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A KAKUTANI–MACKEY-LIKE THEOREM

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Abstract. We give a partial extension of a Kakutani–Mackey theorem for quasi-complemented vector spaces. This can be applied in the representation theory of certain complemented (non-normed) topological algebras. The existence of continuous linear maps, in the context of quasi-complemented vector spaces, is a very important issue in their study. Relative to this, we prove that every Hausdorff quasi-complemented locally convex space has continuous linear maps, under which a certain quasi-complemented locally convex space turns to be pre-Hilbert.

1. Introduction and Preliminaries

In what follows, by the term "a subspace of a vector space" (resp. "a closed subspace of a topological vector space") we shall mean "a vector subspace of a vector space" (resp. "a closed vector subspace of a topological vector space"). Also, $\mathcal{V}_X$ will denote the set of all closed subspaces of a (real or complex) topological vector space X. Moreover, by a *generalized real Hilbert space*, we mean a real Hilbert space which is neither necessarily separable nor infinite-dimensional. In 1944, Shizuo Kakutani and George W. Mackey proved in [7, p. 51, Theorem 1] the following theorem.

Theorem 1.1. (Kakutani–Mackey) *Let X be a real Banach space with dimension at least 3. Suppose there exists an endo-mapping on $\mathcal{V}_X$, say, $\sigma : \mathcal{V}_X \to \mathcal{V}_X$ with $\sigma(M) = M^\sigma$, that satisfies the following conditions:*

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$$If \;\; M_1, M_2 \in \mathcal{V}_X \;\; with \;\; M_1 \subseteq M_2, \;\; then \;\; {M_2}^{\sigma} \subseteq {M_1}^{\sigma}. \tag{1.1}$$

$$If \;\; M \in \mathcal{V}_X, \;\; then \;\; M^{\sigma\sigma} = M. \tag{1.2}$$

$$If \;\; M \in \mathcal{V}_X, \;\; then \;\; M \cap M^{\sigma} = \{0\}. \tag{1.3}$$

Then there is an isomorphism between X and a generalized real Hilbert space, such that σ is finally a correspondence between orthogonal complements in the Hilbert space. In other words, it is possible an inner product $\langle\cdot,\cdot\rangle$ to be defined on X such that the induced norm $|||\cdot|||$, defined by the relation

$$|||x||| = \langle x, x\rangle^{1/2}, x \in X$$

is equivalent with the initial norm $\|\cdot\|$ and, moreover, for any $M \in \mathcal{V}_X$,

$$M^{\sigma} = \{y \in X : \langle x, y\rangle = 0 \;\; for \;\; every \;\; x \in M\}.$$

The complex case of the above theorem was stated for infinite-dimensional spaces in [8, p. 729, Theorem 1]. For later use, we state it here.

Theorem 1.2. (Kakutani–Mackey) *Let X be an infinite-dimensional complex Banach space. Suppose there exists an endo-mapping on $\mathcal{V}_X$, say $\sigma : \mathcal{V}_X \to \mathcal{V}_X$ with $\sigma(M) = M^{\sigma}$, that satisfies the conditions* (1.1), (1.2), *and* (1.3) *of* Theorem 1.1.

Then there is an isomorphism between X and a (not necessarily separated) complex Hilbert space, such that σ is finally a correspondence between orthogonal complements in the Hilbert space. Namely, an inner product $\langle\cdot,\cdot\rangle$ is defined on X such that the induced norm $|||\cdot|||$, defined by the relation $|||x||| = \langle x, x\rangle^{1/2}, x \in X$, is equivalent with the initial norm $\|\cdot\|$, and moreover, for any $M \in \mathcal{V}_X$,

$$M^{\sigma} = \{y \in X : \langle x, y\rangle = 0 \;\; for \;\; every \;\; x \in M\}.$$

An application of Theorem 1.2 is given in section 2 (see Theorem 2.1).

The following definition gathers some types of complemented topological vector spaces, which we deal with in what follows. We also note here that actually Kakutani and Mackey employed topological vector spaces the kind of which alluded in (ii) of the same definition.

In the rest of the paper, all vector spaces are taken over the field $\mathbb{C}$ of complexes.

Definition 1.3. Let X be a topological vector space.

(i) X is called a *semi-quasi-complemented space* if there is a mapping $\sigma : \mathcal{V}_X \to \mathcal{V}_X$ with $\sigma(M) = M^{\sigma}$, satisfying the conditions (1.1) and (1.2).

The mapping σ is called a *semi-quasi-complementor*, and M^{σ} a *semi-quasi-complement* of M.

(ii) X is called a *quasi-complemented space* if it is a semi-quasi-complemented space and satisfies the condition (1.3).

The mapping σ is called a *quasi-complementor*, and M^{σ} a *quasi-complement* of M.

(iii) X is called a *complemented space* if it is a semi-quasi-complemented space and satisfies the condition

$$if \;\; M \in \mathcal{V}_X, \;\; then \;\; X = M \oplus M^\sigma.$$

The mapping σ is called a (*vector*) *complementor on* X, and M^σ a (*vector*) *complement* of M.

In what follows, by (X, σ) we shall denote a semi-quasi-complemented space or a quasi-complemented space or, yet, a complemented (vector) space X with a semi-quasi (or quasi-)complementor or, yet, a complementor σ. Of course, a complemented space is quasi-complemented and semi-quasi-complemented.

A terminology comment. **-** The terms "*quasi-complement*" and "*complement*" have been used in the Banach space context, in connection with the existence of (quasi-)complements of closed subspaces. Here, one concerns about closed subspaces that are "(quasi-)complemented" by other closed subspaces and where the presence of a mapping is not in the proscenium, as in our case (see Definition 1.3). Since 1945, when F.J. Murray introduced quasi-complements [12], the references on the existence-problem of quasi-complements is reach enough. See, for example, [6] for Banach spaces and [2] for F-spaces (complete metrizable topological vector spaces).

Example 1.4. Every Hilbert space is (semi-quasi-)complemented in the sense of Definition 1.3, (iii) (see e.g. [1, p. 201, Theorem 15.1.1 and p. 202, Corollary 15.1.1]). In that case, the vector complementor is defined via orthogonality. Here, we note that in a Banach space, complementation of every closed subspace yields the space to be (isomorphically) a Hilbert one (see [10, p. 263, Theorem 1]).

