \\
&\geq \Big| -\frac{N-p-\alpha}{p}\Big|^p(\sin\theta)^{\alpha-N}(\cos\theta)^p\ \varphi^p \\
&\quad + p\Big| -\frac{N-p-\alpha}{p}\Big|^{p-2}\Big(-\frac{N-p-\alpha}{p}\Big)\Big((\sin\theta)^{\frac{\alpha-N}{p}}\nabla sin\theta\ \varphi\Big)^{p-1}(\sin\theta)^{-\frac{N-p-\alpha}{p}}\nabla\varphi \\
&\quad + c(p)|\mathcal{G}| \\
&\geq \Big(\frac{N-p-\alpha}{p}\Big)^p(\sin\theta)^{\alpha-N}(\cos\theta)^p\ \varphi^p \\
&\quad - \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}(\sin\theta)^{\alpha-N+1}\langle\nabla\sin\theta, \nabla\varphi^p\rangle,
\end{aligned}$$

since $|\mathcal{G}| > 0$, in fact, if $p=2$, $|\mathcal{G}| = (\sin\theta)^{-\frac{N-p-\alpha}{p}}|\nabla\varphi|$. Multiplying through by the weight $(\sin\theta)^{-\alpha}$ and integrating over $\mathbb{S}^N$ yield

$$\begin{aligned}
\int_{\mathbb{S}^N}\frac{|\nabla f|^p}{(\sin\theta)^\alpha}dV &\geq \Big(\frac{N-p-\alpha}{p}\Big)^p\int_{\mathbb{S}^N}\frac{\varphi^p}{(\sin\theta)^N}(\cos\theta)^p dV \\
&\quad - \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}\int_{\mathbb{S}^N}(\sin\theta)^{1-N}\langle\nabla\sin\theta, \nabla\varphi^p\rangle \\
&= \Big(\frac{N-p-\alpha}{p}\Big)^p\int_{\mathbb{S}^N}\frac{\varphi^p}{(\sin\theta)^N}(\cos\theta)^p dV \\
&\quad + \frac{1}{(N-2)}\Big(\frac{N-p-\alpha}{p}\Big)^{p-1}\int_{\mathbb{S}^N}\langle\nabla(\sin\theta)^{-(N-2)}, \nabla\varphi^p\rangle \\
&= \Big(\frac{N-p-\alpha}{p}\Big)^p\int_{\mathbb{S}^N}\frac{\varphi^p}{(\sin\theta)^N}(\cos\theta)^p dV \\
&\quad - \frac{1}{(N-2)}\Big(\frac{N-p-\alpha}{p}\Big)^{p-1}\int_{\mathbb{S}^N}\langle\Delta(\sin\theta)^{-(N-2)}, \varphi^p\rangle,
\end{aligned}$$

where we have used the identity $(\sin\theta)^{1-N}\nabla\sin\theta = -\frac{1}{N-2}\nabla(\sin\theta)^{-(N-2)}$ and integration by parts. We then compute

$$\begin{aligned}
\Delta(\sin\theta)^{-(N-2)} &= (\sin\theta)^{1-N}\frac{d}{d\theta}\Big((\sin\theta)^{N-1}\frac{d}{d\theta}(\sin\theta)^{-(N-2)}\Big) \\
&= -(N-2)(\sin\theta)^{-(N-2)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\mathbb{S}^N}\frac{|\nabla f|^p}{(\sin\theta)^\alpha}dV \quad &\geq \Big(\tfrac{N-p-\alpha}{p}\Big)^p\int_{\mathbb{S}^N}\tfrac{\varphi^p}{(\sin\theta)^N}(\cos\theta)^p dV \\
&\quad -\Big(\tfrac{N-p-\alpha}{p}\Big)^{p-1}\int_{\mathbb{S}^N}\tfrac{|\varphi|^p}{(\sin\theta)^{N-2}}dV.
\end{aligned} \tag{3.1}$$

We then make use of

$$|\cos\theta|^p = |\cos^2\theta|^{\frac{p}{2}} = (1-\sin^2\theta)^{\frac{p}{2}} \geq 1 - \frac{p}{2}\sin^2\theta, \qquad \text{for } p \geq 2,$$

and

$$|\cos\theta|^p \geq \cos^2\theta = 1 - \sin^2\theta, \qquad \text{for } 1 < p < 2,$$

to rewrite inequality (3.1) as

$$\begin{aligned}\int_{\mathbb{S}^N} \frac{|\nabla f|^p}{(\sin\theta)^\alpha} dV \geq & \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{\varphi^p}{(\sin\theta)^N} dV \\ & - \Big[\min\Big\{1, \frac{p}{2}\Big\}\Big(\frac{N-p-\alpha}{p}\Big)^p + \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}\Big] \int_{\mathbb{S}^N} \frac{|\varphi|^p}{(\sin\theta)^{N-2}} dV.\end{aligned}$$

Substituting back the function $f = (\sin\theta)^{-\frac{N-p-\alpha}{p}}\varphi$ yields the desired inequality which is inequality (2.1).

