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PERTURBATION OF MINIMUM ATTAINING OPERATORS

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Abstract. We prove that the minimum attaining property of a bounded linear operator on a Hilbert space H whose minimum modulus lies in the discrete spectrum, is stable under small compact perturbations. We also observe that given a bounded operator with strictly positive essential minimum modulus, the set of compact perturbations which fail to produce a minimum attaining operator is smaller than a nowhere dense set. In fact, it is a porous set in the ideal of all compact operators on H. Further, we try to extend these stability results to perturbations by all bounded linear operators with small norm and obtain subsequent results.

1. Introduction

In this article, we work on a (complex) Hilbert space of arbitrary dimension, which is usually denoted by H. Some perturbation properties of norm attaining operators are discussed by Kover in [14, 15]. Analogous to the norm attaining bounded operators on H, minimum attaining operators are defined and studied by Carvajal and Neves in [7]. Let T be a bounded linear operator on H. The quantity

$$m(T) := \inf\{\|Tx\| : x \in H, \|x\| = 1\}$$

is called the *minimum modulus* of T. The operator T is said to be *minimum attaining* if there exists $x_0 \in H$ with $\|x_0\| = 1$ such that $m(T) = \|Tx_0\|$. Though they have many similarities with the norm attaining operators, their characteristics differ in many ways, for instance the injectivity and closed range properties

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play a major role for the class of minimum attaining operators. This leads to significant changes in the perturbation properties of the minimum attaining operators from those of the norm attaining operators. Our first goal in this article is to study the stability of the minimum attaining property under compact perturbations. In other words, we try to answer the question which compact perturbations of minimum attaining operators on a Hilbert space are again minimum attaining. We also observe that for any fixed bounded linear operator T on H with the strictly positive essential minimum modulus (that is, $m_e(T) > 0$), the set of compact perturbations of T which fail to produce a minimum attaining operator is very small in size, in fact it is a porous set. Our second goal is to extend these results to the class of general bounded operators on H using the connection between the essential spectrum and the Fredholm operators. Much of our work relies on Weyl's theorem, which states that the essential spectrum of a self-adjoint operator remains unchanged under any compact perturbation.

The article is organized as follows. Overall there are four sections. In the second one, we fix the notations and list out some basic definitions and results which are already there in the literature that we are going to use in the forthcoming sections. In the third section, we discuss the main results related to the perturbation of minimum attaining operators by compact operators, and we try to extend some of these results to a more general setting in the fourth section.

2. Notations and Definitions

As usual, we denote the inner product and the induced norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The unit sphere of a Hilbert space H is denoted by S_H. If N is a subset of H, then the closed linear span of N is denoted by $[N]$. The orthogonal projection onto a closed subspace M of the Hilbert space H is denoted by P_M.

The space of all bounded linear operators on H is denoted by $\mathcal{B}(H)$, and the set of all minimum attaining operators on H is denoted by $\mathcal{M}(H)$. Let $T \in \mathcal{B}(H)$. Then the null space and range space of T are denoted by $N(T)$ and $R(T)$, respectively. The adjoint of T is denoted by T^*. The operator T is called normal if $TT^* = T^*T$ and self-adjoint if $T^* = T$. We say T to be positive if $\langle Tx, x \rangle \geq 0$ for all $x \in H$, and we denote it by $T \geq 0$. The set of all self-adjoint and positive operators on H are denoted by $\mathcal{B}^s(H)$ and $\mathcal{B}^+(H)$, respectively. Similarly, the set of all minimum attaining self-adjoint operators and positive operators on H are denoted by $\mathcal{M}^s(H)$ and $\mathcal{M}^+(H)$, respectively. Throughout this article, dim M denotes the Hilbert dimension of the closed subspace M, that is the cardinality of any orthonormal basis of M.

Let M be a closed subspace of H and $T \in \mathcal{B}(H)$. Then M is said to be *invariant* under T if $TM \subseteq M$. Also, M is said to be a *reducing subspace* for T if and only if both M and $M^\perp$ are invariant under T. A pair of closed subspaces M_1 and M_2 is said to be *completely reducing* for T if both M_1 and M_2 are reducing subspaces for T and $H = M_1 \oplus M_2$ (for details see [21]).

Let $T \in \mathcal{B}(H)$ and $T \geq 0$. Then there exists a unique operator $S \in \mathcal{B}(H)$ such that $S \geq 0$ and $T = S^2$. For $T \in \mathcal{B}(H)$, the operator $T^{\frac{1}{2}} := S$ is called the *square root* of T, and the operator $|T| := (T^*T)^{\frac{1}{2}}$ is called the *modulus* of T. Note that if $T \geq 0$, then $|T| = T$. (for details, see [16, 19]).

An operator $T \in \mathcal{B}(H)$ is called *compact* if $\overline{T(B)}$ is compact for every bounded subset B of H. The set of compact operators on H is denoted by $\mathcal{K}(H)$. The set $\{e_n\}_{n=1}^{\infty}$ denotes the standard orthonormal basis or the canonical basis of ℓ^2.

The *spectrum* is defined by $\sigma(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not invertible in } \mathcal{B}(H)\}$, and the *point spectrum* is defined by $\sigma_p(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not injective}\}$.

Definition 2.1. [8, page 30] Let $\{H_i\}_{i=1}^{n}$ be a family of Hilbert spaces such that $H = \overset{n}{\underset{i=1}{\oplus}} H_i$. Let $\{T_i\}_{i=1}^{n}$ be a family of bounded operators such that $T_i \in \mathcal{B}(H_i)$ for all $i = 1, 2, \dots, n$. Then their direct sum, $T = \overset{n}{\underset{i=1}{\oplus}} T_i$, is a bounded operator from H to H defined as $\left(\overset{n}{\underset{i=1}{\oplus}} T_i\right)(x_1, x_2, \dots, x_n) = (T_1x_1, T_2x_2, \dots, T_nx_n)$ for all $(x_1, x_2, \dots, x_n) \in H$.

Remark 2.2. In the above definition, we can observe that $T_i = T|_{H_i}$ for all $i = 1, 2, \dots, n$ and $\|T\| = \underset{1 \leq i \leq n}{max} \|T_i\|$.