The existence of continuous linear maps, in the context of quasi-complemented vector spaces, is a very important issue in the theory of topological vector spaces. In Theorem 3.4, we prove that every Hausdorff quasi-complemented locally convex space has continuous linear maps, under which a certain quasi-complemented locally convex space turns to be a pre-Hilbert space (see Corollary 3.11). The following terminology is employed in Proposition 3.3, where there is defined an appropriate correspondence between the one-dimensional subspaces of a certain semi-quasi-complemented space and that of its dual. The correspondence, in question, ensures among others, the existence of semilinear correspondences (Definition 1.6, Lemma 3.5, and Theorem 3.6).

Definition 1.5. The subspaces $W_1, W_2, \ldots, W_k$ of a vector space X are (linearly) independent, if the relation $a_1 + a_2 + \cdots + a_k = 0$, where $a_i \in W_i, 1 \leq i \leq k$, implies $a_1 = a_2 = \cdots = a_k = 0$.

An extended notion of linearity is that of semilinearity (Definition 1.6) that constitutes a prerequisite in defining the automorphically perfect pairs (Definition 3.8), through which the existence of inner products is succeeded. The problem of the existence of such a type of mappings is faced in infinite-dimensional spaces (Lemma 3.5). In the context of Theorem 3.6, we get semilinear transformations from a vector space to its dual. Under further properties, the vector space is equipped with an inner product, that leads to a representation of a certain topological algebra (see [4]).

Definition 1.6. Let X and Y be vector spaces. A mapping $T : X \to Y$ is named *semilinear* or a *semilinear transformation* (with respect to ϕ), if there exists an automorphism ϕ on $\mathbb{C}$ such that $T(\lambda x + \mu y) = \phi(\lambda)T(x) + \phi(\mu)T(y)$ for any scalars λ, μ and every $x, y \in X$. If, in particular, $\phi(\kappa) = \overline{\kappa}$, where $\overline{\kappa}$ is the complex conjugate of κ, then T is named *conjugate linear.*

In some results, from a family of quasi-complemented spaces (Definition 1.3) there are defined certain locally convex spaces (see Theorem 3.14 and Proposition 3.15) in the sense of the following definition. The term involved here was introduced in [3, p. 27].

Definition 1.7. A *pseudo-H-space* is a vector space X equipped with a family $(\langle \cdot, \cdot \rangle_i)_{i \in I}$ of positive semi-definite (:pseudo-) inner products, such that the induced topology makes X into a locally convex space. The topology of X is then defined by a family $(p_i)_{i \in I}$ of seminorms, such that $p_i(x) = (\langle x, x \rangle_i)^{1/2}$ for each $x \in X$.

For the next example, see [3, p. 27].

Example 1.8. Let I be an arbitrary set of elements. Consider the set $\mathbb{C}^{I \times I}$ of all complex-valued functions a on $I \times I$, such that $\sum_{i,j} |a(i, j)|^2 \in \mathbb{R}_+$. The latter, endowed with "point-wise" defined operations, becomes a vector space. Take a family of real numbers $(t_\alpha)_{\alpha \in \Lambda}$, such that $t_\alpha \geq 1$. For each $\alpha \in \Lambda$, the mapping $\langle \cdot, \cdot \rangle_\alpha : \mathbb{C}^{I \times I} \times \mathbb{C}^{I \times I} \to \mathbb{C}$ given by

$$\langle a, b \rangle_\alpha = t_\alpha \sum_{i,j} a(i, j)\bar{b}(i, j)$$

defines a pseudo-inner product on $\mathbb{C}^{I \times I}$, where "$-$" denotes complex conjugation. Thus $A \equiv (\mathbb{C}^{I \times I}, (\langle \cdot, \cdot \rangle_\alpha)_{\alpha \in \Lambda})$ becomes a pseudo-H-space.

2. Applications of Kakutani–Mackey theorem

An application of Theorem 1.2 is given in the next result, where we apply the following notation. We denote by $\tau_{\langle \cdot, \cdot \rangle}$ or by $\tau_{\| \cdot \|}$ or by τ_p the topology on a topological vector space X induced on it by a (quasi-)inner product $\langle \cdot, \cdot \rangle$ or a norm $\| \cdot \|$ or, yet, a seminorm p, respectively.

Theorem 2.1. *Let $(X_i, \| \cdot \|_i)_{i \in I}$ be a family of infinite-dimensional Banach spaces, and let $\mathcal{V}_i$, $i \in I$, the set of all closed subspaces of X_i. We suppose that each X_i is a quasi-complemented space with a quasi-complementor σ_i.*

Then on $X = \prod_{i \in I} X_i$ is defined a family $([\cdot, \cdot]_i)_{i \in I}$ of pseudo-inner products. Moreover, the locally convex topology, induced on X, via the pseudo-inner products $[\cdot, \cdot]_i$, $i \in I$, coincides with the initial one.

Proof. By Theorem 1.2, on X_i, $i \in I$, an inner product $\langle \cdot, \cdot \rangle_i$ is defined, such that the induced norm $|||\cdot|||_i$ is equivalent with the initial norm $\|\cdot\|_i$ and such that $M^{\sigma_i} = \{y \in X_i : \langle x, y \rangle_i = 0 \text{ for every } x \in M\}$ for any $M \in \mathcal{V}_i$. By the above, we get $\tau_{|||\cdot|||_i} = \tau_{\|\cdot\|_i}$, namely, the topologies on X_i, are defined by the norms $|||\cdot|||_i$ and $\|\cdot\|_i$, coincide. Thus the product topology on X, is defined by the family $(\|\cdot\|_i)_{i\in I}$, coincides with the product topology on X, is defined by the family $(|||\cdot|||_i)_{i\in I}$. Namely, the topological vector spaces $(X, (\|\cdot\|_i)_{i\in I})$ and $(X, (|||\cdot|||_i)_{i\in I})$ have the same topology.