The next is to show that $\left(\frac{N-p-\alpha}{p}\right)^p$ is sharp. One can choose $\alpha = 0$ and then prove that

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^p dV + A(N,p)\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^p} dV} \leq \Big(\frac{N-p}{p}\Big)^p$$

which is similar to what was proved in [20, Theorem 1]. We therefore omit the proof here. □

*Remark* 3.2. Choosing $\alpha = 0$ in Theorem 2.1, we have the $L^p$-Hardy inequality proved in [21, 20]

$$\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^p dV + C\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV \geq \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{S}^N} \Big(\frac{|f|^p}{d(x,q)^p} + \frac{|f|^p}{(\pi - d(x,q))^p}\Big) dV, \tag{3.2}$$

where $d(x,q) = \theta$ is the geodesic distance from $x$ to $q \in \mathbb{S}^N$ and $\left(\frac{N-p}{p}\right)^p$ is the best constant.

For $p \geq 2$, $C = A(N,p) + \sup_{\theta\in(0.\pi)} |V_1(\theta)|$ with $V_1(\theta) > 0$ and $\sup\limits_{\theta\in(0.\pi)} |V_1(\theta)| < \infty$, where

$$V_1(\theta) = \frac{(\sin\theta)^{p-2}}{\theta^p} + \frac{(\sin\theta)^{p-2}}{(\pi-\theta)^p} - \frac{1}{(\sin\theta)^p}.$$

For $1 < p < 2$, $C = A(N,p) + \left(\frac{N-p}{p}\right)^p \sup_{\theta\in(0.\pi)} |V_2(\theta)|$, where

$$V_2(\theta) = \frac{1}{\theta^p} + \frac{1}{(\pi-\theta)^p} - \frac{1}{\sin^2\theta}.$$

The case $p = 2$ was proved by Xiao [21, Theorem 1.1], and the general case $1 < p \leq N$ by Sun and Pan [20, Theorem 1.1]. In fact, it was remarked in [20] that $\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV$ cannot control the right hand side of (3.2) in the case

$1<p<2$, since $(\sin d(x,q))^{p-2}$, $q\in\mathbb{S}^N$ is large when $x$ is very close to $q$. Then, for $1<p<2$, (3.2) is written as

$$\int_{\mathbb{S}^N}|\nabla_{\mathbb{S}^N}f|^p dV + C(N,p)\int_{\mathbb{S}^N}|f|^p dV \geq \Big(\frac{N-p}{p}\Big)^p\int_{\mathbb{S}^N}\Big(\frac{|f|^p}{d(x,q)^p}+\frac{|f|^p}{(\pi-d(x,q))^p}\Big)dV.$$

□

We now prove the weighted $L^p$-Hardy inequality with additional remainder term.

*Proof of Theorem 2.2.* The proof follows from the proof of Theorem 2.1. In order to get the additional remainder term, we only need to bound the term $c(p)\int_{\mathbb{S}^N}\frac{|\nabla\varphi|^p}{(\sin\theta)^{N-p}}dV$. This term would have appeared in (3.1) if we had considered the case $p\geq 2$ only; that is,

$$c(p)\int_{\mathbb{S}^N}\frac{|\mathcal{G}|^p}{(\sin\theta)^\alpha}dV = c(p)\int_{\mathbb{S}^N}\frac{|\nabla\varphi|^p}{(\sin\theta)^{N-p}}dV, \qquad p\geq 2.$$

Let $h=g^{-1/p}\varphi$, where $g\in C^\infty(\mathbb{S}^N)\backslash\{0\}$ and $\varphi$ is defined by $f=(\sin\theta)^{-\frac{N-p-\alpha}{p}}\varphi$. We compute

$$\begin{aligned}
|\nabla\varphi|^p &= \Big|\frac{1}{p}hg^{1/p-1}\nabla g + g^{1/p}\nabla h\Big|^p\\
&\geq \frac{|h|^p}{p^p}g^{1-p}|\nabla g|^p + p\Big(\frac{h}{p}g^{1/p-1}\nabla g\Big)^{p-2}\cdot\frac{h}{p}g^{1/p-1}\nabla g\cdot g^{1/p}\nabla h\\
&= \frac{|h|^p}{p^p}\frac{|\nabla g|^p}{g^{p-1}} + p^{1-p}|\nabla g|^{p-2}g^{2-p}\langle\nabla g,\nabla h^p\rangle\\
&\geq \frac{|h|^p}{p^p}\frac{|\nabla g|^p}{g^{p-1}},
\end{aligned}$$