Definition 2.3. [11, page 184] A bounded linear operator $T \colon H_1 \to H_2$, is called a *Fredholm operator* if its range, $R(T)$, is closed and the numbers

$$n(T) = \dim\, N(T),\ d(T) = \dim R(T)^{\perp}$$

are finite. In this case $\operatorname{ind}(T) = n(T) - d(T)$ is said to be the *index* of T.

We will use the spectral decomposition notion from Reed and Simon [19], which we summarize in the following theorem.

Theorem 2.4. *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then the spectrum $\sigma(T)$ of T decomposes as the disjoint union of the discrete spectrum $\sigma_{disc}(T)$ of T, and the essential spectrum $\sigma_{ess}(T)$ of T. The* discrete spectrum *is the set of all eigenvalues with finite multiplicity which are isolated from the rest of the spectrum of T and the* essential spectrum *is the set of all $\lambda \in \sigma(T)$ that satisfy at least one of the following:*

1. *λ is an eigenvalue with infinite multiplicity,*
2. *λ is a limit point of $\sigma_p(T)$,*
3. *$\lambda \in \sigma_c(T)$, the continuous spectrum of T; that is $T - \lambda I$ is one to one but not onto.*

Let $T \in \mathcal{B}(H)$. By the definition, we have $|T|^2 = T^*T$. Then, for every $x \in H$, we have $\langle T^*Tx, x\rangle = \langle |T|^2x, x\rangle$. This implies that $\|Tx\| = \||T|x\|$ for all $x \in H$. Consequently, $m(T) = m(|T|)$ and $\|T\| = \||T|\|$. We use this fact frequently in the forthcoming sections of this article. We set $\mathcal{M}_d(H) := \{T \in \mathcal{B}(H) \colon m(T) \in \sigma_{disc}(|T|)\}$. Similarly, we define $\mathcal{M}_d^+(H) := \{T \in \mathcal{B}^+(H) \colon m(T) \in \sigma_{disc}(T)\}$ and $\mathcal{M}_d^s(H) := \{T \in \mathcal{B}^s(H) \colon m(T) \in \sigma_{disc}(|T|)\}$.

The following version of the Weyl's theorem will be used frequently in what follows.

Theorem 2.5 (Weyl's theorem)**.** [14, Theorem 2] *Let $S, T \in \mathcal{B}(H)$ be self-adjoint. Then*

1. *$\sigma_{ess}(S) = \sigma_{ess}(T)$ if and only if $S - T$ is compact.*
2. *$\lambda \in \sigma(T)$ if and only if there exists a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that $\|(T - \lambda I)u_n\| \to 0$ as $n \to \infty$.*
3. *$\lambda \in \sigma_{ess}(T)$ if and only if there exists an orthonormal sequence of vectors $\{u_n\}_{n=1}^{\infty}$ such that $\|(T - \lambda I)u_n\| \to 0$ as $n \to \infty$.*

Lemma 2.6. [14, Lemma 3] *Let $T \in \mathcal{B}(H)$. Then,*

$$\lambda \in \sigma_{ess}(|T|) \text{ if and only if } \lambda^2 \in \sigma_{ess}(|T|^2).$$

Definition 2.7. [1] Let $T \in \mathcal{B}(H)$. Then the *Weyl spectrum* of T is

$$\{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not a Fredholm operator of index } 0\}.$$

The Weyl spectrum of a linear operator is the set of elements in the spectrum which are not eigenvalues of finite multiplicity. For self-adjoint operators the Weyl spectrum is the remainder of the spectrum once the isolated eigenvalues of finite multiplicity are removed (see [15], for details).

Remark 2.8. For a self-adjoint operator the Weyl spectrum and the essential spectrum are the same.

Let $T \colon H_1 \to H_2$ be a Fredholm operator. Then the $N(T)$ and $R(T)$ are complemented in H_1 and H_2 by subspaces W_1 and W_2, respectively, where W_2 is finite dimensional. Then we can define a bijection $\widetilde{T} \colon W_1 \times W_2 \to H_2 = R(T) \times W_2$ by

$$\widetilde{T}(x_0, y_0) = (Tx_0, y_0).$$

This is called the bijection *associated with the Fredholm operator T* (For details, see [11]).

Theorem 2.9. [11] *Suppose that $T \colon H_1 \to H_2$ is a Fredholm operator, and let $\widetilde{T}$ be the bijection associated with T. If $S \colon H_1 \to H_2$ is a bounded linear operator with $\|S\| < \|\widetilde{T}^{-1}\|^{-1}$, then $S + T$ is Fredholm and*

(1) $n(S + T) \leq n(T)$,
(2) $d(S + T) \leq d(T)$,
(3) $ind(S + T) = ind(T)$.

Here we list out some important results from the literature that we will use frequently in the next coming sections.

Proposition 2.10. [9, Proposition 3.1] *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then $T \in \mathcal{M}(H)$ if and only if either $m(T)$ or $-m(T)$ is an eigenvalue of T. In particular, when $T \geq 0$, we have $T \in \mathcal{M}(H)$ if and only if $m(T)$ is an eigenvalue of T.*

Proposition 2.11. [9, Proposition 3.2] *Let $T \in \mathcal{B}(H)$. Then the following statements are equivalent:*

(1) $T \in \mathcal{M}(H)$;

(2) $|T| \in \mathcal{M}(H)$;
(3) $T^*T \in \mathcal{M}(H)$.

Lemma 2.12. [10, Lemma 2.2] *Let* $T_1, T_2 \in \mathcal{B}(H)$. *Then*

$$|m(T_1) - m(T_2)| \leq \|T_1 - T_2\|.$$

Proposition 2.13. [7, Proposition 2.2] *Let* $T \in \mathcal{B}^+(H)$. *Then*

$$m(T) = \inf_{x \in S_H} \|Tx\| = \inf\{\langle Tx, x\rangle : x \in S_H\}.$$

Proposition 2.14. [18, Proposition 2.2(1)] *Let* $T \in \mathcal{B}(H)$ *be normal. Then*

$$m(T) = d(0, \sigma(T)) = \inf\{|\lambda| : \lambda \in \sigma(T)\}.$$

Remark 2.15. If $T \in \mathcal{B}^+(H)$, then, by the compactness of $\sigma(T)$, we have

$$m(T) = \min\{\lambda : \lambda \in \sigma(T)\}.$$

Let H be a complex separable Hilbert space, and let $T \in \mathcal{B}(H)$. For $n \in \mathbb{N} \cup \{0, \infty\}$ we have the following definition

$$\rho_n(T) = \inf\{\|T - S\| : S \in \mathcal{B}(H), \dim N(S) = n\}.$$

Theorem 2.16. [4, Theorem 2] *Assume that* $n < \dim N(T)$. *We have*

(1) if $n \geq ind(T)$, *then* $\rho_n(T) = 0$
(2) if $n < ind(T)$, *then* $\rho_n(T) = m(T^*)$.