Define the mapping

$$[\cdot,\cdot]_i : \prod_{j\in I} X_j \times \prod_{j\in I} X_j \longrightarrow \mathbb{C} \text{ with } [(x_j),(y_j)]_i = \langle x_i, y_i \rangle_i, i \in I.$$

It is easily seen that the latter is a quasi-inner product on $X = \prod_{i\in I} X_i$. Thus the family $([\cdot,\cdot]_i)_{i\in I}$ on X gives a family of seminorms $(p_i)_{i\in I}$ with

$$p_i((x_j)_{j\in I}) = ([(x_j)_{j\in I},(x_j)_{j\in I}]_i)^{1/2} = (\langle x_i, x_i\rangle_i)^{1/2} = |||x_i|||_i, i \in I. \tag{2.1}$$

It is known that a basis of the system of the neighborhoods of 0 in the topological vector space $(X, (p_i)_{i\in I})$ consists of sets of the form $\bigcap_{i=1}^n V_{p_i}(\varepsilon_i)$ where

$$V_{p_i}(\varepsilon_i) = \{x = (x_j) \in \prod_{j\in I} X_j : p_i(x) < \varepsilon_i, \varepsilon_i > 0\},$$

namely, (see also (2.1))

$$V_{p_i}(\varepsilon_i) = \{x = (x_j) \in \prod_{j\in I} X_j : |||x_i|||_i < \varepsilon_i, \varepsilon_i > 0\}.$$

On the other hand, a basis of the system of the neighborhoods of 0 in $X = \prod_{i\in I} X_i$ with respect to the product topology, is described as follows. If $U_i = \{x \in X_i : |||x|||_i < \varepsilon, \varepsilon > 0\}$ is a neighborhood of 0 in X_i, then the set $V_i = pr_i^{-1}(U_i)$ is a neighborhood of 0 in the product space X with respect to the product topology, say, $\tau_{\Pi X_i}$. Thus the topologies of the vector spaces $(X, (p_i)_{i\in I})$ and $(X, \tau_{\Pi X_i})$ have the same basis of the system of neighborhoods of 0 and therefore they coincide.

□

The following two corollaries are direct consequences of Theorem 2.1.

Corollary 2.2. *Let $(X_i, \|\cdot\|_i)_{i\in I}$ be a family of infinite-dimensional complemented Banach spaces. Then on $X = \prod_{i\in I} X_i$ a family $([\cdot,\cdot]_i)_{i\in I}$ of pseudo-inner products is defined. Moreover, the locally convex topology, induced on X, via the pseudo-inner products $[\cdot,\cdot]_i, i \in I$, coincides with the initial one.*

For the next result, we remind that $\varprojlim_{i\in I} X_i$, as a subspace of the locally convex space $\prod_{i\in I} X_i$, is locally convex too. Here, the relative topology is involved.

Corollary 2.3. *Let $((X_i, \|\cdot\|_i)_{i\in I}, f_{ij}, i, j \in I)$ be a projective system of infinite-dimensional (quasi-)complemented Banach spaces. Then on $X = \varprojlim_{i\in I} X_i$ a family $([\cdot,\cdot]_i)_{i\in I}$ of pseudo-inner products is defined. Moreover, the locally convex* *topology, induced on X, via the pseudo-inner products $[\cdot,\cdot]_i, i \in I$, coincides with the initial one.*

3. Existence of continuous linear mappings in quasi-complemented spaces

Throughout, X' stands for the topological dual of a topological vector space X. Namely $X' = \{f : X \to \mathbb{C} \text{ with } f \text{ linear and continuous}\}$. Moreover, if S is a subset of a vector space X, then $[S]$ stands for the subspace of X, generated by S. We also use the symbol $[S]_X$, in the case, we want to indicate the vector space X. If, in particular, $S = \{x\}, x \in X$, we shall use the symbol $[x]$ or, yet, $[x]_X$. We start with a useful result, concerning the analysis of quasi-complemented vector spaces via the kernels of continuous linear maps.

Proposition 3.1. *Let (X, σ) be a Hausdorff semi-quasi-complemented vector space. If M is an one-dimensional subspace of X, then M^σ is a maximal closed subspace of X. If, moreover, (X, σ) is a quasi-complemented vector space, and there exists $f \in X'$ with $\ker f = M^\sigma$, then $X = M \oplus M^\sigma$.*

Proof. M, as an one-dimensional subspace of the Hausdorff topological vector space X, is closed. Namely, $M \in \mathcal{V}_X$. Claim that M^σ is a maximal closed subspace of X. Indeed, if $N \in \mathcal{V}_X$ with $M^\sigma \subseteq N \subseteq X$, then

$$\{0\} = X^\sigma \subseteq N^\sigma \subseteq M^{\sigma\sigma} = M.$$

Thus, by the one-dimensionality of M, we get either $\{0\} = N^\sigma$ or $N^\sigma = M$, that yields either $N = X$ or $N = M^\sigma$, assuring the maximality of M^σ.

If $M = [x]$ (with $x \neq 0$), then $M^\sigma \neq [M^\sigma \cup \{x\}] = M \oplus M^\sigma$, and by the maximality of M^σ, we get $\overline{M \oplus M^\sigma} = X$. Take $z \in X$; then $z = \lim\limits_\delta z_\delta$ with $z_\delta = \lambda_\delta x + y_\delta, \delta \in \Delta$, where $y_\delta \in M^\sigma$ and $\lambda_\delta \in \mathbb{C}$, for every $\delta \in \Delta$. Then $f(z_\delta) = f(\lambda_\delta x + y_\delta) = \lambda_\delta f(x)$. Since $x \notin M^\sigma$, $f(x) \neq 0$, and thus $\lambda_\delta = (1/f(x))f(z_\delta)$. By the continuity of f, $\lim\limits_\delta \lambda_\delta = f(z)/f(x), z \in X$. By putting $\lambda = f(z)/f(x)$, we get $\lambda_\delta x \to \lambda x \in M$. Moreover,

$$y_\delta = z_\delta - \lambda_\delta x \underset{\delta}{\longrightarrow} z - \lambda x \text{ with } y_\delta \in \overline{M^\sigma} = M^\sigma.$$

Thus $z - \lambda x \in M^\sigma$, and therefore $z = \lambda x + (z - \lambda x) \in M \oplus M^\sigma$ that yields the assertion.

□

Proposition 3.2. *Let (X, σ) be a Hausdorff semi-quasi-complemented vector space. If M is an one-dimensional subspace of X and there exists $f \in X'$ with $\ker f = M^\sigma$, then the subspace $M^\tau = \{f \in X' : M^\sigma \subseteq \ker f\}$ of X' is one-dimensional.*

Proof. By Proposition 3.1, M^σ is a maximal closed subspace of X. To prove that the subspace M^τ is one-dimensional; take $f \in M^\tau$, and then the subspace $\ker f$ of X is closed, since f is continuous and contains M^σ. The maximality of M^σ yields either $\ker f = M^\sigma$ or $\ker f = X$. By hypothesis, there exists $f \in X'$ with $\ker f = M^\sigma$. But, $M^\sigma \neq X$. Thus $\ker f \neq X$, and hence $f \neq 0$. Namely,