since $\langle\nabla g,\nabla h^p\rangle = ph^{p-1}\langle\nabla g,\nabla h\rangle\geq 0$ follows from the given assumption. Substituting back $h=g^{-1/p}(\sin\theta)^{\frac{N-p-\alpha}{p}}f$, we have

$$\begin{aligned}
C(p)\int_{\mathbb{S}^N}\frac{|\nabla\varphi|^p}{(\sin\theta)^{N-p}}dV &\geq \frac{c(p)}{p^p}\int_{\mathbb{S}^N}\frac{|h|^p}{(\sin\theta)^{N-p}}\frac{|\nabla g|^p}{g^{p-1}}dV\\
&= \frac{c(p)}{p^p}\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^\alpha}\frac{|\nabla g|^p}{g^p}dV.
\end{aligned}$$

Putting this into (3.1) in the proof of Theorem 2.1 we arrived at the desired result which is (2.2). This completes the proof. □

### 3.2. **Proof of Rellich inequality.**

*Proof of Theorem 2.3.* For $\epsilon > 0$, define $f_\epsilon := (|f|^2 + \epsilon^2)^{p/2} - \epsilon^p \in C^\infty(\mathbb{S}^N)$ with the same support as $f$. We have

$$\begin{aligned}\Delta f_\epsilon &= p(|f|^2+\epsilon^2)^{p/2-1}|\nabla f|^2 + p(p-2)(|f|^2+\epsilon^2)^{p/2-2}f^2|\nabla f|^2 \\ &\quad + p(|f|^2+\epsilon^2)^{p/2-1} f\Delta f \\ &\geq p(p-1)(|f|^2+\epsilon^2)^{p/2-2}f^2|\nabla f|^2 + p(|f|^2+\epsilon^2)^{p/2-1}f\Delta f \\ &\geq \frac{4(p-1)}{p}|\nabla g_\epsilon| + p(|f|^2+\epsilon^2)^{p/2-1}f\Delta f,\end{aligned}$$

where we have defined a function $g_\epsilon := (|f|^2+\epsilon^2)^{p/4} - \epsilon^{p/2}$ and used

$$(|f|^2+\epsilon^2)^{p/2-2}f^2|\nabla f|^2 = \frac{4}{p}|\nabla g_\epsilon|.$$

Therefore

$$-p(|f|^2+\epsilon^2)^{p/2-1}f\Delta f \geq \frac{4(p-1)}{p}|\nabla g_\epsilon|^2 - \Delta f_\epsilon. \tag{3.3}$$

Multiplying (3.3) through by the weight $(\sin\theta)^{-\alpha}$ and integrating over $\mathbb{S}^N$ yield

$$-p\int_{\mathbb{S}^N}\frac{(|f|^2+\epsilon^2)^{p/2-1}}{(\sin\theta)^\alpha}f\Delta f dV \geq \frac{4(p-1)}{p}\int_{\mathbb{S}^N}\frac{|\nabla g_\epsilon|^2}{(\sin\theta)^\alpha}dV - \int_{\mathbb{S}^N}\frac{\Delta f_\epsilon}{(\sin\theta)^\alpha}dV.$$

By integration by parts, we have

$$\begin{aligned}-\int_{\mathbb{S}^N}\frac{\Delta f_\epsilon}{(\sin\theta)^\alpha}dV &= -\int_{\mathbb{S}^N} f_\epsilon\Delta(\sin\theta)^{-\alpha}dV \\ &= -\int_{\mathbb{S}^N} f_\epsilon(\sin\theta)^{1-N}\frac{d}{d\theta}\Big((\sin\theta)^{N-1}\frac{d}{d\theta}(\sin\theta)^{-(N-2)}\Big) \\ &= -\alpha\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^\alpha}dV + \alpha(N-\alpha-2)\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha+2}}\cos^2\theta dV.\end{aligned}$$

Therefore

$$\begin{aligned}-p\int_{\mathbb{S}^N}\frac{(|f|^2+\epsilon^2)^{p/2-1}}{(\sin\theta)^\alpha}f\Delta f dV \geq{}& \frac{4(p-1)}{p}\int_{\mathbb{S}^N}\frac{|\nabla g_\epsilon|^2}{(\sin\theta)^\alpha}dV \\ &- \alpha\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^\alpha}dV \\ &+ \alpha(N-\alpha-2)\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha+2}}\cos^2\theta dV.\end{aligned}$$

