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ORTHOGONALITY OF BOUNDED LINEAR OPERATORS ON COMPLEX BANACH SPACES

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Abstract. We study Birkhoff–James orthogonality of compact linear operators on complex reflexive Banach spaces and obtain its characterization. By means of introducing new definitions, we illustrate that it is possible in the complex case, to develop a study of orthogonality of compact linear operators, analogous to the real case. Furthermore, earlier operator theoretic characterizations of Birkhoff–James orthogonality in the real case, can be obtained as simple corollaries to our present study. In fact, we obtain more than one equivalent characterizations of Birkhoff–James orthogonality of compact linear operators in the complex case, in order to distinguish the complex case from the real case.

1. Introduction.

The notion of Birkhoff–James orthogonality (B–J orthogonality) plays a very important role in the geometry of Banach spaces. In [7], James illustrated the role of B–J orthogonality in characterizing geometric properties like smoothness, strict convexity, and other properties of the space. It is quite straightforward to observe that the notion of B–J orthogonality extends to the space of all bounded linear operators on a Banach space. The role of B–J orthogonality in the study of geometry of Banach spaces has been explored by several researchers, from various points of view. We refer the readers to [1, 3, 4, 5, 6, 8, 14, 15], and the references therein, for a detailed study in this regard. Recently, in [10], Sain characterized

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B–J orthogonality of linear operators on finite-dimensional real Banach spaces. Although B–J orthogonality can be defined for either real or complex Banach spaces, till now most of the operator theoretic study of B–J orthogonality [10, 12] has been conducted exclusively in the context of real Banach spaces. In this paper, our aim is to initiate an analogous study of B–J orthogonality of linear operators in the complex case and to obtain its characterization. It is interesting to observe that the results already known in the context of real Banach spaces follow quite easily from these new results. It is in this sense, that our present study can be considered as an extension of our earlier studies [10]. Without further ado, let us establish the relevant notations and terminologies to be used throughout the paper.

Let $\mathbb{X}$ and $\mathbb{Y}$ be complex Banach spaces. Let $B_{\mathbb{X}} = \{x \in \mathbb{X}\colon \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X}\colon \|x\| = 1\}$ be the unit ball and the unit sphere of $\mathbb{X}$, respectively. Let $\mathbb{L}(\mathbb{X},\mathbb{Y})(\mathbb{K}(\mathbb{X},\mathbb{Y}))$ denote the Banach space of all bounded (compact) linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. We write $\mathbb{L}(\mathbb{X},\mathbb{Y}) = \mathbb{L}(\mathbb{X})$ and $\mathbb{K}(\mathbb{X},\mathbb{Y}) = \mathbb{K}(\mathbb{X})$ if $\mathbb{X} = \mathbb{Y}$.

For any two elements $x, y \in \mathbb{X}$, x is said to be B–J orthogonal to y [2, 7], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

Similarly, for any two elements $T, A \in \mathbb{L}(\mathbb{X})$, T is said to be B–J orthogonal to A, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{C}$.

For a linear operator T defined on a Banach space $\mathbb{X}$, let M_T denote the collection of all unit vectors in $\mathbb{X}$ at which T attains norm; that is,

$$M_T = \{x \in S_{\mathbb{X}}\colon \|Tx\| = \|T\|\}.$$

In order to characterize B–J orthogonality of bounded linear operators on finite-dimensional real Banach spaces, Sain [10] introduced the notions of x^+ and x^- in the following way:
For any two elements x and y in a real Banach space $\mathbb{X}$, let us say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Following similar motivations, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. Using these notions, Sain [10] characterized B–J orthogonality of linear operators defined on finite-dimensional real Banach spaces.

Theorem 1.1. [10, Theorem 2.2] *Let $\mathbb{X}$ be a finite-dimensional real Banach space. Let $T, A \in \mathbb{L}(\mathbb{X})$. Then $T \perp_B A$ if and only if there exist $x, y \in M_T$ such that $Ax \in Tx^+$ and $Ay \in Ty^-$.*

In this paper, in order to obtain an analogous result for complex Banach spaces, let us introduce the following notions:

Let $x \in \mathbb{X}$ and $U = \{\alpha \in \mathbb{C} : |\alpha| = 1,\ \arg\alpha \in [0, \pi)\}$. For $\alpha \in U$ define

$$x_\alpha^+ = \{y \in \mathbb{X} : \|x + \lambda y\| \geq \|x\| \ \forall\ \lambda = t\alpha, t \geq 0\},$$

$$x_\alpha^- = \{y \in \mathbb{X} : \|x + \lambda y\| \geq \|x\| \ \forall\ \lambda = t\alpha, t \leq 0\},$$

$$x_\alpha^\perp = \{y \in \mathbb{X} : \|x + \lambda y\| \geq \|x\| \ \forall\ \lambda = t\alpha, t \in \mathbb{R}\}.$$

If $\beta = e^{i\pi}\alpha$, then we define $x_\beta^+ = x_\alpha^-$, $x_\beta^- = x_\alpha^+$, and $x_\beta^\perp = x_\alpha^\perp$.
If $y \in x_\alpha^\perp$, then we write $x \perp_\alpha y$. Let us define the notions of x^+, x^-, and $x^\perp$ in a complex Banach space in the following way:

$$x^+ = \bigcap\{x_\alpha^+ : \alpha \in U\},\ x^- = \bigcap\{x_\alpha^- : \alpha \in U\},\ \text{and } x^\perp = \bigcap\{x_\alpha^\perp : \alpha \in U\}.$$

If the space $\mathbb{X}$ is a real Banach space, then we must have, $\alpha \in U$ implies that $\alpha = 1$. Therefore, $x_\alpha^+ = x^+$, $x_\alpha^- = x^-$, and $x_\alpha^\perp = x^\perp$.