$M^\tau \neq \{0\}$, and hence the choice of f above is meaningful. Choose $x \in X$ with $f(x) = 1$. Consider $g \in M^\tau$. If $g = 0$, we have nothing to prove. If $g \neq 0$, then $\ker f = M^\sigma = \ker g$. Since $f(x) = 1$ and $\ker f = M^\sigma$, we get that $x \notin M^\sigma$. Thus, by the maximality of the closed subspace M^σ, we get $X = \overline{[M^\sigma \cup \{x\}]}$. Therefore, for arbitrary $z \in X$, there are nets $(y_\delta)_{\delta\in\Delta}$ and $(\lambda_\delta)_{\delta\in\Delta}$ with $y_\delta \in M^\sigma$ and $\lambda_\delta \in \mathbb{C}$, for each $\delta \in \Delta$, such that if $z_\delta = y_\delta + \lambda_\delta x, \delta \in \Delta$, then $z = \lim_\delta z_\delta$. We have $g(z_\delta) = g(y_\delta + \lambda_\delta x) = \lambda_\delta g(x)$. Since $g(x) \neq 0$, we get $\lambda_\delta = (1/g(x))g(z_\delta)$, and by the continuity of g, $\lambda_\delta = \lim_\delta((1/g(x))g(z_\delta)) = g(z)/g(x)$. By putting $\lambda = \lim_\delta \lambda_\delta$, we get $g(z) = \lambda g(x)$. Moreover,

$$f(z) = \lim_\delta f(z_\delta) = (\lim_\delta \lambda_\delta) f(x) = \lambda f(x) = \lambda = g(z)/g(x).$$

Put $g(x) = \mu$. Then $g(z) = \mu f(z), z \in X$, and $g = \mu f$, which completes the proof.

□

The use of the symbol M^τ, in Proposition 3.2, is evident in the next proposition. Moreover, in the same proposition, (3.1) is, for instance, fulfilled in Banach spaces (see [8, p. 729, Theorem 1]). According to this condition, the quasi-complements of one-dimensional subspaces of a semi-quasi-complemented space are realized by the kernels of continuous linear forms. The same condition is the gist for the results of Theorems 3.6 and 3.10 that follow.

Proposition 3.3. *Let (X,σ) be a Hausdorff semi-quasi-complemented vector space. Suppose that the following condition holds.*

$$\begin{gathered} If\ \ M\ \ is\ \ an\ \ one-dimensional\ \ subspace\ \ of\ \ X, \\ then\ \ there\ \ exists\ \ f \in X'\ \ with\ \ \ker f = M^\sigma. \end{gathered} \tag{3.1}$$

Consider the correspondence τ from the one-dimensional subspaces of X into one-dimensional subspaces of X' with $\tau(M) = M^\tau$, where M is an one-dimensional subspace of X and $M^\tau = \{f \in X' : M^\sigma \subseteq \ker f\}$ of X'. Then τ is well defined, $1-1$, and onto. If, in particular, (X,σ) is quasi-complemented, then τ preserves the linearly independence.

Proof. From Proposition 3.2, τ is well defined. Let M_1 and M_2 be one-dimensional subspaces of X, ${M_1}^\sigma$ and ${M_2}^\sigma$ be their images through σ, and ${M_1}^\tau$ and ${M_2}^\tau$ be the respective one-dimensional subspaces in X' (see Proposition 3.2). Suppose that ${M_1}^\tau = {M_2}^\tau$, and take $f \in {M_1}^\tau$ with $f \neq 0$. Then ${M_1}^\sigma \subseteq \ker f$. Since $f \neq 0$ and the closed subspace ${M_1}^\sigma$ is maximal (see Proposition 3.1), we get ${M_1}^\sigma = \ker f$. Moreover, $f \in {M_2}^\tau$, whence ${M_2}^\sigma = \ker f$ and ${M_1}^\sigma = {M_2}^\sigma$. The latter gives $M_1 = M_2$. Therefore τ is $1-1$. Concerning the ontoness of τ, take an one-dimensional subspace $N = \{\lambda f : 0 \neq f \in X', \lambda \in \mathbb{C}\}$ of X'. Put $K = \ker f$, which obviously is a proper closed subspace of X. Claim that K is maximal. Choose $x \in X$ with $f(x) = 1$ and take $z \in X$. Put $y = z - f(z)x$, then $f(y) = 0$. Therefore $y \in K$, and since $x \notin K$, we get $z \in K \oplus [x]$. Therefore $X = K \oplus [x]$, assuring the maximality of K. Moreover, the closed subspace $M = K^\sigma$ of X is one-dimensional. Indeed, consider a finite-dimensional subspace M_1 of X with

$\{0\} \subseteq M_1 \subseteq M$. Then $K = M^\sigma \subseteq M_1^\sigma \subseteq X$. The maximality of K implies either $K = M_1^\sigma$ or $M_1^\sigma = X$. Hence $M_1 = K^\sigma = M$ or $M_1 = \{0\}$, which means that M is one-dimensional. Obviously, $M^\tau = N$ and the correspondence τ is onto.

By [8, p. 729, Theorem 1], τ preserves the linearly independence, in the sense of Definition 1.5. We present here the proof for two subspaces. Consider two linearly independent (one-dimensional) subspaces $M_1 = [x]$ and $M_2 = [y]$ of X, with $x, y \in X$ linearly independent. By the above proof, there are $f \neq 0$ and $g \neq 0$ with $M_1^\tau = [f]$ and $M_2^\tau = [g]$. Claim that M_1^τ and M_2^τ are linearly independent, otherwise, f and g would be linearly dependent elements of X'. Hence $\ker f = \ker g$, namely, $M_1^\sigma = M_2^\sigma$, whence $M_1^{\sigma\sigma} = M_2^{\sigma\sigma}$ and $M_1 = M_2$, which is a contradiction. □

It is true that, if X is a locally convex space, F a closed subspace of X and $x \notin F$, then there is $f \in X'$ with $f(x) = 1$ and $f(y) = (0)$ for every $y \in F$ (see e.g. [9, p. 233, §20, 1. (3)]). In the same vein, we get the next, as a realization of (3.1). Namely, the quasi-complements of one-dimensional subspaces are the kernels of continuous linear forms.