Applying Theorem 2.1 (i.e., the Hardy inequality for $p = 2$), we have

$$
\begin{aligned}
-p\int_{\mathbb{S}^N} \frac{(|f|^2+\epsilon^2)^{p/2-1}}{(\sin\theta)^\alpha} f\Delta f dV \geq{}& \frac{(p-1)(N-\alpha-2)^2}{p}\int_{\mathbb{S}^N}\frac{|g_\epsilon|^2}{(\sin\theta)^{\alpha+2}}dV\\
&-\frac{4(p-1)A(N.\alpha,p)}{p}\int_{\mathbb{S}^N}\frac{|g_\epsilon|^2}{(\sin\theta)^{\alpha}}dV\\
&-\alpha\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha}}dV\\
&+\alpha(N-\alpha-2)\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha+2}}(1-\sin^2\theta)dV.
\end{aligned}
$$

Hence

$$
\begin{aligned}
-p\int_{\mathbb{S}^N} \frac{(|f|^2+\epsilon^2)^{p/2-1}}{(\sin\theta)^\alpha} f\Delta f dV \geq{}& \frac{(p-1)(N-\alpha-2)^2}{p}\int_{\mathbb{S}^N}\frac{|g_\epsilon|^2}{(\sin\theta)^{\alpha+2}}dV\\
&-\frac{4(p-1)A(N.\alpha,p)}{p}\int_{\mathbb{S}^N}\frac{|g_\epsilon|^2}{(\sin\theta)^{\alpha}}dV\\
&+\alpha(N-\alpha-2)\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha+2}}dV\\
&-\alpha(N-\alpha-1)\int_{\mathbb{S}^N}\frac{f_\epsilon}{(\sin\theta)^{\alpha}}dV.
\end{aligned}
$$

Letting $\epsilon\to 0$, we have, by Lebesgue dominated convergence theorem,

$$
\begin{aligned}
p\int_{\mathbb{S}^N}\frac{|f|^{p-1}\Delta f}{(\sin\theta)^\alpha}dV \geq{}& \Big(\frac{(N-\alpha-2)[(p-1)(N-2)+\alpha]}{p}\Big)\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2}}dV\\
&-\Big(\frac{4(p-1)A(N,\alpha,p)+\alpha(N-\alpha-1)}{p}\Big)\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha}}dV.
\end{aligned}
$$

By Hölder's inequality

$$
\int_{\mathbb{S}^N}\frac{|f|^{p-1}\Delta f}{(\sin\theta)^\alpha}dV \leq \Big(\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2}}dV\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\Big)^{\frac{1}{p}},
$$

we have

$$
\begin{aligned}
p\Big(\int_{\mathbb{S}^N}\tfrac{|\Delta f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\Big)^{\frac{1}{p}} \geq{}& \Big(\tfrac{(N-\alpha-2)[(p-1)(N-2)+\alpha]}{p}\Big)\Big(\int_{\mathbb{S}^N}\tfrac{|f|^p}{(\sin\theta)^{\alpha+2}}dV\Big)^{\frac{1}{p}}\\
&-\Big(\tfrac{4(p-1)A(N,\alpha,p)+\alpha(N-\alpha-1)}{p}\Big)\Big(\int_{\mathbb{S}^N}\tfrac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Denote by

$$
\mathcal{B}(N,\alpha,p)=\Big(\frac{4(p-1)A(N,\alpha,p)+\alpha(N-\alpha-1)}{p^2}\Big)^p
$$

and

$$
\mathcal{C}(N,\alpha,p)=\Big(\frac{(N-\alpha-2)[(p-1)(N-2)+\alpha]}{p^2}\Big)^p.
$$

We arrive at

$$\mathcal{C}(N,\alpha,p)\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2}}dV \le \int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{\alpha+2-2p}}dV + \mathcal{B}(N,\alpha,p)\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}dV. \tag{3.4}$$