In this paper, we completely characterize B–J orthogonality of compact linear operators from a complex reflexive Banach space to a complex Banach space. In order to illustrate the importance of our study, we show that earlier characterizations of operator B–J orthogonality [10, 13] in the real case follow as simple corollaries to our present study.

2. Main Results

Let us begin with two easy propositions, that would be useful in obtaining the desired characterization of B–J orthogonality of bounded linear operators between complex Banach spaces.

Proposition 2.1. *Let $\mathbb{X}$ be a complex Banach space, and consider $x, y \in \mathbb{X}$ and $\alpha \in U$. Then the following are true:*

(i) *Either $y \in x_\alpha^+$ or $y \in x_\alpha^-$.*
(ii) *$x \perp_\alpha y$ if and only if $y \in x_\alpha^+$ and $y \in x_\alpha^-$.*
(iii) *$y \in x_\alpha^+$ implies that $\eta y \in (\mu x)_\alpha^+$ for all $\eta, \mu > 0$.*
(iv) *$y \in x_\alpha^+$ implies that $-y \in x_\alpha^-$ and $y \in (-x)_\alpha^-$.*
(v) *$y \in x_\alpha^-$ implies that $\eta y \in (\mu x)_\alpha^-$ for all $\eta, \mu > 0$.*
(vi) *$y \in x_\alpha^-$ implies that $-y \in x_\alpha^+$ and $y \in (-x)_\alpha^+$.*
(vii) *$y \in x_\alpha^+$ implies that $\beta y \in (\beta x)_\alpha^+$ for all $\beta \in \mathbb{C}$.*
(viii) *$y \in x_\alpha^-$ implies that $\beta y \in (\beta x)_\alpha^-$ for all $\beta \in \mathbb{C}$.*

Proof. (i) If $y \notin x_\alpha^+$, then we show that $y \in x_\alpha^-$. Since $y \notin x_\alpha^+$, we have $\|x + \lambda_0 y\| < \|x\|$ for some $\lambda_0 = t_0\alpha$ with $t_0 > 0$. Let $\lambda = t\alpha$ with $t < 0$. Then there exists $s \in [0, 1]$ such that

$$\begin{aligned} & x && = s(x + \lambda_0 y) + (1 - s)(x + \lambda y) \\ \Rightarrow\ & \|x\| && \leq s\|x + \lambda_0 y\| + (1 - s)\|x + \lambda y\| \\ \Rightarrow\ & \|x\| && < s\|x\| + (1 - s)\|x + \lambda y\| \\ \Rightarrow\ & \|x\| && < \|x + \lambda y\|. \end{aligned}$$

Therefore, $\|x\| \leq \|x + \lambda y\|\ \forall \lambda = t\alpha$ with $t \leq 0 \Rightarrow y \in x_\alpha^-$.

The proofs of (ii)–(viii) can be easily completed using similar approach. □

Proposition 2.2. *Let $\mathbb{X}$ be a complex Banach space, and let $x, y \in \mathbb{X}$. Then the following are true:*

(i) *$x \perp_B y$ if and only if $y \in x^+$ and $y \in x^-$.*
(ii) *$y \in x^+$ implies that $\eta y \in (\mu x)^+$ for all $\eta, \mu > 0$.*
(iii) *$y \in x^+$ implies that $-y \in x^-$ and $y \in (-x)^-$.*

(iv) $y \in x^-$ *implies that* $\eta y \in (\mu x)^-$ *for all* $\eta, \mu > 0$.
(v) $y \in x^-$ *implies that* $-y \in x^+$ *and* $y \in (-x)^+$.

Proof. (i) The proof follows from the definitions of x^+ and x^-.

(ii) Let $y \in x^+$. Then $y \in x^+_\alpha$ for each α with $\arg\alpha \in [0, \pi)$. We show that $\eta y \in (\mu x)^+_\alpha$ for each α with $\arg\alpha \in [0, \pi)$. Now,

$$\|\mu x + (t\alpha)\eta y\| = |\mu|\|x + (\frac{t\eta}{\mu})\alpha y\| \geq |\mu|\|x\| = \|\mu x\|, \qquad \text{for all } t, \mu, \eta > 0,$$

and so $\eta y \in (\mu x)^+_\alpha$ for all $\mu, \eta > 0$. Thus, $\eta y \in (\mu x)^+$ for all $\mu, \eta > 0$.

(iii) Suppose that $y \in x^+$. Then for each α with $\arg\alpha \in [0, \pi)$, $\|x + t\alpha y\| \geq \|x\|$ for all $t \geq 0$. So $\|x + (-t)\alpha(-y)\| \geq \|x\|$ for all $t \geq 0$. This shows that $-y \in x^-_\alpha$ for each α with $\arg\alpha \in [0, \pi)$, and so $-y \in x^-$.