Theorem 3.4. *Every Hausdorff quasi-complemented locally convex space (X, σ) satisfies the condition (3.1).*

Proof. Let $M = [x]$ be an one-dimensional subspace of X. Since $M \cap M^\sigma = \{0\}$, we get that $x \notin M^\sigma$. By the closedness of M^σ and the comment preceding the statement, there exists $f \in X'$ with $f(x) = 1$ and $f(y) = 0$ for every $y \in M^\sigma$. The latter yields to $M^\sigma \subseteq \ker f$. But, M^σ is a maximal closed subspace of X (see Proposition 3.1) while, by the continuity of f, the subspace $\ker f$ is closed. Moreover, since $f \neq 0$, $\ker f \neq X$, and therefore $M^\sigma = \ker f$. □

Lemma 3.5. (Existence) *Let X and Y be infinite-dimensional vector spaces, and let $A \mapsto A'$ an $1-1$ correspondence between the one-dimensional subspaces of X and Y, respectively, that "respects" the linearly independence. Then there exists an $1-1$ semilinear correspondence $T : X \to Y$, such that if $A_x = [x]_X$, then $A_{T(x)} = A'_x$, where $A_{T(x)} = [T(x)]_Y$ and $A'_x = ([x]_X)'$.*

Proof. Apply [11, p. 245, Lemma A], adapted in the complex case. See also the proof of Theorem 1 in [8, p. 729]. □

Note. - In what follows, we make use of the following fact. *Let X be a vector space, and let $M = [x]$, $x \neq 0$, be an one-dimensional subspace of X that has a vector complement N in X, namely, $X = M + N$. If $x \notin N$, then the subspace N of X is maximal.* Indeed, suppose that K is a subspace of X with $N \subsetneq K$. If $x \in K$, then $M + K \subseteq K$. But $M + N \subseteq M + K$, and thus $K = X$. If $x \notin K$, then for $z \in K$, there are $\lambda \in \mathbb{C}, y \in N$ with $z = \lambda x + y$. Since $N \subseteq K$, $\lambda x = z - y \in K$, and since $x \notin K$, we get $\lambda = 0$. Therefore $z = y$, and hence $z \in N$. Namely, $K \subseteq N$, and finally $K = N$.

The next result generalizes and improves respective assertions in the proof of [ibid p. 729, Theorem 1].

Theorem 3.6. *Let (X, σ) be a Hausdorff quasi-complemented vector space that satisfies (3.1). Consider the correspondence τ from the one-dimensional subspaces* *of X to those of X' with $\tau(M) = M^\tau$, where M is an one-dimensional subspace of X and $M^\tau = \{f \in X' : M^\sigma \subseteq \ker f\}$. Then there exists an $1-1$ semilinear correspondence $T : X \to X'$, such that $T(M) = M^\tau$. In that case, T is onto, the subspace $T(M^\sigma)$ is a closed maximal subspace of X', and the following hold:*

$$T(M^\sigma) = \{f \in X' : M \subseteq \ker f\} \text{ and } X' = T(M) \oplus T(M^\sigma). \tag{3.2}$$

Moreover,

$$T(M \oplus M^\sigma) = T(M) \oplus T(M^\sigma). \tag{3.3}$$

Proof. By Proposition 3.3, τ is well defined $1-1$ and onto, preserving the linearly independence. Thus (Lemma 3.5) there is defined an $1-1$ semilinear correspondence, say, $T : X \to X'$. We prove the "onto" of the semilinear mapping T. Take $g \in X'$. If $g = 0$, then $g = T(0)$. Suppose that $g \neq 0$, and let $K = \ker g$ be the closed proper subspace of X, since $g \neq 0$. Choose $w \in X$, such that $g(w) = 1$. Take arbitrary $z \in X$, and put $y = z - g(z)w$. Then $g(y) = g(z) - g(z)g(w) = 0$, and hence $y \in K$. Therefore $z = y + g(z)w \in K \oplus [w]$, and thus $X = K \oplus [w]$. Hence K is a maximal subspace of X (see the note preceding Theorem 3.6). It is obvious that the subspace K^σ is one-dimensional (see also the proof of Proposition 3.3), so there exists $0 \neq x \in X$ with $K^\sigma = [x] = M_x$. Then, by Lemma 3.5, $[T(x)]_{X'} = \tau([x]_X) = [x]_X^\tau = M_x^\tau$. But, $T([x]_X) = T(\{\lambda x : \lambda \in \mathbb{C}\}) = \{\phi(\lambda)T(x) : \lambda \in \mathbb{C}\} = [T(x)]_{X'}$ where ϕ is the automorphism on $\mathbb{C}$ that corresponds to the semilinear map T. Therefore $T(M_x) = M_x^\tau$. Then (see also the condition (3.1)) $g \in M_x^\tau = T(M_x) \subseteq T(X)$. Therefore $X' \subseteq T(X)$ and hence $T(X) = X'$.

In what follows, put $M = M_x = [x], x \neq 0$. Take $y \in M_x^\sigma$ with $y \neq 0$, and put $T(y) = g \in X'$. Since $x \neq 0$, $T(x) \neq 0$. From

$$T(x) \in T(M_x) = [T(x)]_{X'} = M_x^\tau = \{f \in X' : M_x^\sigma \subseteq \ker f\}$$

and since M_x^σ is a maximal closed subspace of X, we get

$$T(x) \in \{f \in X' : M_x^\sigma = \ker f\}$$

, and thus $\ker T(x) = M_x^\sigma$. Now, from $T(x) \in M_x^\tau$, we get $T(x)(y) = 0$. Thus, if $M_y = [y]$, then $T(x)(M_y) = 0$, and

$$M_y \subseteq \ker T(x) = M_x^\sigma \text{ or } M_x = M_x^{\sigma\sigma} \subseteq M_y^\sigma.$$

But $\ker T(x) = M_x^\sigma$ is true for all $x \neq 0$; so, since $y \neq 0$, we also get the relation $\ker T(y) = M_y^\sigma$. Thus, by the above reasoning, $M_x \subseteq \ker T(y)$ (the last relation also holds when $y = 0$). Therefore $T(y) \in \{f \in X' : M_x \subseteq \ker f\}$. But, $T(y) \in T(M_x^\sigma)$ for all $y \in M_x^\sigma$, so we get $T(M_x^\sigma) \subseteq \{f \in X' : M_x \subseteq \ker f\}$.