3. Perturbation by compact operators

We know that the norm satisfies the triangle inequality but this is not valid for the minimum modulus, not even for the class of positive operators. For example, let $H = \ell^2, M_1 = [e_{2n-1} : n \in \mathbb{N}]$, and $M_2 = [e_{2n} : n \in \mathbb{N}]$, and consider $T_1 = P_{M_1}, T_2 = P_{M_2}$. But the following inequality is true for the class of positive operators $\mathcal{B}^+(H)$.

Proposition 3.1. *Let* $T_1, T_2 \in \mathcal{B}^+(H)$. *Then,*

$$m(T_1 + T_2) \geq m(T_1) + m(T_2).$$

Proof. Since $T_1, T_2 \geq 0$, we have $T_1 + T_2 \geq 0$. Now the proof follows directly from Proposition 2.13. ◻

Remark 3.2. The above Proposition is not valid in general for the class of all bounded operators $\mathcal{B}(H)$, for instance, consider $T_1 = I$ and $T_2 = -I$ where I is the Identity operator on ℓ^2.

The following theorem is crucial in proving the stability of minimum attaining property under small compact perturbations.

Theorem 3.3. *Let* $T \in \mathcal{B}(H)$ *and* $K \in \mathcal{K}(H)$ *be such that* $m(T+K) \notin \sigma_{ess}(|T|)$. *Then* $m(T+K) \in \sigma_{disc}(|T+K|)$ *and* $T + K \in \mathcal{M}(H)$.

Proof. Let us consider the operator,

$$|T+K|^2 = (T+K)^*(T+K) = T^*T + T^*K + K^*T + K^*K = |T|^2 + C,$$

where $C = T^*K + K^*T + K^*K \in \mathcal{K}(H)$. Since $|T+K|^2$ and $|T|^2$ are self-adjoint, by Theorem 2.5, it follows that

$$\sigma_{ess}(|T+K|^2) = \sigma_{ess}(|T|^2).$$

Now, Lemma 2.6 gives that $\sigma_{ess}(|T+K|) = \sigma_{ess}(|T|)$. Hence, $m(T+K) \notin \sigma_{ess}(|T+K|)$. As $|T+K| \geq 0$, by Lemma 2.15, we can conclude that $m(T+K) \in \sigma_{disc}(|T+K|)$. Consequently, $T+K \in \mathcal{M}(H)$ by Proposition 2.10. □

The above theorem yields the following stability result.

Corollary 3.4. *Let $T \in \mathcal{B}(H)$, and let $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $K \in \mathcal{K}(H)$ with $\|K\| < \epsilon$, we have $T+K \in \mathcal{M}(H)$.*

Proof. By the definition of the discrete spectrum,

$$d = d\left(m(T), \sigma_{ess}(|T|)\right) = \inf\left\{|\lambda - m(T)| : \lambda \in \sigma_{ess}\left(|T|\right)\right\} > 0.$$

Now choose an $\epsilon \in (0, d)$. By Lemma 2.12, for any $K \in \mathcal{K}(H)$ with $\|K\| < \epsilon$ we have

$$|m(T+K) - m(T)| \leq \|T+K-T\| = \|K\| < \epsilon.$$

This implies that $m(T+K) \notin \sigma_{ess}\left(|T|\right) = \sigma_{ess}\left(|T+K|\right)$. By Theorem 3.3, we have $T+K \in \mathcal{M}(H)$. □

Remark 3.5. Note that the condition $m(T) \in \sigma_{disc}(|T|)$ is necessary in Corollary 3.4. For instance, we have an example below.

Example 3.6. For every $n \in \mathbb{N}$, let $D_n : \ell^2 \to \ell^2$ be defined by

$$D_n(e_k) = \frac{e_k}{n+k-1}, \forall k \geq 1.$$

Clearly, $D_n \in \mathcal{K}(\ell^2)$ and $\|D_n\| = \frac{1}{n}$ for all $n \in \mathbb{N}$. Next, we have $I + D_n \geq 0$ and $I + D_n \notin \mathcal{M}(\ell^2)$ for all $n \in \mathbb{N}$. Note that $m(I) = 1 \in \sigma_{ess}(I)$.

The following corollary will be used frequently in proving many compact perturbation results that come later.

Corollary 3.7. *Let $T \in \mathcal{B}(H)$, and let $K \in \mathcal{K}(H)$. If $m(T+K) < m(T)$, then $m(T+K) \in \sigma_{disc}(|T+K|)$ and $T+K \in \mathcal{M}_d(H)$.*

Proof. Lemma 2.15 and the spectral radius formula [2, Theorem1] imply that $\sigma(|T|) \subseteq [m(T), \|T\|]$ and so $m(T+K) \notin \sigma_{ess}(|T|) = \sigma_{ess}(|T+K|)$. Consequently, $m(T+K) \in \sigma_{disc}(|T+K|)$ and $T+K \in \mathcal{M}_d(H)$. □

Remark 3.8. Note that Corollary 3.7 is meaningful only for all $T \in \mathcal{B}(H)$ with $m(T) > 0$; because $m(T+K) < 0$ does not hold true for any $K \in \mathcal{K}(H)$.

Remark 3.9. Let $T \in \mathcal{B}(H)$, and let $K \in \mathcal{K}(H)$. Suppose that $m(T+K) \geq m(T)$; then $T+K$ may or may not be minimum attaining. The following example illustrates this.