In order to get our result we can write (3.4) as

$$\begin{aligned}\mathcal{C}(N,\alpha,p)&\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}\Big(\frac{1}{(\sin\theta)^{2p}}-\frac{(\sin\theta)^{\alpha+2-2p}}{\theta^p}-\frac{(\sin\theta)^{\alpha+2-2p}}{(\pi-\theta)^p}\Big)dV\\ &+\mathcal{C}(N,\alpha,p)\int_{\mathbb{S}^N}\Big(\frac{|f|^p}{\theta^p}+\frac{|f|^p}{(\pi-\theta)^p}\Big)dV \le \int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\\ &+\mathcal{B}(N,\alpha,p)\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}dV,\end{aligned}$$

which implies

$$\begin{aligned}\mathcal{C}\int_{\mathbb{S}^N}\Big(\frac{|f|^p}{\theta^p}+\frac{|f|^p}{(\pi-\theta)^p}\Big)dV \le& \int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\\ &+\mathcal{B}\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}dV\\ &+\mathcal{C}\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha+2-2p}}V(\theta)dV\end{aligned}$$

where

$$V(\theta)=\frac{(\sin\theta)^{\alpha+2-2p}}{\theta^p}+\frac{(\sin\theta)^{\alpha+2-2p}}{(\pi-\theta)^p}-\frac{1}{(\sin\theta)^{2p}}.$$

Following the approach in [20, 21] we have $0<\sup_\theta|V(\theta)|<+\infty$. Inequality (2.3) follows immediately.

Now we prove that the constant $\mathcal{C}(N,\alpha,p)=\Big(\frac{(N-2-\alpha)[(p-1)(N-2)+\alpha]}{p^2}\Big)^p$ is sharp. To prove this it is enough to set $\alpha=0$ and show that

$$\inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}}\frac{\int_{\mathbb{S}^N}\frac{|\Delta_{\mathbb{S}^N}f|^p}{(\sin\theta)^{\alpha-2-2p}}dV+\mathcal{D}\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{\alpha-2-2p}}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{d(x,q)^p}dV}\le\Big(\frac{(N-2)^2(p-1)}{p^2}\Big)^p$$

Let $\phi(t)\in[0,1]$ be the cut-off function such that $\phi(t)=1$, for $|t|\le 1$, and $\phi(t)\equiv 0$ for $|t|>2$. Set $H(t)=1-\phi(t)$ and

$$f_\varepsilon(\theta)=\begin{cases}H\Big(\dfrac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}} & \text{for } 0<\theta\le\pi,\\ 0 & \text{for } \theta=0.\end{cases}$$

Without loss of generality, we assume that $0 < \varepsilon < 1/2$ and that $f_\varepsilon(\theta)$ is a smooth radial function on $\mathbb{S}^N$. Then we have

$$\begin{aligned}\int_{\mathbb{S}^N} \frac{f_\varepsilon^p}{\theta^p} dV &= \int_{\mathbb{S}^{N-1}} d\sigma \int_\varepsilon^\pi H^p\Big(\frac{\theta}{\varepsilon}\Big)\theta^{2-N-p}(\sin\theta)^{N-1}d\theta \\ &= Vol\Big(\mathbb{S}^{N-1}\Big)\int_\varepsilon^\pi H^p\Big(\frac{\theta}{\varepsilon}\Big)\theta^{2-N-p}(\sin\theta)^{N-1}d\theta \\ &\geq Vol\Big(\mathbb{S}^{N-1}\Big)\int_{2\varepsilon}^\pi \theta^{2-N-p}(\sin\theta)^{N-1}d\theta\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{S}^N} \frac{f_\varepsilon^p}{(\sin\theta)^{2-2p}} dV &= \int_{\mathbb{S}^{N-1}} d\sigma \int_\varepsilon^\pi H^p\Big(\frac{\theta}{\varepsilon}\Big)\theta^{2-N}(\sin\theta)^{N-3+2p}d\theta \\ &= Vol\Big(\mathbb{S}^{N-1}\Big)\int_\varepsilon^\pi H^p\Big(\frac{\theta}{\varepsilon}\Big)\theta^{2p-1}d\theta \\ &\leq Vol\Big(\mathbb{S}^{N-1}\Big)\int_\varepsilon^\pi \theta^{2p-1}d\theta = Vol\Big(\mathbb{S}^{N-1}\Big)\frac{\pi^{2p}-\varepsilon^{2p}}{2p}.\end{aligned}$$