To prove the inverse relation, we consider $f \in X'$ with $f \neq 0$ and $M_x \subseteq \ker f$. If $K = \ker f$, then K is a maximal closed subspace of X and the subspace K^σ is one-dimensional (see also the proof of Proposition 3.3). If $K^\sigma = M_z = [z]$ with $z \neq 0$, then $K = \ker f = M_z^\sigma$ and $T(M_z) = M_z^\tau = [f]$. Without loss of generality, we consider $T(z) = f$. Namely, $M_x \subseteq \ker f = M_z^\sigma$, that yields $M_z \subseteq M_x^\sigma$. Hence $z \in M_x^\sigma$. Therefore $f = T(z) \in T(M_x^\sigma)$, proving that

$$\{f \in X' : M_x \subseteq \ker f\} \subseteq T(M_x^\sigma).$$

So, finally

$$T(M^\sigma) = T(M_x^\sigma) = \{f \in X' : M_x \subseteq \ker f\}.$$

The subspace $T(M_x^\sigma)$ is closed. Indeed, take a net $(g_\delta)_{\delta\in\Delta} \subseteq T(M_x^\sigma)$ with $\lim_\delta g_\delta = g$, $g \in X'$. Since $M_x \subseteq \ker g_\delta$, for every $\delta \in \Delta$, we get $g_\delta(x) = 0$. Thus, we take in turn $g(x) = 0$, $x \in \ker g$, and $M_x \subseteq \ker g$, that yields $g \in T(M_x^\sigma)$. We claim that $T(M_x^\sigma)$ is maximal. To this end, consider $h \in X'$. Since T is onto, there exists $z \in X$ with $T(z) = h$. Since $X = M_x \oplus M_x^\sigma$ (see Proposition 3.1), there exist $\lambda \in \mathbb{C}$ and $y \in M_x^\sigma$, such that $z = \lambda x + y$. Therefore

$$h = T(z) = T(\lambda x + y) = T(\lambda x) + T(y) \in T(M_x) + T(M_x^\sigma).$$

Moreover, $X' \subseteq T(M_x) + T(M_x^\sigma)$, and thus $X' = T(M_x) + T(M_x^\sigma)$. The latter sum is direct, since if $f \in T(M_x) \cap T(M_x^\sigma)$, then $M_x^\sigma \subseteq \ker f$ and $M_x \subseteq \ker f$. Thus $M_x + M_x^\sigma \subseteq \ker f$. But $X = M_x \oplus M_x^\sigma$, whence $X = \ker f$, namely, $f = 0$, and so the aforementioned sum is direct. Namely, $X' = T(M_x) \oplus T(M_x^\sigma)$, and since $T(M_x)$ is one-dimensional, the subspace $T(M_x^\sigma)$ is maximal (see the Note, preceding Theorem 3.6). Besides, the ontoness of T gives $X' = T(X)$, and thus $X' = T(M_x \oplus M_x^\sigma)$, and finally, $T(M_x \oplus M_x^\sigma) = T(M_x) \oplus T(M_x^\sigma)$. Here, we remind that we have employed the symbol $M = M_x$. □

As a consequence of Theorems 3.4 and 3.6 we get the next.

Corollary 3.7. *Let (X, σ) be a Hausdorff quasi-complemented locally convex space. If τ is a mapping as in* Theorem 3.6*, then there exists an $1-1$ semilinear correspondence $T : X \to X'$, such that $T(M) = M^\tau$, which is onto, the subspace $T(M^\sigma)$ is a closed maximal subspace (of X'), and the equalities (3.2) and (3.3) hold true.*

For the following notion, we also refer to Lemma 3.5 and Definition 1.6.

Definition 3.8. Let X be an infinite-dimensional topological vector space and X' its dual. Suppose there exists an $1-1$ correspondence τ between the one-dimensional subspaces of X and X' that respects the linearly independence. If for any semilinear mapping $T : X \to X'$ with $\tau([x]_X) = [T(x)]_{X'}$ the automorphism ϕ, that corresponds to T, is continuous, then the pair (X, τ) is called *automorphically perfect*.

The previous definition is realized in the context of Banach vector spaces (see [8, p. 728, Lemma 2 and p. 729, the proof of Theorem 1]).

Remark 3.9. In the proof of Theorem 3.10 below, by the phrase "*adapting the operator T*", we mean the following. If $x \in X$ with $x \neq 0$ and $M_x = [x]$, then $M_x^\tau = \{f \in X' : M_x^\sigma \subseteq \ker f\}$ and for T we have $T([x]) = M_x^\tau$. If $T(x) = f$ and $f(x) = \alpha \in \mathbb{C} - \{0\}$, we consider the semilinear transformation $T_\alpha = \frac{1}{\alpha}T$, which has all the properties of T, and further

$$T_\alpha(x)(x) = \frac{1}{\alpha}T(x)(x) = \frac{1}{\alpha}f(x) = 1.$$

This adaptation satisfies the requirement $\langle x, x\rangle \in \mathbb{R}$.

We state now one of the main results which is a partial generalization of the Kakutani–Mackey Theorem (see Theorem 1.2) in the context of Banach complemented (in some sense) vector spaces. This result is a key to state an important result concerning continuous representations of appropriate complemented algebras (see [4]).

Theorem 3.10. *Let (X,σ) be a Hausdorff quasi-complemented vector space that satisfies (3.1). Suppose that the pair (X,τ) is automorphically perfect, where τ is the correspondence between the one-dimensional subspaces of the spaces X and X', as defined in* Proposition 3.3*. Then, on X, there is defined an inner product $\langle\cdot,\cdot\rangle$; that is, the pair $(X,\langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Moreover, σ is, finally, a correspondence between orthogonal complements, namely, for every closed subspace M of X, the following holds.*

$$M^\sigma = \{x\in X : \langle x,y\rangle = 0 \ \textit{for every}\ y\in M\}. \tag{3.4}$$

Proof. According to Proposition 3.3, τ is defined between the one-dimensional subspaces of the spaces X and X', respectively. Thus, in view of Lemma 3.5, there exists a semilinear 1-1 correspondence

$$T : X \to X', \text{ and such that } \tau([x]) = [T(x)], x\in X.$$

By the automorphically perfectness of the pair (X,τ), the respective automorphism (as in Definition 1.6) is continuous. But, the only continuous automorphisms of the field of complexes are the identity and the conjugate (see, e.g., [13, p. 5]). Thus T is linear or conjugate linear. Claim that the first case is impossible. Otherwise, if x and y are linearly independent elements in X, $T(x)=f$, and $T(y)=g$, then f and g are linearly independent elements in X', as well (see also Proposition 3.3). For every $\mu\in\mathbb{C}, x,y\in X$, we have $T(x+\mu y)=f+\mu g$ and

$$(f+\mu g)(x+\mu y) = f(x)+\mu(g(x)+f(y))+\mu^2 g(y).$$

Since $x\neq 0$, if $M_x=[x]$, then $T(M_x)=M_x^\tau$ and, according to (3.1), $f\in M_x^\tau$ and $\ker f = M_x^\sigma$. But, $M_x\cap M_x^\sigma=\{0\}$. So $x\notin M_x^\sigma$, and thus $f(x)\neq 0$. In analogy, we have $g(y)\neq 0$, since $0\neq y\in M_y=[y]$ and $T(y)=g\in M_y^\tau$. Again, in the same fashion, $0\neq x+\mu y\in M_{x+\mu y}$ and $T(x+\mu y)=f+\mu g\in M^\tau_{x+\mu y}$, that yield $(f+\mu g)(x+\mu y)\neq 0$. Since $g(y)\neq 0$, the equation

$$f(x)+\mu(g(x)+f(y))+\mu^2 g(y)=0$$

has, with respect to μ, a solution (in $\mathbb{C}$). Therefore $(f+\mu g)(x+\mu y)=0$, which is a contradiction, and thus T is conjugate linear.