Example 3.10. Let $T : \ell^2 \to \ell^2$ be defined by

$$T(e_n) = \begin{cases} 0 & \text{if } n = 1, \\ \left(1 + \frac{1}{n}\right) e_n & \text{if } n \geq 2. \end{cases}$$

Clearly, $m(T) = 0$. Let $P_{[e_1]}$ be the orthogonal projection onto $[e_1]$. Then, $P_{[e_1]} \in \mathcal{K}(\ell^2)$. For every $\mu \in \mathbb{C}$, consider the operator $T + \mu P_{[e_1]} \in \mathcal{B}(H)$. We have always $m(T + \mu P_{[e_1]}) \geq m(T)$. But, $T + \mu P_{[e_1]} \in \mathcal{M}(\ell^2)$ whenever $|\mu| \leq 1$, and in the other case, $T + \mu P_{[e_1]} \notin \mathcal{M}(\ell^2)$.

The following lemma will be used frequently in proving the theorems coming later and the proof is essentially contained in the proof of [17, Theorem 3.4]. We provide the details for the sake of completeness.

Lemma 3.11. *Let $T \in \mathcal{B}^+(H)$, and let $m(T) > 0$. Then there exists a sequence of positive finite rank operators $\{R_n\}_{n \geq 1}$ with $\|R_n\| = \frac{1}{n}$ such that $T_n := T - R_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $T_n := T - R_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the lemma in the following two cases separately.

Case(I) $T \in \mathcal{M}_d^+(H)$: In this case, the result follows trivially by taking $T_n = T$ and $R_n = 0$ for all $n \in \mathbb{N}$.

Case(II) $T \notin \mathcal{M}_d^+(H)$: We have $m_e(T) = m(T) > 0$. By Proposition 2.13, for every $n \in \mathbb{N}$, there exists a $x_n \in S_H$ such that,

$$m(T) \leq \langle T x_n, x_n \rangle < m(T) + \frac{1}{2n}. \tag{3.1}$$

For a fixed $n \in \mathbb{N}$, we set $R_n x := \frac{1}{n} \langle x, x_n \rangle x_n$ for all $x \in H$. Clearly, R_n is a positive rank one operator for all $n \in \mathbb{N}$ and $\|R_n\| = \frac{1}{n}$. Without loss of generality, we can assume that $\frac{1}{n} < m(T)$ for all $n \in \mathbb{N}$. Let $T_n := T - R_n$ for all $n \in \mathbb{N}$. Then for every $x \in S_H$, we have

$$\begin{aligned} \langle T_n x, x \rangle &= \langle T x, x \rangle - \frac{1}{n} |\langle x, x_n \rangle|^2 \\ &\geq \langle T x, x \rangle - \frac{1}{n} \quad \text{(by Cauchy–Schwarz inequality)} \\ &\geq m(T) - \frac{1}{n}. \end{aligned}$$

Consequently, $T_n \in \mathcal{B}^+(H)$ for all $n \in \mathbb{N}$. Again, by Proposition 2.13, we have

$$\begin{aligned} m(T_n) &\leq \langle T_n x_n, x_n \rangle \\ &\leq \langle T x_n, x_n \rangle - \frac{1}{n} \\ &< \left(m(T) + \frac{1}{2n} \right) - \frac{1}{n} \quad \text{(by Equation (3.1))} \\ &< m(T) - \frac{1}{2n} \\ &< m(T). \end{aligned}$$

Next, by Corollary 3.7, we have $T_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$. $\square$

Definition 3.12. [3, 5] Let $T \in \mathcal{B}(H)$. Then the quantity,

$$m_e(T) = \inf\{\lambda\colon \lambda \in \sigma_{ess}(|T|)\},$$

is called the *essential minimum modulus* of T.

For a fixed $T \in \mathcal{B}(H)$, let us define $A_T = \{K \in \mathcal{K}(H)\colon T + K \notin \mathcal{M}(H)\}$ and $S_T = \{K \in \mathcal{K}(H)\colon m(T + K) = m_e(T)\}$.

The next lemma gives the relationship between these two sets.

Lemma 3.13. *Let $T \in \mathcal{B}(H)$. Then $A_T \subseteq S_T$.*

Proof. First, note that $\sigma_{ess}(|T + K|) = \sigma_{ess}(|T|)$. Therefore,

$$m_e(T) = \inf\{\lambda\colon \lambda \in \sigma_{ess}(|T + K|)\} \qquad \text{for all } K \in \mathcal{K}(H);$$

That is, for a fixed T, $m_e(T)$ is constant under all compact perturbations of T.

Suppose $A_T = \emptyset$, the result is trivial. Assume $A_T \neq \emptyset$. Let $K \in A_T$. Since $T + K \notin \mathcal{M}(H)$ we know that $m(T + K) \notin \sigma_{disc}(|T + K|)$. Since, $m(|T + K|) = m(T + K)$ and $|T + K| \geq 0$, we have $m(T + K) \in \sigma(|T + K|)$. It follows that $m(T + K) \in \sigma_{ess}(|T + K|) = \sigma_{ess}(|T|)$. By Remark 2.15, we can conclude that $m_e(T) = m(T + K)$ and $A_T \subseteq S_T$. ∎

Remark 3.14. Note that for $T \in \mathcal{B}(H)$ such that $m(T) \in \sigma_{ess}(|T|)$, we have $A_T \subseteq S_T$.

Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior. Equivalently, a subset is nowhere dense if and only if the complement of its closure is dense (see, [7, page 132]).

We are now ready to prove a theorem, which is one of our main goals of this article. We use the previous lemma to characterize the size of A_T.

Theorem 3.15. *Let $T \in \mathcal{B}(H)$, and let $m_e(T) > 0$. Then A_T is nowhere dense in $\mathcal{K}(H)$.*

Proof. By Lemma 3.13, we have $A_T \subseteq S_T$. To conclude that A_T is nowhere dense, it is sufficient to show that S_T is nowhere dense or equivalently it suffices to prove that $\overline{S_T}^c = \mathcal{K}(H) \setminus \overline{S_T}$ is dense in $\mathcal{K}(H)$. Using Lemma 2.12, it is easy to observe that $\overline{S_T} = S_T$.