Again, for each α with $\arg\alpha \in [0, \pi)$,

$$\|x + t\alpha y\| = \|(-x) + (-t)\alpha(y)\| \geq \|x\| = \| - x\|, \qquad \text{for all } t \geq 0.$$

This shows that $y \in (-x)^-_\alpha$ for each α with $\arg\alpha \in [0, \pi)$, and therefore $y \in (-x)^-$.

(iv) Follows similarly as (ii).

(v) Follows similarly as (iii). ☐

Let us now obtain the promised characterization theorem, the proof of which follows the same line of argument given in [13, Th. 2.1]. For the sake of completeness of the paper we give the proof in details here.

Theorem 2.3. *Let* $\mathbb{X}$ *be a reflexive complex Banach space, and let* $\mathbb{Y}$ *be any complex Banach space. Let* $T, A \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$. *Then* $T \perp_B A$ *if and only if for each* $\alpha \in U$ *there exist* $x = x(\alpha),\ y = y(\alpha) \in M_T$ *such that* $Ax \in (Tx)^+_\alpha$ *and* $Ay \in (Ty)^-_\alpha$.

Proof. Let us first prove the sufficient part. Suppose, for each $\alpha \in U$, there exist $x = x(\alpha),\ y = y(\alpha) \in M_T$ such that $Ax \in (Tx)^+_\alpha$ and $Ay \in (Ty)^-_\alpha$. Let $\lambda \in \mathbb{C}$. Then there exist $t \in \mathbb{R}$ and $\alpha \in U$ such that $\lambda = t\alpha$. If $t \geq 0$, then $\|T + \lambda A\| = \|T + t\alpha A\| \geq \|(T + (t\alpha)A)x\| \geq \|Tx\| = \|T\|$, and if $t \leq 0$, then $\|T + \lambda A\| = \|T + t\alpha A\| \geq \|(T + (t\alpha)A)y\| \geq \|Ty\| = \|T\|$. Hence, $T \perp_B A$. This completes the proof of the sufficient part of the theorem.

Let us now prove the necessary part. Suppose that $T \perp_B A$. Let $\alpha \in U$. Then for each $n \in \mathbb{N}$, the operator $(T + \frac{\alpha}{n}A)$, being compact on a reflexive complex normed linear space, attains norm. Therefore, there exists $x_n \in S_{\mathbb{X}}$ such that $\|(T + \frac{\alpha}{n}A)\| = \|(T + \frac{\alpha}{n}A)x_n\|$. Now, since $\mathbb{X}$ is reflexive, $B_{\mathbb{X}}$ is weakly compact. Therefore, $\{x_n\}$ has a subsequence, say, $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ weakly converges to $x = x(\alpha)$ (say) in $B_{\mathbb{X}}$. Without loss of generality we assume that $\{x_n\}$ weakly converges to x. Then T and A being compact, $Tx_n \to Tx$ and $Ax_n \to Ax$. Since $T \perp_B A$, we have $\|T + \frac{\alpha}{n}A\| \geq \|T\|$ for all $n \in \mathbb{N}$. Hence

$$\|Tx_n + \frac{\alpha}{n}Ax_n\| \geq \|T\| \geq \|Tx_n\| \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ we have $\|Tx\| \geq \|T\| \geq \|Tx\|$. Therefore, $x \in M_T$. Finally, we show that $Ax \in (Tx)^+_\alpha$.

For any $t > \frac{1}{n} > 0$, we claim that $\|Tx_n + t\alpha Ax_n\| \geq \|Tx_n\|$. If possible, suppose that $\|Tx_n + t\alpha Ax_n\| < \|Tx_n\|$. Then

$$\begin{aligned}
Tx_n + \frac{\alpha}{n}Ax_n &= (1 - \frac{1}{nt})Tx_n + \frac{1}{nt}(Tx_n + t\alpha Ax_n) \\
\Rightarrow \|Tx_n + \frac{\alpha}{n}Ax_n\| &\leq (1 - \frac{1}{nt})\|Tx_n\| + \frac{1}{nt}\|(Tx_n + t\alpha Ax_n)\| \\
\Rightarrow \|Tx_n + \frac{\alpha}{n}Ax_n\| &< (1 - \frac{1}{nt})\|Tx_n\| + \frac{1}{nt}\|Tx_n\| \\
\Rightarrow \|Tx_n + \frac{\alpha}{n}Ax_n\| &< \|Tx_n\|,
\end{aligned}$$

a contradiction. This proves our claim.
Now, for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \frac{1}{n_0}$. Hence, for all $n \geq n_0$, we have,

$$\|Tx_n + t\alpha Ax_n\| \geq \|Tx_n\|.$$

Letting $n \longrightarrow \infty$, we have,

$$\|Tx + t\alpha Ax\| \geq \|Tx\|.$$

Therefore, $Ax \in (Tx)^+_\alpha$.
Similarly, considering the operator $(T - \frac{\alpha}{n}A)$, for each $n \in \mathbb{N}$, we obtain $y = y(\alpha)$ in M_T such that $Ay \in (Ty)^-_\alpha$. This completes the proof of the theorem. □

In particular, if $\mathbb{X}$ and $\mathbb{Y}$ are finite-dimensional complex Banach spaces, then we have the following corollary.