Since $f_\varepsilon(\theta)$ is radial, we compute

$$\begin{aligned}\Delta_{\mathbb{S}^N} f_\varepsilon(\theta) &= \Big(\frac{d^2}{d\theta^2} + (N-1)\cot\theta\frac{d}{d\theta}\Big)\Big(H\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}}\Big) \\ &= \Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)H\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N-2p}{p}} \\ &\quad + \frac{2}{\varepsilon}\Big(\frac{2-N}{p}\Big)H'\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N-p}{p}} + \frac{1}{\varepsilon^2}H''\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}} \\ &\quad + (N-1)\cot\theta\left(\Big(\frac{2-N}{p}\Big)H\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N-p}{p}} + \frac{1}{\varepsilon}H'\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}}\right) \\ &= H\Big(\frac{\theta}{\varepsilon}\Big)\left(\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)\theta^{\frac{2-N-2p}{p}}\right. \\ &\qquad\qquad \left. + (N-1)\Big(\frac{2-N}{p}\Big)\theta^{\frac{2-N-p}{p}}\cot\theta\right) \\ &\quad + \frac{1}{\varepsilon}H'\Big(\frac{\theta}{\varepsilon}\Big)\left(\frac{2(2-N)}{p}\theta^{\frac{2-N-2}{p}} + (N-1)\theta^{\frac{2-N}{p}}\cot\theta\right) \\ &\quad + \frac{1}{\varepsilon^2}H''\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}}.\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{S}^N} \frac{|\Delta f_\varepsilon|^p}{(\sin\theta)^{2-2p}} dV \\
&\quad \le Vol(\mathbb{S}^{N-1}) \int_\varepsilon^\pi H^p\Big(\frac{\theta}{\varepsilon}\Big) \Big| \Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)\theta^{\frac{2-N-2p}{p}} \\
&\qquad\qquad + (N-1)\Big(\frac{2-N}{p}\Big)\theta^{\frac{2-N-p}{p}} \cot\theta \Big|^p (\sin\theta)^{N-3+2p} d\theta \\
&\quad + Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^p} \int_\varepsilon^{2\varepsilon} \Big|H'\Big(\frac{\theta}{\varepsilon}\Big)\Big|^p \Big| \frac{2(2-N)}{p}\theta^{\frac{2-N-2}{p}} \\
&\qquad\qquad + (N-1)\theta^{\frac{2-N}{p}} \cot\theta \Big|^p (\sin\theta)^{N-3+2p} d\theta \\
&\quad + Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^{2p}} \int_\varepsilon^{2\varepsilon} \Big|H''\Big(\frac{\theta}{\varepsilon}\Big)\Big|^p \theta^{2-N} (\sin\theta)^{N-3+2p} d\theta \\
&\quad := \mathrm{I} \quad + \quad \mathrm{II} \quad + \quad \mathrm{III}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\mathrm{I} \le Vol(\mathbb{S}^{N-1}) \int_\varepsilon^\pi \Big| &\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big) \\
&+ (N-1)\Big(\frac{2-N}{p}\Big)\cos\theta \Big|^p \theta^{2-N-p} (\sin\theta)^{N-3+2p} d\theta,
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &\le Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^p} \Big(\max_{t\in[0,2]} H'(t)\Big)^p \int_\varepsilon^{2\varepsilon} \Big| \frac{2(2-N)}{p}\sin\theta \\
&\qquad\qquad + (N-1)\theta\cos\theta \Big|^p \theta^{2-N-p} (\sin\theta)^{N-3+2p} d\theta \\
&\le Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^p} \Big(\max_{t\in[0,2]} H'(t)\Big)^p \int_\varepsilon^{2\varepsilon} \Big| \frac{2(2-N)}{p}\theta + (N-1)\theta \Big|^p \theta^{2-N-p}\theta^{N-3+2p} d\theta \\
&= Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^p} \Big(\max_{t\in[0,2]} H'(t)\Big)^p \Big(\frac{2(2-N)+(N-1)p}{p}\Big) \int_\varepsilon^{2\varepsilon} \theta^{2p-1} d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{III} &\le Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^{2p}} \Big(\max_{t\in[0,2]} H''(t)\Big)^p \int_\varepsilon^{2\varepsilon} \theta^{2-N}\theta^{N-3+2p} d\theta \\
&= Vol(\mathbb{S}^{N-1}) \frac{1}{\varepsilon^{2p}} \Big(\max_{t\in[0,2]} H''(t)\Big)^p \int_\varepsilon^{2\varepsilon} \theta^{2p-1} d\theta.
\end{aligned}$$

Thus

$$\inf_{f\in C^\infty(\mathbb{S}^N)} \frac{\int_{\mathbb{S}^N} \frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{\theta^p}dV}$$

$$\leq \inf_{f\in C^\infty(\mathbb{S}^N)} \frac{\int_{\mathbb{S}^N} \frac{|\Delta f_\varepsilon|^p}{(\sin\theta)^{2-2p}}dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f_\varepsilon|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N} \frac{|f_\varepsilon|^p}{\theta^p}dV}$$

$$\leq \frac{\mathrm{I} \quad + \quad \mathrm{II} \quad + \quad \mathrm{III}}{Vol\Big(\mathbb{S}^{N-1}\Big)\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta} + \frac{\mathcal{D}\int_\varepsilon^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$\leq \frac{\int_\varepsilon^\pi \left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big) + (N-1)\Big(\frac{2-N}{p}\Big)\cos\theta\right|^p \theta^{2-N-p}(\sin\theta)^{N-3+2p}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$+ \frac{\frac{1}{\varepsilon^p}\Big(\max_{t\in[0,2]} H'(t)\Big)^p \Big(\frac{2(2-N)+(N-1)p}{p}\Big)\int_\varepsilon^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$+ \frac{\frac{1}{\varepsilon^{2p}}\Big(\max_{t\in[0,2]} H''(t)\Big)^p \int_\varepsilon^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta} + \frac{\mathcal{D}\int_\varepsilon^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}.$$