Suppose $S_T = \emptyset$, the result is trivial. Assume $S_T \neq \emptyset$. Let $K \in S_T$. Then $m(T+K) = m_e(T)$. Let $T+K = V|T+K|$ be the Polar decomposition of $T+K$. Note that V is an isometry because $m(T+K) > 0$. Since, $|T+K| \in \mathcal{B}^+(H)$ and $m(|T+K|) = m(T+K) > 0$, by Lemma 3.11, there exists a sequence of positive rank one operators $\{R_n\}_{n\geq 1}$ such that $S_n := |T+K| - R_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $S_n \to |T+K|$ in norm as $n \to \infty$. Denote by $T_n := VS_n$ and $K_n := K - VR_n$ for all $n \in \mathbb{N}$. Then, we have $T_n = T + K_n$ and $|T_n| = (S_n V^* V S_n)^{\frac{1}{2}} = S_n$ for all $n \in \mathbb{N}$. Consequently, $T_n \in \mathcal{M}_d(H)$ and $m(T + K_n) \notin \sigma_{ess}(|T + K_n|)$ for all $n \in \mathbb{N}$. By Lemma 2.6, we have $\sigma_{ess}(|T + K_n|) = \sigma_{ess}(|T|)$. Therefore, $m(T + K_n) \notin \sigma_{ess}(|T|)$ and $m(T + K_n) < m_e(T)$ for all $n \in \mathbb{N}$. Consequently, $K_n \notin S_T$ for all $n \in \mathbb{N}$.

Next, we have $K_n \to K$, since $R_n \to 0$ as $n \to \infty$. Therefore, we can conclude that $S_T^c = \mathcal{K}(H) \setminus S_T$ is dense and S_T is nowhere dense in $\mathcal{K}(H)$. ∎

Remark 3.16. In case $m_e(T) = 0$, we have $S_T = \mathcal{K}(H)$ and it cannot be nowhere dense. But A_T may be nowhere dense or may not be. Below we illustrate this.

Firstly, by the spectral theorem it is easy to observe that every compact operator on a nonseparable Hilbert space has nontrivial kernel and hence is not injective.

Example 3.17. Let H be a nonseparable complex Hilbert space and $T \equiv 0$. Then, $A_0 = \{K \in \mathcal{K}(H) \colon K \text{ is injective}\} = \emptyset$ and hence nowhere dense in $\mathcal{K}(H)$.

We need to prove the following basic tools to provide an example for $T \in \mathcal{B}(H)$ such that $m_e(T) = 0$ and A_T is not nowhere dense.

Lemma 3.18. *Let H be separable, and let $F \in \mathcal{K}^+(H)$ be a finite rank operator. Then there exists a sequence $\{K_n\} \subseteq \mathcal{K}(H)$ such that K_n is injective for all $n \in \mathbb{N}$ and $K_n \to F$ in norm as $n \to \infty$.*

Proof. Note that $N(F)$ is an infinite dimensional Hilbert space. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal set such that $N(F) = [e_j : j \in \mathbb{N}]$. Let $\{f_j : 1 \leq j \leq n\}$ be another orthormal set such that $R(F) = [f_j \colon 1 \leq j \leq n]$. From the projection theorem, F is positive. Hence $H = N(F) \oplus R(F)$. Let us define the linear map $D \colon H \to H$ by $De_i := \frac{e_i}{i}$ for all $i \in \mathbb{N}$ and $Df_j = 0$ for all $1 \leq j \leq n$. Now, it is easy to verify that $K_n := F + \frac{D}{n}$ is an injective compact operator for all $n \in \mathbb{N}$ and $K_n \to F$ as $n \to \infty$ in norm. ☐

The above lemma leads to the following result on the denseness of injective compact operators.

Theorem 3.19. *Let H be separable. Then the set of all injective compact operators is dense in $\mathcal{K}(H)$.*

Proof. Since the set of finite rank operators is dense in $\mathcal{K}(H)$, it is enough to prove that given any finite rank operator F, there exists a sequence $\{C_n\} \subseteq \mathcal{K}(H)$ such that C_n is injective for all $n \in \mathbb{N}$ and $C_n \to F$ in norm as $n \to \infty$.

Both the subspaces $N(F)$ and $N(F^*)$ are infinite dimensional and separable. Therefore, dim $N(F)$= dim $N(F^*)$. Consequently, there exists an isometry V such that $F = V|F|$ (For details, see [12, Problem 135]). Since $|F|$ is a positive finite rank operator, by Lemma 3.18, there exists a sequence $\{K_n\} \subseteq \mathcal{K}(H)$ such that K_n is injective for all $n \in \mathbb{N}$ and $K_n \to |F|$ in norm as $n \to \infty$. Now, consider $C_n := VK_n$ for all $n \in \mathbb{N}$. Since V is an isometry, C_n is injective for all $n \in \mathbb{N}$ and $C_n \to F$ in norm as $n \to \infty$. ☐

Now, we are in a position to construct many examples of T such that the set A_T is not nowhere dense.

Example 3.20. Let H be separable, and let $T \equiv 0$. Then, from Theorem 3.19, $A_0 = \{K \in \mathcal{K}(H)/K \text{is injective}\}$ is a dense set, and so it cannot be a nowhere dense set. In fact, for every $C \in \mathcal{K}(H)$, we have $A_C = \{K \in \mathcal{K}(H)/C + K \text{is injective}\}$ is a dense set and, so it cannot be a nowhere dense set. This is because $-C + A_0 \subseteq A_C$. Note that $m_e(C) = 0$ for every $C \in \mathcal{K}(H)$.

From many equivalent definitions of *porosity* that can be found in the literature (See, [22, 14, 13]), we choose the following one which is used by Kover in [14].

Let X be a Banach space. An open ball with center x and radius r will be denoted by $B(x,r)$; that is, $B(x,r) = \{y \in X : \|y - x\| < r\}$.

Definition 3.21. [14, Definition 11] A set E in a Banach space X is called *porous* if there is a number $0 < \lambda < 1$ with the following property: For every $x \in E$ and for every $r > 0$ there is a $y \in B(x,r)$ such that $B(y, \lambda\|x - y\|) \cap E = \emptyset$.

It is easy to observe that every porous set is nowhere dense. Zajíček [22] showed that a porous set is smaller than a nowhere dense set. He proved that even in $\mathbb{R}^n$ there exists a closed nowhere dense set which is not porous.