Corollary 2.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be finite-dimensional complex Banach spaces. Let $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Then $T \perp_B A$ if and only if, for each $\alpha \in U$, there exist $x = x(\alpha)$ and $y = y(\alpha)$ in M_T such that $Ax \in (Tx)^+_\alpha$ and $Ay \in (Ty)^-_\alpha$.*

Proof. Since every finite-dimensional complex Banach space is reflexive and every linear operator on a finite-dimensional complex Banach space is compact, the proof of the corollary follows from Theorem 2.3. □

We would further like to comment that the proofs of the corresponding characterization theorems in the real case are now obvious.

Corollary 2.5. [13, Theorem 2.1] *Let $\mathbb{X}$ be a reflexive real Banach space, and let $\mathbb{Y}$ be any real Banach space. Let $T, A \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$. Then $T \perp_B A$ if and only if there exist $x, y \in M_T$ such that $Ax \in (Tx)^+$ and $Ay \in (Ty)^-$.*

Proof. Let $T \perp_B A$. Since in real Banach space, $\alpha \in U$ implies that $\alpha = 1$, by Theorem 2.3, there exist $x, y \in M_T$ such that $Ax \in (Tx)^+$ and $Ay \in (Ty)^-$. □

Corollary 2.6. *(Theorem 2.2 of* [10]*) Let $\mathbb{X}, \mathbb{Y}$ be finite-dimensional real Banach spaces. Let $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Then $T \perp_B A$ if and only if there exist $x, y \in M_T$ such that $Ax \in (Tx)^+$ and $Ay \in (Ty)^-$.*

Proof. Since every finite-dimensional complex Banach space is reflexive and every linear operator on a finite-dimensional complex Banach space is compact, the proof of the corollary follows from Corollary 2.5. □

In spite of being a complete characterization of B–J orthogonality of compact linear operators on a reflexive complex Banach space, Theorem 2.3 does not capture the full strength of the complex number system. Indeed, in our opinion, Theorem 2.3 should be regarded as a stepping stone towards our next theorem, that also distinguishes the complex case from the real case. First we need the following geometric lemma.

Lemma 2.7. *Let $\mathbb{X}$ be a complex Banach space. Let $x,\ y \in \mathbb{X}$ and $\alpha = e^{i\theta}$, where $\theta \in [0, \pi]$. If $y \in x^+_\alpha$, then either $y \in x^+_\beta$ for all β with $\arg\beta \in [0, \theta]$ or $y \in x^+_\beta$ for all β with $\arg\beta \in [\theta, \pi]$.*

Proof. Suppose that $y \notin x^+_{\alpha_1}$ for some α_1 with $\arg\alpha_1 \in [0, \theta]$. Then there exists $t_1 > 0$ such that $\|x + t_1\alpha_1 y\| < \|x\|$. We claim that $y \in x^+_\beta$ for all β with $\arg\beta \in [\theta, \pi]$. If possible suppose that $y \notin x^+_{\alpha_2}$ for some α_2 with $\arg\alpha_2 \in [\theta, \pi]$. Then there exists $t_2 > 0$ such that $\|x + t_2\alpha_2 y\| < \|x\|$. Then it is easy to verify that there exist $0 < s < 1$ and $t > 0$ such that $(1-s)t_1\alpha_1 + st_2\alpha_2 = t\alpha$. Therefore, $(1-s)[x + t_1\alpha_1 y] + s[x + t_2\alpha_2 y] = x + t\alpha y$. This implies that

$$\|x + t\alpha y\| \leq (1-s)\|x + t_1\alpha_1 y\| + s\|x + t_2\alpha_2 y\| < (1-s)\|x\| + s\|x\| = \|x\|,$$

which is a contradiction. This proves our claim. $\square$

Let us now prove the following characterization theorem, that improves the necessary part of Theorem 2.3.

Theorem 2.8. *Let $\mathbb{X}$ be a reflexive complex Banach space, and let $\mathbb{Y}$ be any complex Banach space. Let $T, A \in K(\mathbb{X}, \mathbb{Y})$. Then $T \perp_B A$ if and only if there exist $x,\ y,\ z,$ and w in M_T and $\phi_1, \phi_2 \in [0, \pi]$ such that*

(i) $Ax \in (Tx)^+_\alpha\ \forall\ \alpha$ *with* $\arg\alpha \in [0, \phi_1]$,

(ii) $Ay \in (Ty)^+_\alpha\ \forall\ \alpha$ *with* $\arg\alpha \in [\phi_1, \pi]$,

(iii) $Az \in (Tz)^-_\alpha\ \forall\ \alpha$ *with* $\arg\alpha \in [0, \phi_2]$,

(iv) $Aw \in (Tw)^-_\alpha\ \forall\ \alpha$ *with* $\arg\alpha \in [\phi_2, \pi]$.

Proof. We first prove the easier sufficient part. Suppose that there exist $x,\ y,\ z,$ and w in M_T and $\phi_1, \phi_2 \in [0, \pi]$ such that all the conditions in (i), (ii), (iii), and (iv) are satisfied. Let $\lambda \in \mathbb{C}$. Then one of the following conditions hold:
(a)There exist $t_1 \geq 0$ and α_1 with $\arg\alpha_1 \in [0, \phi_1]$ such that $\lambda = t_1\alpha_1$.
(b)There exist $t_2 \geq 0$ and α_2 with $\arg\alpha_2 \in [\phi_1, \pi]$ such that $\lambda = t_2\alpha_2$.
(c)There exist $t_3 \leq 0$ and α_3 with $\arg\alpha_3 \in [0, \phi_2]$ such that $\lambda = t_3\alpha_3$.
(d)There exist $t_4 \leq 0$ and α_4 with $\arg\alpha_4 \in [\phi_2, \pi]$ such that $\lambda = t_4\alpha_4$.