Since $\lim_{\varepsilon\to 0^+}\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta \to +\infty$, we then pass to the limit as $\varepsilon \to 0^+$, and we have

$$\inf_{f\in C^\infty(\mathbb{S}^N)} \frac{\int_{\mathbb{S}^N} \frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV + \mathcal{D}\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{\theta^p}dV}$$

$$\leq \lim_{\varepsilon\to 0^+} \frac{\int_\varepsilon^\pi \left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big) + (N-1)\Big(\frac{2-N}{p}\Big)\cos\theta\right|^p \theta^{2-N-p}(\sin\theta)^{N-3+2p}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$= \lim_{\varepsilon\to 0^+} \frac{-\left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big) + (N-1)\Big(\frac{2-N}{p}\Big)\cos\varepsilon\right|^p (\varepsilon)^{2-N-p}(\sin\varepsilon)^{N-3+2p}}{-(2\varepsilon)^{2-N-p}(\sin 2\varepsilon)^{N-1}}$$

$$= \Big(\frac{(2-N)^2(p-1)}{p^2}\Big)^p,$$

where we have applied L'Hopital rule. Similarly, if we use $f_\varepsilon(\pi-\theta)$ as the test function and follow the above proof we can also establish $\Big(\frac{(2-N)^2(p-1)}{p}\Big)^p$ as a sharp constant. The proof is complete. □

We now turn our attention to an improved weighted $L^p$-Rellich type inequality which connects the first and second order derivatives.

**Theorem 3.3.** *Let $N \geq 3$ and $\alpha + N - p - 2 > 0$, $2 \leq p < \infty$. The following inequality is valid, for all $f \in C^\infty(\mathbb{S}^N)\backslash\{0\}$,*

$$\int_{\mathbb{S}^N} (\sin\theta)^\alpha |\Delta f|^p dV + C_1(N,\alpha,p) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p-2} |f|^p dV \\ \geq C_2(N,\alpha,p) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |f|^p dV + p^2(p-1) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |\nabla f|^p dV,$$

*where*

$$C_1(N,\alpha,p) = p\Big(\frac{p-1}{p} + (\alpha-p)(\alpha+N-p-2)\Big)$$

*and*

$$C_2(N,\alpha,p) = (\alpha-p)(\alpha+N-p-1).$$

*Proof.* As usual a straightforward computation gives

$$\Delta(\sin\theta)^{\alpha-p} = (\alpha-p)(\alpha+N-p-2)(\sin\theta)^{\alpha-p-2}\cos^2\theta - (\alpha-p)(\sin\theta)^{\alpha-p}.$$

Therefore

$$(\alpha-p)(\alpha+N-p-2) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p-2}\cos^2\theta |f|^p dV - (\alpha-p) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |f|^p dV \\ = \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} \Delta f^p dV \\ = \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} \Big[p(p-1)|f|^{p-2} f |\nabla f|^2 + p|f|^{p-1}\Delta f\Big] dV \\ \geq p(p-1) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |\nabla f|^2 dV + p \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |f|^{p-1} \Delta f dV$$

which implies

$$\begin{aligned} p(p-1) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |\nabla f|^2 dV \leq & - p \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |f|^{p-1} \Delta f dV \\ & + \Big[(\alpha-p)(\alpha+N-p-2) \\ & \quad \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p-2} \cos^2\theta |f|^p dV\Big] \qquad (3.5) \\ & - (\alpha-p) \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} dV. \end{aligned}$$