Now, we prove that the set A_T is porous in $\mathcal{K}(H)$.

Theorem 3.22. *Let $T \in \mathcal{B}(H)$, and let $m_e(T) > 0$. Then, A_T is porous in $\mathcal{K}(H)$.*

Proof. Let $X = \mathcal{K}(H)$ and $E = A_T$ be as in the definition of porous set. We prove that $\lambda = \frac{1}{2}$ is one such scalar that satisfies all the requirements for A_T to be porous. Let $K \in A_T$. From Lemma 2.6, we have $m(T + K) = m_e(T)$. Let $T + K = V|T + K|$ be the Polar decomposition of $T + K$. Note that V is an isometry because $m(T + K) > 0$.

Let $r > 0$ be arbitrary. We can choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \min\{m_e(T), \frac{r}{2}\}$. Since $|T + K| \in \mathcal{B}^+(H)$ and $m(|T + K|) > 0$, by proceeding similarly like in Lemma 3.11, we can find a positive rank one operator R_n with $\|R_n\| = \frac{1}{n}$ such that

$$m(|T + K| - R_n) \leq m_e(T) - \frac{1}{2n}. \tag{3.2}$$

Let $K_n := K - VR_n$. Then, $\|K - K_n\| = \frac{1}{n}$. Hence, $K_n \in B(K,r)$. It remains to prove that $B(K_n, \frac{1}{2n}) \cap A_T = \emptyset$. Let $C \in B(K_n, \frac{1}{2n})$. Then, by Lemma 2.12, we have $|m(T + C) - m(T + K_n)| \leq \|T + C - T - K_n\| < \frac{1}{2n}$. It follows that

$$\begin{aligned}
m(T + C) &< m(T + K_n) + \frac{1}{2n} \\
&< m(V|T + K| - VR_n)) + \frac{1}{2n} \\
&< m(|T + K| - R_n) + \frac{1}{2n} \ \text{(since V is an isometry)} \\
&< m_e(T) - \frac{1}{2n} + \frac{1}{2n} \ \text{(from Equation (3.2))} \\
&< m_e(T).
\end{aligned}$$

Therefore, $C \notin S_T$. It follows that $C \notin A_T$ because $A_T \subseteq S_T$. Consequently, $B(K_n, \frac{1}{2n}) \cap A_T = \emptyset$. ☐

4. Perturbation by minimum attaining operators

After proving the compact perturbation results in the previous section, it is natural to ask to what extent those results can be generalized. In this section, answering this question will be our main concern. For this purpose, we build upon the ideas of compact perturbations used in the last section.

Firstly, we will try to extend the stability results for minimum attaining operators under compact perturbations to a more general setting, by making use of the connection between Fredholm operators and the essential spectrum.

Theorem 4.1. *Let $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $S \in \mathcal{B}(H)$ if $\|S - T\| < \epsilon$, then $m(S) \in \sigma_{disc}(|S|)$. In particular, if $A \in \mathcal{B}(H)$ with $\|A\| < \epsilon$ then $m(A + T) \in \sigma_{disc}(|A + T|)$.*

Proof. Suppose this is not true. Then there exists $\{T_n\} \subseteq \mathcal{B}(H)$ such that $T_n \to T$ and $m(T_n) \notin \sigma_{disc}(|T_n|)$ for all $n \in \mathbb{N}$. Now $m(T) \in \sigma_{disc}(|T|)$, so $m(T)I - |T|$ must be a Fredholm operator of index 0, by the definition of the Weyl spectrum. Let $\widetilde{T}$ be the bijection associated with the Fredholm operator $m(T)I - |T|$. Let $\epsilon = \|\widetilde{T}^{-1}\|^{-1}$. Since $T_n \to T$ we know that $|T_n| \to |T|$ [19, Problem 15(a), page 217]. Fix n_0 large enough so that $\||T_n| - |T|\| < \frac{\epsilon}{2}$ and $\|T_n - T\| < \frac{\epsilon}{2}$ for all $n \geq n_0$. By using Lemma 2.12,

$$\begin{aligned}
\|(m(T_n)I - |T_n|) - (m(T)I - |T|)\| &\leq |m(T_n) - m(T)| + \||T_n| - |T|\| \\
&\leq \|T_n - T\| + \||T_n| - |T|\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Now, by Theorem 2.9, it follows that $m(T_n)I - |T_n|$ is a Fredholm operator of index 0 for all $n \geq n_0$. Since, Weyl's spectrum and essential spectrum are same for the case of self-adjoint operators, we have $m(T_n) \notin \sigma_{ess}(|T_n|)$ for all $n \geq n_0$. Consequently, $m(T_n) \in \sigma_{disc}(|T_n|)$ for all $n \geq n_0$. This contradicts our assumption, and hence the theorem is true. The particular case holds true if we consider $S := A + T$ in the main theorem. $\square$

Remark 4.2. Note that for $T \in \mathcal{B}(H)$ and ϵ as above in the Theorem 4.1, we have $B(T, \epsilon) \cap \mathcal{K}(H) = \emptyset$.

The following result extends the Corollary 3.4 from small compact perturbations to perturbations by all bounded linear operators of small norm.

Corollary 4.3. *Let $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $S \in \mathcal{B}(H)$ if $\|S - T\| < \epsilon$ then $S \in \mathcal{M}(H)$. In particular, if $\|A\| < \epsilon$, then $T + A \in \mathcal{M}(H)$.*

Proof. The proof follows directly from Theorem 4.1; once we observe that for any $S \in \mathcal{B}(H)$ we have $\sigma_{disc}(|S|) \subseteq \sigma_{pt}(|S|)$, and $|S| \in \mathcal{M}(H)$ implies $S \in \mathcal{M}(H)$. $\square$

Next, we measure the size of the set $\mathcal{M}_d^+(H)$ in $\mathcal{B}^+(H)$.

Theorem 4.4. *Let $T \in \mathcal{B}^+(H)$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, $\overline{\mathcal{M}_d^+(H)} = \mathcal{B}^+(H)$.*

Proof. We prove the result in two cases separately.

Case(I) $m(T) > 0$: In this case, the result follows directly from Lemma 3.11.