Now $\lambda = t_1\alpha_1$ implies that

$$\|T + \lambda A\| = \|T + t_1\alpha_1 A\| \geq \|Tx + t_1\alpha_1 Ax\| \geq \|Tx\| = \|T\|.$$

Similarly, in the other cases, it can be shown that $\|T + \lambda A\| \geq \|T\|$. Hence $T \perp_B A$.
For the necessary part, suppose that $T \perp_B A$. Then from Theorem 2.3, we have,

for each α with $\arg\alpha \in [0,\pi]$, there exists $x_\alpha \in M_T$ such that $Ax_\alpha \in (Tx_\alpha)^+_\alpha$. Now, consider

$$V_1 = \{\theta \in [0,\pi] : \exists\, x \in M_T \ s.t.\ Ax \in (Tx)^+_\alpha \ \forall\ \alpha \ with\ \arg\alpha \in [0,\theta]\},$$

$$V_2 = \{\theta \in [0,\pi] : \exists\, x \in M_T \ s.t.\ Ax \in (Tx)^+_\alpha \ \forall\ \alpha \ with\ \arg\alpha \in [\theta,\pi]\}.$$

Clearly, $0 \in V_1$ and $\pi \in V_2$, and therefore, V_1 and V_2 are nonempty. Moreover, V_1 and V_2 are bounded. Suppose that $\xi = \sup V_1$ and that $\eta = \inf V_2$. Now, we claim that $\xi \geq \eta$.
If possible suppose that $\xi < \eta$. Then consider $\zeta = \frac{\xi+\eta}{2} \in [0,\pi]$. Now, from Theorem 2.3, we have, for $\alpha = e^{i\zeta}$ there exists $x_\alpha \in M_T$ such that $Ax_\alpha \in (Tx_\alpha)^+_\alpha$. Using Lemma 2.7, we have, either $Ax_\alpha \in (Tx_\alpha)^+_\beta$ for all β with $\arg\beta \in [0,\zeta]$ or $Ax_\alpha \in (Tx_\alpha)^+_\beta$ for all β with $\arg\beta \in [\zeta,\pi]$. But $Ax_\alpha \in (Tx_\alpha)^+_\beta$ for all β with $\arg\beta \in [0,\zeta]$ implies that $\zeta \in V_1$. This contradicts that $\xi = \sup V_1$.
Again, $Ax_\alpha \in (Tx_\alpha)^+_\beta$ for all β with $\arg\beta \in [\zeta,\pi]$ implies that $\zeta \in V_2$. This contradicts that $\eta = \inf V_2$. Hence $\xi \geq \eta$.

Now, there exist sequences $\{\xi_n\} \subseteq V_1$ and $\{\eta_n\} \subseteq V_2$ such that $\{\xi_n\}$ and $\{\eta_n\}$ converge to ξ and η, respectively. Since $\xi_n \in V_1$ and $\eta_n \in V_2$, there exist x_n and y_n in M_T such that $Ax_n \in (Tx_n)^+_\alpha$ for all α with $\arg\alpha \in [0,\xi_n]$ and $Ay_n \in (Ty_n)^+_\alpha$ for all α with $\arg\alpha \in [\eta_n,\pi]$. Since $\mathbb{X}$ is reflexive, $\{x_n\}$ and $\{y_n\}$ have weakly convergent subsequences. Without loss of generality assume that $\{x_n\}$ and $\{y_n\}$ weakly converge to x and y, respectively. Since T and A are compact operators, $Tx_n \longrightarrow Tx,\ Ty_n \longrightarrow Ty,\ Ax_n \longrightarrow Ax$, and $Ay_n \longrightarrow Ay$. Clearly, $x, y \in M_T$. Now,

$$\begin{aligned} \|Tx_n + t\alpha Ax_n\| &\geq \|T\|, && \text{for all } t \geq 0 \text{ and for all } \alpha \text{ with } \arg\alpha \in [0,\xi_n],\\ \Rightarrow \|Tx + t\alpha Ax\| &\geq \|T\|, && \text{for all } t \geq 0 \text{ and for all } \alpha \text{ with } \arg\alpha \in [0,\xi]. \end{aligned}$$

Similarly,

$$\begin{aligned} \|Ty_n + t\alpha Ay_n\| &\geq \|T\|, && \text{for all } t \geq 0 \text{ and for all } \alpha \text{ with } \arg\alpha \in [\eta_n,\pi],\\ \Rightarrow \|Ty + t\alpha Ay\| &\geq \|T\|, && \text{for all } t \geq 0 \text{ and for all } \alpha \text{ with } \arg\alpha \in [\eta,\pi]. \end{aligned}$$