Using the trigonometry identity $\cos^2\theta = 1 - \sin^2\theta$ and the inequality

$$-p \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p} |f|^{p-1} \Delta f dV \leq \frac{1}{p} \int_{\mathbb{S}^N} (\sin\theta)^\alpha |\Delta f|^p dV \\ + \frac{p-1}{p} \int_{\mathbb{S}^N} (\sin\theta)^{\alpha-p-2} |f|^p dV$$

obtained from Young's inequality into (3.5) yield

$$
\begin{aligned}
&[(\alpha-p)+(\alpha-p)(\alpha+N-p-2)]\int_{\mathbb{S}^N}(\sin\theta)^{\alpha-p}|f|^p dV\\
&\quad +p(p-1)\int_{\mathbb{S}^N}(\sin\theta)^{\alpha-p}|\nabla f|^2 dV\\
&\qquad \le \frac{1}{p}\int_{\mathbb{S}^N}(\sin\theta)^{\alpha}|\Delta f|^p dV\\
&\qquad\quad +(\alpha-p)(\alpha+N-p-2)\int_{\mathbb{S}^N}(\sin\theta)^{\alpha-p-2}|f|^p dV\\
&\qquad\quad +\frac{p-1}{p}\int_{\mathbb{S}^N}(\sin\theta)^{\alpha-p-2}|f|^p dV,
\end{aligned}
$$

which completes the proof. □

## 4. Uncertainty principle

The uncertainty principle can be mathematically formulated on Euclidean space $\mathbb{R}^N$ as follows:

$$
\Big(\int_{\mathbb{R}^N}|x|^2|f(x)|^2dx\Big)\Big(\int_{\mathbb{R}^N}|\nabla f(x)|^2dx\Big)\ge\Big(\frac{N-2}{2}\Big)^2\Big(\int_{\mathbb{R}^N}|f(x)|^2dx\Big)^2,
$$

for all $f\in L^2(\mathbb{R}^N)$, with inequality being attained when $f$ is Gaussian like function $f(x)=A\exp(-\lambda|x|^2)$ for some $A\in\mathbb{R}$ and $\lambda>0$. Uncertainty principle firstly arose from quantum mechanics, where it says that both position and momentum of a particle cannot be simultaneously determined. As a theorem in Euclidean harmonic analysis, it expresses impossibility of simultaneously smallness of a nonzero function $f$ and its Fourier transform $\hat{f}$ (where $\hat{f}(y)=(2\pi)^{-N/2}\int_{\mathbb{R}^N}f(x)e^{i\langle x,y\rangle}dx$ and $\|f\|_2=1=\|\hat{f}\|_2$).

In [15], the authors proved Heisenberg uncertainty principle inequalities on complete noncompact Riemannian manifolds and determined an explicit constant in the case of hyperbolic space. In this section we establish an analogue of Heisenberg uncertainty principle inequality (2.4) on the geodesic sphere. Our result is a natural consequence of the weighted $L^p$-Hardy inequality (2.1). Here is the proof.

*Proof of Theorem 2.4.* Choosing $\alpha=0$ in (2.1), we have the weighted $L^p$-Hardy inequality

$$
\begin{aligned}
&\int_{\mathbb{S}^N}|\nabla_{\mathbb{S}^N}f|^p dV+C\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{p-2}}dV\\
&\qquad\ge\Big(\frac{N-p}{p}\Big)^p\int_{\mathbb{S}^N}\Big(\frac{|f|^p}{d(x,\xi)^p}+\frac{|f|^p}{(\pi-d(x,\xi))^p}\Big)dV,
\end{aligned}
\tag{4.1}
$$

where

$$C = \min\Big\{1, \frac{p}{2}\Big\}\Big(\frac{N-p}{p}\Big)^p + \Big(\frac{N-p}{p}\Big)^{p-1} + \Big(\frac{N-p}{p}\Big)^p \sup_{\theta\in(0,\pi)} |V_1(\theta)|$$

and $d(x,\xi)$ is the geodesic distance between $x$ and $\xi \in \mathbb{S}^N$. Notice that if $f(-x) = f(x)$ for all $x \in \mathbb{S}^N$, then (see [21, Remark 1])

$$\int_{\mathbb{S}^N} \frac{|f|^p}{d(x,\xi)^p} dV = \int_{\mathbb{S}^N} \frac{|f|^p}{(\pi - d(x,\xi))^p} dV.$$

Here we assume that $f(-x) = f(x)$ for all $x$ and write (4.1) as

$$\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^p dV + C\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV \geq \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{d(x,\xi)^p} dV. \tag{4.2}$$

By Hölder's inequality, we have

$$\int_{\mathbb{S}^N} \frac{|f|^p}{d(x,\xi)^p} dV \geq \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p \Big(\int_{\mathbb{S}^N} |d(x,\xi)^q||f|^p dV\Big)^{-p/q}, \tag{4.3}$$

where $1/p + 1/q = 1$. Combining (4.2) and (4.3), we obtain

$$\begin{aligned}\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^p dV &+ C\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{p-2}} dV \\ &\geq \Big(\frac{N-p}{p}\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p \Big(\int_{\mathbb{S}^N} |d(x,\xi)^q||f|^p dV\Big)^{-p/q}\end{aligned}$$

from where the result follows at once. □

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