Case(II) $m(T) = 0$: For each $n \in \mathbb{N}$, let us consider $S_n = T + \frac{1}{n}I$. Clearly, $m(S_n) > 0$ for all $n \in \mathbb{N}$. Then, by Lemma 3.11, there exists a $T_n \in \mathcal{M}_d^+(H)$ such that $\|S_n - T_n\| < \frac{1}{n}$.
Next, for every $n \in \mathbb{N}$, we have

$$\|T_n - T\| \leq \|T_n - S_n\| + \|S_n - T\| \leq \frac{2}{n}.$$

Therefore, $T_n \to T$ in norm as $n \to \infty$.

Combining both the cases, we can conclude that $\overline{\mathcal{M}_d^+(H)} = \mathcal{B}^+(H)$. □

The following result is an easy consequence of the above theorem.

Corollary 4.5. *Let $T \in \mathcal{B}^+(H)$. Then, the set of minimum attaining positive operators is dense in $\mathcal{B}^+(H)$; that is, $\overline{\mathcal{M}^+(H)} = \mathcal{B}^+(H)$.*

Proof. Since $\mathcal{M}_d^+(H) \subseteq \mathcal{M}^+(H)$, the proof follows directly from Theorem 4.4. □

To prove similar results for the class of self-adjoint operators, we need the following lemma.

Lemma 4.6. *Let (M_1, M_2) be a completely reducing pair for $T \in \mathcal{B}(H)$. Let $T_1 = T|_{M_1}$, and let $T_2 = T|_{M_2}$, and let $m(T_1) < m(T_2)$. Then $T \in \mathcal{M}_d(H)$ if and only if $T_1 \in \mathcal{M}_d(M_1)$.*

Proof. We have $T = T_1 \oplus T_2$. Then $T^* = T_1^* \oplus T_2^*$. It follows that $|T| = |T_1| \oplus |T_2|$. From [21, Theorem 5.4, page 289], we have

$$\sigma(|T|) = \sigma(|T_1|) \cup \sigma(|T_2|). \tag{4.1}$$

Using Remark 2.15, we can conclude that

$$m(T) = \min\{m(T_1), m(T_2)\}. \tag{4.2}$$

Therefore, $m(T) = m(T_1)$. Let $T \in \mathcal{M}_d(H)$. Then $m(T) \in \sigma_{disc}(|T|)$. That means, $m(T)$ is an eigenvalue for $|T|$ with finite multiplicity, which is also an isolated point of $\sigma(|T|)$. By Remark 2.15, $m(T) \notin \sigma(|T_2|)$. From [21, Theorem 5.4, page 289], we have

$$\sigma_p(|T|) = \sigma_p(|T_1|) \cup \sigma_p(|T_2|). \tag{4.3}$$

Therefore, we can conclude that $m(T_1) = m(T) \in \sigma_p(|T_1|)$. Clearly, $m(T_1)$ is an isolated point of $\sigma(|T_1|)$, since it is isolated in a bigger set $\sigma(|T|)$.
Next, the multiplicity of $m(T_1)$ is finite because M_2 does not contribute anything to the multiplicity of $m(T_1)$ as $m(T_1) \notin \sigma_p(|T_2|)$. So, we can conclude that $m(T_1) \in \sigma_{disc}(|T_1|)$. Consequently, $T_1 \in \mathcal{M}_d(M_1)$.

Conversely, let $T_1 \in \mathcal{M}_d(M_1)$. Then, $m(T_1)$ is an eigenvalue for $|T_1|$ with finite multiplicity, which is also an isolated point of $\sigma(|T_1|)$. Now, Equation (4.2) implies that $m(T) = m(T_1)$. From Equation (4.3), we have $m(T) \in \sigma_p(|T|)$. By Remark 2.15, $m(T) \notin \sigma(|T_2|)$. Now, the fact that $\sigma(|T_2|)$ is a closed set implies that $m(T)$ is not a limit point of $\sigma(|T_2|)$. Already, it is not a limit point of $\sigma(|T_1|)$. Consequently, it is an isolated point of $\sigma(|T|)$. Its multiplicity is finite, because $m(T) \notin \sigma_p(|T_2|)$. Therefore, $m(T) \in \sigma_{disc}(|T|)$ and $T \in \mathcal{M}_d(H)$. □

Remark 4.7. Suppose $m(T_2) < m(T_1)$ in Lemma 4.6; then it still holds true with the roles of T_1 and T_2 interchanged.

Remark 4.8. Suppose $m(T_1) = m(T_2)$ in Lemma 4.6, then it need not hold true. For instance, we have an example below.

Example 4.9. Let $M_1 = [e_n\colon 1 \leq n \leq 5, n \in \mathbb{N}]$, and let $M_2 = [e_n\colon n > 5, n \in \mathbb{N}]$. Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(e_n) = \begin{cases} (n-1)e_n & \text{if } 1 \leq n \leq 5, \\ \frac{e_n}{n} & \text{if } n > 5. \end{cases}$$

Clearly, (M_1, M_2) is a completely reducing pair for T. Let $T_1 = T|_{M_1}$, and let $T_2 = T|_{M_2}$. We have $T_1 \in \mathcal{M}_d(M_1)$ but $T \notin \mathcal{M}_d(H)$. Notice that $m(T_1) = m(T_2) = 0$.

The following theorem proves that $\mathcal{M}_d^s(H)$ is a very large set in $\mathcal{B}^s(H)$; in fact it is dense.

Theorem 4.10. *Let $T \in \mathcal{B}^s(H)$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d^s(H)$ for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, $\overline{\mathcal{M}_d^s(H)} = \mathcal{B}^s(H)$.*

Proof. We prove the result in two cases separately.

Case(I): $m(T) > 0$: Let $T = V|T|$ be the polar decomposition of T. Since $T^* = T$, we have $V^* = V$. The Hilbert space H has the decomposition

$$H = H_+ \oplus H_-,$$

where $H_+ = N(I - V)$ and $H_- = N(I + V)$. Also, (H_+, H_-) is a completely reducing pair for T and $|T|$. Let $T_1 = T|_{H_+} = |T|\big|_{H_+}$, and let $T_2 = T|_{H_-} = -|T|\big|_{H_-}$. Then we have

$$T = T_1 \oplus T_2. \tag{4.4}$$

Moreover, T_1 is strictly positive and T_2 is strictly negative (for details, see [20, Example 7.1, page 139]).