Since $\xi \geq \eta$, $\|Ty + t\alpha Ay\| \geq \|T\|$ for all $t \geq 0$ and for all α with $\arg\alpha \in [\xi,\pi]$. Let $\xi = \phi_1$. Then $Ax \in (Tx)^+_\alpha$ for all α with $\arg\alpha \in [0,\phi_1]$ and $Ay \in (Ty)^+_\alpha$ for all α with $\arg\alpha \in [\phi_1,\pi]$.
Similarly, the fact that for each α with $\arg\alpha \in [0,\pi]$, there exists $z_\alpha \in M_T$ such that
$Az_\alpha \in (Tz_\alpha)^-_\alpha$ gives that there exist $\phi_2 \in [0,\pi]$ and $z, w \in M_T$ such that $Az \in (Tz)^-_\alpha$ for all α with $\arg\alpha \in [0,\phi_2]$ and $Aw \in (Tw)^-_\alpha$ for all α with $\arg\alpha \in [\phi_2,\pi]$. □

Sain and Paul proved in [12] that if T is a linear operator on a finite-dimensional real Banach space $\mathbb{X}$, with $M_T = \pm D$ (D being a closed connected subset of $S_{\mathbb{X}}$), then $T \perp_B A$ if and only if there exists $x \in D$ such that $Tx \perp_B Ax$. In the following theorem we prove an analogous result for complex Banach spaces. Before proving the theorem, let us observe that if $\mathbb{X}$ is a complex Banach space, $T \in \mathbb{L}(\mathbb{X})$ and D is a closed connected subset of $S_{\mathbb{X}}$ such that $D \subset M_T$, then we must have, $\bigcup_{\theta\in[0,2\pi)} e^{i\theta} D \subset M_T$ and $\bigcup_{\theta\in[0,2\pi)} e^{i\theta} D$ is also a connected subset of $S_{\mathbb{X}}$. Note that, it is not true in general, if $\mathbb{X}$ is a real Banach space. This explains the change in the statement of Theorem 2.9, compared to the corresponding real case.

Theorem 2.9. *Let $\mathbb{X}$ be a finite-dimensional complex Banach space, and let $T \in \mathbb{L}(\mathbb{X})$ be such that M_T is a closed connected subset of $S_{\mathbb{X}}$. Then for $A \in \mathbb{L}(\mathbb{X})$, $T \perp_B A$ if and only if for each $\alpha \in U$, there exists $x = x(\alpha) \in M_T$ such that $Tx \perp_\alpha Ax$.*

Proof. The sufficient part of the theorem follows trivially. For the necessary part, suppose that $T \perp_B A$. Let $\alpha \in U$. Consider two sets $W_{1\alpha}$ and $W_{2\alpha}$, where,

$$W_{1\alpha} = \{x \in M_T : \ Ax \in (Tx)^+_\alpha\},$$
$$W_{2\alpha} = \{x \in M_T : \ Ax \in (Tx)^-_\alpha\}.$$

Now, let $x \in M_T$. Then by Proposition 2.1 (i), we have, either $Ax \in (Tx)^+_\alpha$ or $Ax \in (Tx)^-_\alpha$. Thus, $x \in W_{1\alpha} \cup W_{2\alpha}$. This implies that $M_T \subseteq W_{1\alpha} \cup W_{2\alpha} \subseteq M_T$. Hence, $M_T = W_{1\alpha} \cup W_{2\alpha}$. Now, by applying Corollary 2.4, it follows that there exist $x = x(\alpha)$ and $y = y(\alpha)$ in M_T such that $Ax \in (Tx)^+_\alpha$ and $Ay \in (Ty)^-_\alpha$. Therefore, $x \in W_{1\alpha}$ and $y \in W_{2\alpha}$. Hence, $W_{1\alpha} \neq \emptyset$ and $W_{2\alpha} \neq \emptyset$.
Next, we show that $W_{1\alpha}$ is closed. Let $\{x_n\}$ be a sequence in $W_{1\alpha}$ converging to x. Clearly, $x \in M_T$. Now, $Ax_n \in (Tx_n)^+_\alpha$ for all $n \in \mathbb{N}$. Therefore, for any $t \geq 0$, $\|Tx_n + t\alpha Ax_n\| \geq \|Tx_n\|$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have, $\|Tx + t\alpha Ax\| \geq \|Tx\|$. Hence, $Ax \in (Tx)^+_\alpha$, and so $x \in W_{1\alpha}$. Thus, $W_{1\alpha}$ is closed. Similarly, we can show that $W_{2\alpha}$ is closed.
Now, since M_T is connected, we must have, $W_{1\alpha} \cap W_{2\alpha} \neq \emptyset$. Let $u \in W_{1\alpha} \cap W_{2\alpha}$. Then $Au \in (Tu)^+_\alpha$ and $Au \in (Tu)^-_\alpha$. This implies that $Tu \perp_\alpha Au$. This establishes the theorem. $\square$

Once again, in contrast to the real case, we would like to sharpen the necessary part of Theorem 2.9 in the complex case. First we need the following lemma.