Using the conjugate linearity of T, we define, on X, an inner product as follows: for any $x,y\in X$, we define

$$\langle x,y\rangle = f(x), \text{ where } f=T(y).$$

It is easily seen that $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{C}$ is well defined, linear in x, and conjugate linear in y. Claim now that $\langle\cdot,\cdot\rangle$ satisfies $\overline{\langle x,y\rangle}=\langle y,x\rangle$ for all $x,y\in X$. If $\langle x,y\rangle=0$, then $\langle y,x\rangle=0$. Indeed, in case $x=0$ or $y=0$, we have nothing to prove. If $x \neq 0$, $y \neq 0$, and M_x and M_y are the one-dimensional subspaces of X generated by x, y, respectively, we get

$$\langle x, y\rangle = 0 \Leftrightarrow f(x) = 0, f = T(y) \Leftrightarrow T(y)(x) = 0 \Leftrightarrow x \in \ker T(y) \Leftrightarrow M_x \subseteq M_y^\sigma,$$

whence

$$M_y \subseteq M_x^\sigma \Leftrightarrow y \in \ker T(x) \Leftrightarrow T(x)(y) = 0 \Leftrightarrow \langle y, x\rangle = 0.$$

Now, take $x \in X$ with $x \neq 0$. We have

$$M_x = \{\lambda x : \lambda \in \mathbb{C}\} \quad \text{and} \quad T(M_x) = M_x^\tau.$$

Since M_x^σ is a closed maximal subspace of X (see Proposition 3.1) we have

$$M_x^\tau = \{f \in X' : \ker f = X \ \textit{ or } \ \ker f = M_x^\sigma\}.$$

If $\langle x, x\rangle = 0$, then

$$f(x) = 0, f = T(x) \Leftrightarrow T(x)(x) = 0 \Leftrightarrow x \in \ker T(x) \Leftrightarrow M_x \subseteq M_x^\sigma.$$

Therefore $M_x \subseteq M_x \cap M_x^\sigma = \{0\}$, a contradiction. The previous argumentation yields $\langle x, x\rangle = 0$ if and only if $x = 0$.

Now let x and y be arbitrary elements in X with $\langle x, y\rangle \neq 0$. Then there exist $\lambda, \mu \in \mathbb{C} - \{0\}$ such that the following hold.

$$\lambda\langle x, x\rangle + \langle x, y\rangle = 0 \ \text{ and} \tag{3.5}$$

$$\mu\langle y, y\rangle + \langle x, y\rangle = 0. \tag{3.6}$$

From (3.5) and the fact that $\langle x, y\rangle = 0$ if and only if $\langle y, x\rangle = 0$, we have

$$\langle x, \overline{\lambda}x\rangle + \langle x, y\rangle = 0 \Leftrightarrow \langle x, \overline{\lambda}x + y\rangle = 0 \Leftrightarrow \langle \overline{\lambda}x + y, x\rangle = 0 \Leftrightarrow \overline{\lambda}\langle x, x\rangle + \langle y, x\rangle = 0.$$

Therefore

$$\lambda\overline{\langle x, x\rangle} + \overline{\langle y, x\rangle} = 0. \tag{3.7}$$

Similarly, from (3.6), it follows

$$\mu\overline{\langle y, y\rangle} + \overline{\langle y, x\rangle} = 0. \tag{3.8}$$

By the relations (3.5)-(3.8), we get $\lambda\langle x, x\rangle = \mu\langle y, y\rangle$ and $\lambda\overline{\langle x, x\rangle} = \mu\overline{\langle y, y\rangle}$. Therefore $\langle x, x\rangle / \overline{\langle x, x\rangle} = \langle y, y\rangle / \overline{\langle y, y\rangle}$, which means that, if $\langle z, z\rangle$ is a real number for some $z \in X$, then the number $\langle w, w\rangle$ is real, as well, for every $w \in X$ with $\langle z, w\rangle \neq 0$. Take $z \neq 0$ with $\langle z, z\rangle \in \mathbb{R}$ and $\langle z, w\rangle = 0$. Then $\langle w, z\rangle = 0$, and thus $T(z)(w) = 0$ if and only if $w \in \ker T(z)$. But, $\ker T(z) = M_z^\sigma$, and thus $w \in M_z^\sigma$, whence $z + w \notin M_z^\sigma$, otherwise, $z \in M_z^\sigma$, which is a contradiction, since $z \neq 0$. Hence

$$T(z)(z + w) \neq 0 \Leftrightarrow \langle z + w, z\rangle \neq 0,$$

and thus $\langle z, z + w\rangle \neq 0$. The latter yields $\langle z + w, z + w\rangle \in \mathbb{R}$. Therefore $\langle z + w, z + w\rangle - \langle z, z\rangle \in \mathbb{R}$, namely, $\langle w, w\rangle \in \mathbb{R}$. By adapting the operator T, in the sense of Remark 3.9, we can choose $z \in X$ with $z \neq 0$ and $T(z)(z) \in \mathbb{R}$, that means $\langle z, z\rangle \in \mathbb{R}$. Thus we may suppose that $\langle x, x\rangle \in \mathbb{R}$, for every $x \in X$. We prove that the numbers $\langle x, x\rangle$ have the same sign for every nonzero element

$x \in X$. Indeed, suppose there exist nonzero elements $x, y \in X$ with $\langle x, x\rangle > 0$ and $\langle y, y\rangle < 0$. For every real number λ, we have

$$\langle \lambda x + y, \lambda x + y\rangle = \lambda^2\langle x, x\rangle + \lambda\langle x, y\rangle + \lambda\langle y, x\rangle + \langle y, y\rangle.$$