Lemma 2.10. *Let $\mathbb{X}$ be a complex Banach space. Let $x, y \in \mathbb{X}$ and $\alpha \in U$ with $\arg \alpha = \theta$ such that $x \perp_\alpha y$. Then either $y \in (x)^+_\beta$ for all β with $\arg \beta \in [\theta - \pi, \theta]$ or $y \in (x)^+_\beta$ for all β with $\arg \beta \in [\theta, \theta + \pi]$.*

Proof. Let $x \perp_\alpha y$. Suppose that $y \notin (x)^+_{\beta_1}$ for some β_1 with $\arg \beta_1 \in [\theta - \pi, \theta]$. Then there exists $t_1 > 0$ such that $\|x + t_1\beta_1 y\| < \|x\|$. If possible suppose that $y \notin (x)^+_{\beta_2}$ for some β_2 with $\arg \beta_2 \in [\theta, \theta + \pi]$. Then there exists $t_2 > 0$ such that $\|x + t_2\beta_2 y\| < \|x\|$. Then it is easy to verify that there exist $0 < s < 1$ and $t \in \mathbb{R}$ such that

$$\begin{aligned}
t\alpha &= (1-s)t_1\beta_1 + st_2\beta_2 \\
\Rightarrow x + t\alpha y &= (1-s)(x + t_1\beta_1 y) + s(x + t_2\beta_2 y) \\
\Rightarrow \|x + t\alpha y\| &\leq (1-s)\|(x + t_1\beta_1 y)\| + s\|(x + t_2\beta_2 y)\| \\
\Rightarrow \|x + t\alpha y\| &< (1-s)\|x\| + s\|x\| \\
&= \|x\|,
\end{aligned}$$

this leads to a contradiction, and so $y \in (x)^+_\beta$ for all β with $\arg \beta \in [\theta, \theta + \pi]$. This proves the lemma. $\square$

Now, we prove the promised theorem.

Theorem 2.11. *Let $\mathbb{X}$ be a finite-dimensional complex Banach space, and let $T \in \mathbb{L}(\mathbb{X})$ be such that M_T is a closed connected subset of $S_{\mathbb{X}}$. Then, $T \perp_B A$, for $A \in \mathbb{L}(\mathbb{X})$, if and only if there exist some $\theta \in [0, \pi]$ and $x, y \in M_T$ such that $Ax \in (Tx)^+_\alpha$ for all α with $\arg \alpha \in [\theta - \pi, \theta]$ and $Ay \in (Ty)^+_\alpha$ for all α with $\arg \alpha \in [\theta, \theta + \pi]$.*

Proof. Let us first prove the sufficient part of the theorem. Suppose there exist some $\theta \in [0, \pi]$ and $x, y \in M_T$ such that $Ax \in (Tx)^+_\alpha$ for all α with $\arg \alpha \in [\theta - \pi, \theta]$ and $Ay \in (Ty)^+_\alpha$ for all α with $\arg \alpha \in [\theta, \theta + \pi]$. Let $\lambda \in \mathbb{C}$. Then either there exist $t_1 \geq 0$ and α_1 with $\arg \alpha_1 \in [\theta - \pi, \theta]$ such that $\lambda = t_1 \alpha_1$ or there exist $t_2 \geq 0$ and α_2 with $\arg \alpha_2 \in [\theta, \theta + \pi]$ such that $\lambda = t_2 \alpha_2$. Now, $\lambda = t_1 \alpha_1$ implies that $\|T + \lambda A\| = \|T + t_1 \alpha_1 A\| \geq \|(T + t_1 \alpha_1 A)x\| \geq \|Tx\| = \|T\|$ and $\lambda = t_2 \alpha_2$ implies that $\|T + \lambda A\| = \|T + t_2 \alpha_2 A\| \geq \|(T + t_2 \alpha_2 A)y\| \geq \|Ty\| = \|T\|$. Therefore, $T \perp_B A$. This completes the proof of the sufficient part of the theorem.
For the necessary part, suppose that $T \perp_B A$. Let us consider the following two sets

$$V_1 = \{\theta \in [0, \pi] : \exists \ x \in M_T \ s.t. \ Ax \ \in (Tx)^+_\alpha \ \forall \ \alpha \ with \ \arg \alpha \in [\theta - \pi, \theta]\},$$

$$V_2 = \{\theta \in [0, \pi] : \exists \ x \in M_T \ s.t. \ Ax \ \in (Tx)^+_\alpha \ \forall \ \alpha \ with \ \arg \alpha \in [\theta, \theta + \pi]\}.$$

We first show that $[0, \pi] = V_1 \cup V_2$. Let $\theta \in [0, \pi]$ and $\alpha = e^{i\theta}$. Since $T \perp_B A$, by Theorem 2.9, we have, there exists $x = x(\alpha) \in M_T$ such that $Tx \perp_\alpha Ax$. Therefore, applying Lemma 2.10, we have, either $Ax \in (Tx)^+_\beta$ for all β with $\arg \beta \in [\theta - \pi, \theta]$; that is, $\theta \in V_1$ or $Ax \in (Tx)^+_\beta$ for all β with $\arg \beta \in [\theta, \theta + \pi]$; that is, $\theta \in V_2$. Hence $[0, \pi] = V_1 \cup V_2$.
We claim that $V_1 \neq \emptyset$. Let $0 \notin V_1$. Then $0 \in V_2$. Hence there exists $z \in M_T$ such that $Az \ \in (Tz)^+_\beta$ for all β with $\arg \beta \in [0, \pi]$. This implies that $\pi \in V_1$. Hence $V_1 \neq \emptyset$. Similarly, it can be shown that $V_2 \neq \emptyset$.
We next show that V_1 is closed. Let $\{\theta_n\}$ be a sequence in V_1 such that $\{\theta_n\}$ converges to θ. Let $\beta = e^{i\theta}$. Then there exists $x_n \in M_T$ such that $Ax_n \ \in (Tx_n)^+_\alpha$ for all α with $\arg \alpha \in [\theta_n - \pi, \theta_n]$. Since $\mathbb{X}$ is finite-dimensional, $\{x_n\}$ has a convergent subsequence. Without loss of generality assume that $\{x_n\}$ converges to x (say). Clearly, $x \in M_T$. Now, $Ax_n \in (Tx_n)^+_\alpha$ for all α with $\arg \alpha \in [\theta_n - \pi, \theta_n]$ gives that $\|Tx_n + te^{i\theta_n} Ax_n\| \geq \|T\|$ for all $t \geq 0$. Letting $n \longrightarrow \infty$, we have $\|Tx + te^{i\theta} Ax\| = \|Tx + t\beta Ax\| \geq \|T\| \Rightarrow Ax \in (Tx)^+_\beta$. Similarly, $Ax \in (Tx)^+_\gamma$, where $\arg \gamma = \theta - \pi$. Now, let $\theta - \pi < \phi < \theta$. If possible suppose that there does not exist any $n_0 \in \mathbb{N}$ such that $\phi \in [\theta_n - \pi, \theta_n]$ for all $n \geq n_0$. Then without loss of generality we may assume that $\phi > \theta_n$ for all $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, we have $\phi \geq \theta$, a contradiction. Hence there exists $n_0 \in \mathbb{N}$ such that $\phi \in [\theta_n - \pi, \theta_n]$ for all $n \geq n_0$. This implies that $\|Tx_n + te^{i\phi} Ax_n\| \geq \|T\|$ for all $t \geq 0$ and for all $n \geq n_0$. Therefore, as $n \longrightarrow \infty$, we have $\|Tx + te^{i\phi} Ax\| \geq \|T\|$ for all $t \geq 0$. This implies that $Ax \in (Tx)^+_\delta$, where $\delta = e^{i\phi}$. Thus, $Ax \in (Tx)^+_\alpha$ for all α with $\arg \alpha \in [\theta - \pi, \theta]$. Hence $\theta \in V_1$. Thus, V_1 is closed. Similarly, it can be shown that V_2 is closed.
Now, since $[0, \pi]$ is connected, $V_1 \cap V_2 \neq \emptyset$. Let $\theta \in V_1 \cap V_2$. Then there exist

$x, y \in M_T$ such that $Ax \in (Tx)^+_\alpha$ for all α with $\arg \alpha \in [\theta - \pi, \theta]$ and $Ay \in (Ty)^+_\alpha$ for all α with $\arg \alpha \in [\theta, \theta + \pi]$. This establishes the theorem. □

Next, in the context of complex Banach spaces we explore the structure of M_T in connection with B–J orthogonality. We would like to invite the reader to have a look at Theorem 2.2 and Corollary 2.2.1 of [9], for an analogous result in the real case.

Theorem 2.12. *Let $\mathbb{X}$ be a complex Banach space, $0 \neq T \in \mathbb{L}(\mathbb{X})$ and $x \in M_T$.*

(i) *If $y \in \mathbb{X}$ is such that $Tx \perp_B Ty$, then $x \perp_B y$.*

(ii) $T(x^+_\alpha \setminus x^\perp_\alpha) \subset (Tx)^+_\alpha \setminus (Tx)^\perp_\alpha$ *for $\alpha \in U$.*

(iii) $T(x^-_\alpha \setminus x^\perp_\alpha) \subset (Tx)^-_\alpha \setminus (Tx)^\perp_\alpha$ *for $\alpha \in U$.*

(iv) $\ker T \subset \bigcap_{x \in M_T} x^\perp$.

Proof. (i) Suppose $Tx \perp_B Ty$. Then

$$\|T\|\|x\| = \|Tx\| \leq \|Tx + \lambda Ty\| \leq \|T\|\|x + \lambda y\| \qquad \forall \lambda \in \mathbb{C}.$$

This implies that $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$. Therefore, $x \perp_B y$.

(ii) Let $y \in x^+_\alpha \setminus x^\perp_\alpha$. Then there exists $t < 0$ such that $\|x + t\alpha y\| < \|x\|$. Now, $\|Tx + t\alpha Ty\| \leq \|T\|\|x + t\alpha y\| < \|T\|\|x\| = \|Tx\|$. This implies that $Ty \notin (Tx)^-_\alpha$. It now follows from Proposition 2.1 that $Ty \in (Tx)^+_\alpha \setminus (Tx)^\perp_\alpha$.

(iii) Follows similarly as (ii).

(iv) If $M_T = \phi$, then the theorem follows trivially. Let us assume that $M_T \neq \phi$. Let $y \in \ker T$. Then for any $x \in M_T$, we have $Ty \in (Tx)^\perp$, since $Ty = 0$. From (i) it follows that $y \in x^\perp$. This implies that $\ker T \subset \bigcap_{x \in M_T} x^\perp$. □

Remark 2.13. In addition to $x \in M_T$, if x and Tx are smooth points in $\mathbb{X}$, then, for any $y \in \mathbb{X}$, we have $x \perp_B y \Rightarrow Tx \perp_B Ty$. This can be proved following the same line of arguments, as in Lemma 2.1 of [11]. As a matter of fact, a closer inspection reveals that only the smoothness of x in $\mathbb{X}$ suffices in both the cases. We thank the referee for this nice observation.

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