But, by (3.5) and (3.7), we get $\overline{\langle x, y\rangle} = \langle y, x\rangle$, whence it is easy to see that the trinomial

$$\lambda^2\langle x, x\rangle + \lambda(\langle x, y\rangle + \overline{\langle x, y\rangle}) + \langle y, y\rangle$$

has (a real) root. Namely, there exists nonzero $\lambda \in \mathbb{R}$ with $\langle \lambda x + y, \lambda x + y\rangle = 0$. So, as we have mentioned $y = -\lambda x$. Therefore $\langle y, y\rangle = \langle -\lambda x, -\lambda x\rangle = \lambda^2\langle x, x\rangle$, which obviously, is a contradiction. After the adaptation of T, if $\langle z, z\rangle < 0$ then we choose as T, the operator $-T$, and thus we have $\langle z, z\rangle > 0$, and so $\langle x, x\rangle > 0$ for every nonzero $x \in X$, which completes the assertion that $\langle\cdot,\cdot\rangle$ is a hermitian inner product. We finally prove that σ is a correspondence between orthogonal complements on $\mathcal{V}_X$. For $M \in \mathcal{V}_X$, we have

$$\begin{aligned} &\{x \in X : \langle x, y\rangle = 0 \text{ for every } y \in M\} \\ &= \{x \in X : T(y)(x) = 0 \text{ for every } y \in M\} \\ &= \bigcap_{y\in M}\{x \in X : T(y)(x) = 0\} = \bigcap_{y\in M}\ker T(y) = \bigcap_{y\in M} M_y^\sigma. \end{aligned}$$

Claim that $\bigcap_{y\in M} M_y^\sigma = M^\sigma$. Indeed, for $y \in M$ we have $M_y \subseteq M$, whence $M^\sigma \subseteq \bigcap_{y\in M} M_y^\sigma$. For the inverse inclusion, take $x \in \bigcap_{y\in M} M_y^\sigma$, then $M_x \subseteq M_y^\sigma$ for every $y \in M$, and thus $M_y \subseteq M_x^\sigma$ for every $y \in M$. Namely, $y \in M_x^\sigma$ for every $y \in M$ or $M \subseteq M_x^\sigma$. So, we have $M_x \subseteq M^\sigma$ and $x \in M^\sigma$, and, finally, $\bigcap_{y\in M} M_y^\sigma \subseteq M^\sigma$. The previous argument shows that

$$M^\sigma = \{x \in X : \langle x, y\rangle = 0 \text{ for every } y \in M\}.$$

□

As a direct consequence of Theorems 3.4 and 3.10, we get the next.

Corollary 3.11. *Let (X, σ) be a Hausdorff quasi-complemented locally convex space, such that (X, τ) is automorphically perfect, where τ is the correspondence, as defined in* Proposition 3.3*. Then an inner product $\langle\cdot,\cdot\rangle$ is defined on X; that is, the pair $(X, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Moreover, σ is a correspondence between orthogonal complements, namely,* (3.4) *holds.*

Based on Theorem 3.10 and laying on Remark 3.9, we are in position to define a family of appropriate semilinear mappings from a quasi-complemented (vector) space to its dual. The last family is further used to equip the pseudo-complemented space with a family of inner products, making it a locally convex one. Namely, we have the next.

Proposition 3.12. *Let (X, σ) and (X, τ) be pairs as in* Theorem 3.10. *Then a family $(\langle\cdot,\cdot\rangle_i)_{i\in I}$ of inner products is defined on X, such that I is the set of positive real numbers and, for every $i \in I$, the pair $(X, \langle\cdot,\cdot\rangle_i)$ is a pre-Hilbert* *space and σ is a correspondence between orthogonal complements, namely, the respective (3.4) holds with respect to the inner product $\langle\cdot,\cdot\rangle_i$.*

Moreover, the topological space $(X, (\|\cdot\|_i)_{i\in I})$ is locally convex, where $\|\cdot,\cdot\|_i$ is the norm on X, induced from the inner product $\langle\cdot,\cdot\rangle_i$, $i \in I$.

Proof. In view of Remark 3.9, there exists a conjugate semilinear transformation $T_\alpha : X \to X'$ and $x \in X$ with $T_\alpha(x)(x) = 1$. For any positive real number i, we consider the semilinear transformation

$$T_i : X \to X' \text{ with } T_i = iT_\alpha.$$

For $i \in I$, an inner product $\langle\cdot,\cdot\rangle_i$ is defined on X by $\langle x, y\rangle_i = T_i(y)(x)$. From Theorem 3.10, it follows that σ is a correspondence between orthogonal complements, namely the respective (3.4) holds with respect to the inner product $\langle\cdot,\cdot\rangle_i$. Of course, the topological space $(X, (\|\cdot\|_i)_{i\in I})$ is locally convex. □

Theorem 3.4 and Proposition 3.12 give the next.

Corollary 3.13. *Let (X, σ) be a Hausdorff quasi-complemented space. Consider the assertions.*

(1) *(X, σ) is a locally convex space.*
(2) *(X, σ) satisfies (3.1).*

Then (1) implies (2). If moreover, the pair (X, τ) is automorphically perfect, where τ is the correspondence, as in Proposition 3.3, *then (2) implies (1).*

Theorem 3.14. *Let $(X_i, \sigma_i)_{i\in I}$ be a family of Hausdorff quasi-complemented spaces, each member of which satisfies (3.1). Suppose that the pair (X_i, τ_i) is automorphically perfect, where τ_i is the correspondence between the one-dimensional subspaces of the spaces X_i and X_i', $i \in I$, as defined in* Proposition 3.3. *Then on $X = \prod_{i\in I} X_i$, a family $([\cdot,\cdot]_i)_{i\in I}$ of pseudo-inner products is defined, such that X is a pseudo-H-space.*

Proof. By Theorem 3.10, for each $i \in I$, an inner product, say, $\langle\cdot,\cdot\rangle_i$, is defined on X_i, such that σ_i is turn to be a correspondence between orthogonal complements. We define the mapping

$$[\cdot,\cdot]_i : \prod_{i\in I} X_i \times \prod_{i\in I} X_i \to \mathbb{C} \text{ with } [(x_j), (y_j)]_i = \langle x_i, y_i\rangle_i,$$

which, as it is easily seen, is a pseudo-inner product on X. Thus we have a family $([\cdot,\cdot]_i)_{i\in I}$ of pseudo-inner products, that make X a pseudo-H-space (see Definition 1.7). □

For the notion of a *projective system of topological vector spaces*, see for example [5, p. 155].

Proposition 3.15. *Let $\{((X_i, \sigma_i), f_{ij})\}_{i\in I}$ be a projective system of Hausdorff quasi-complemented vector spaces, where each (X_i, σ_i) satisfies (3.1). Suppose that the pair (X_i, τ_i), $i \in I$, is as in* Theorem 3.14. *Then on $X = \varprojlim_i X_i$, a family $([\cdot,\cdot]_i)_{i\in I}$ of pseudo-inner products is defined, such that X is a pseudo-H-space.*

Proof. The proof goes along the same fashion as that of Theorem 3.14. □

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