First we consider, the case $m(T_1) \leq m(T_2)$. Then Equation (4.2) gives that $m(T) = m(T_1)$. So, $m(T_1) > 0$. Since $T_1 \in \mathcal{B}^+(H)$, from Lemma 3.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $S_n \to T_1$ in norm as $n \to \infty$. Moreover, $m(S_n) < m(T_1)$ for all $n \in \mathbb{N}$.
Now, consider the sequence of operators $\{T_n\}_{n\geq 1}$ where $T_n := S_n \oplus T_2$ for all $n \in \mathbb{N}$. Being the direct sum of two self-adjoint operators implies that T_n is self-adjoint for all $n \in \mathbb{N}$. By applying Lemma 4.6, we can conclude that $T_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$.

Next consider the case $m(T_2) < m(T_1)$. Then, $m(T) = m(T_2) > 0$. Now,the fact that T_2 is strictly negative implies $-T_2$ is strictly positive. Also, we have $m(-T_2) = m(T_2) > 0$. From Lemma 3.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $S_n \to -T_2$ in norm as $n \to \infty$. Moreover, $m(S_n) < m(-T_2)$ for all $n \in \mathbb{N}$. Let us consider the sequence of operators $\{T_n\}_{n\geq 1}$, where $T_n := T_1 \oplus -S_n$ for all $n \in \mathbb{N}$. The rest of the proof is same as above.

Case(II) $m(T) = 0$: For each $n \in \mathbb{N}$, let us consider $S_n = T + \frac{1}{n} P_{N(T)}$. Then, $m(S_n) > 0$ for all $n \in \mathbb{N}$. By the Case (I) above, there exists a $T_n \in \mathcal{M}_d^s(H)$ such that $\|S_n - T_n\| < \frac{1}{n}$. Next, for every $n \in \mathbb{N}$, we have

$$\|T_n - T\| \leq \|T_n - S_n\| + \|S_n - T\| \leq \frac{2}{n}.$$

Therefore, $T_n \to T$ in norm as $n \to \infty$.

Combining both the cases, we can conclude that $\overline{\mathcal{M}_d^s(H)} = \mathcal{B}^s(H)$. □

Corollary 4.11. *Let $T \in \mathcal{B}^s(H)$. Then, the set of minimum attaining self-adjoint operators is dense in $\mathcal{B}^s(H)$. That is, $\overline{\mathcal{M}^s(H)} = \mathcal{B}^s(H)$.*

Proof. Since $\mathcal{M}_d^s(H) \subseteq \mathcal{M}^s(H)$, the proof follows directly from Theorem 4.10. □

It follows from [17, Theorem 3.5] that $\mathcal{M}(H)$ is dense in $\mathcal{B}(H)$. Along the similar lines, one expects $\overline{\mathcal{M}_d(H)} = \mathcal{B}(H)$. But it is not the case. We will observe this in the following results.

First we prove that specific operators in $\mathcal{B}(H)$ can be approximated by the operators in $\mathcal{M}_d(H)$.

Theorem 4.12. *Let $T \in \mathcal{B}(H)$, and let $m_e(T) > 0$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the theorem in the following two cases separately.

Case(I) $m(T) = 0$: Since $m_e(T) > 0$, we have $m(T) = m(|T|) = 0 \in \sigma_{disc}(|T|)$ and $T \in \mathcal{M}_d(H)$. Now, the result follows trivially by taking $T_n = T$ for all $n \in \mathbb{N}$.

Case (II) $m(T) > 0$: Let $T = V|T|$ be the Polar decomposition of T. Since $m(T) > 0$, we have T is injective and V is an isometry. We have $|T| \in \mathcal{B}^+(H)$ and $m(|T|) = m(T) > 0$. By Lemma 3.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $S_n \to |T|$ in norm as $n \to \infty$. Put $T_n := V S_n$ for all $n \in \mathbb{N}$.

Next, $|T_n| = (S_n V^* V S_n)^{\frac{1}{2}} = S_n$ implies that $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$. □

The lemma below is an important tool in proving the next theorem.

Lemma 4.13. *Let $T \in \mathcal{B}(H)$. Then there exists a sequence of closed range operators $\{T_n\}_{n\geq 1} \subseteq \mathcal{B}(H)$ such that $N(T_n) = N(T)$ for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, the set of all closed range operators are dense in $\mathcal{B}(H)$.*

Proof. Let $T = V|T|$ be the Polar decomposition of T. Put $T_n := V(|T|+\frac{1}{n}P_{\overline{R(|T|)}})$ for all $n \in \mathbb{N}$. Then, for every $x \in H$ we have

$$\begin{aligned}
\|T_n x\|^2 &= \left\| V\left(|T| + \frac{1}{n}P_{\overline{R(T)}}\right) x \right\|^2 \\
&= \left\langle V\left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}\right) x, V\left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}x\right)\right\rangle \\
&= \left\langle \left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}\right) x, \left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}x\right)\right\rangle \text{ (since } V^*V = I) \\
&= \|Tx\|^2 + \frac{2}{n}\langle |T|x, x\rangle + \frac{1}{n^2}\left\|P_{\overline{R(|T|)}}x\right\|^2.
\end{aligned}$$

From this equation it follows that $N(T_n) = N(T)$ for all $n \in \mathbb{N}$.
For every $x \in N(T_n)^\perp = N(T)^\perp = N(|T|)^\perp = \overline{R(|T|)}$, we have

$$\|T_n x\| \geq \frac{1}{n}\|x\|. \tag{4.5}$$

Therefore, the reduced minimum modulus $\gamma(T_n) = \inf\{\|T_n x\| \colon x \in N(T_n)^\perp\} > 0$, and hence $R(T_n)$ is closed [8, page 363, Proposition 6.1] for all $n \in \mathbb{N}$. □

The next result is again about approximation of specific kind of operators on a separable Hilbert space by the operators in $\mathcal{M}_d(H)$.

Theorem 4.14. *Let H be a separable infinite dimensional complex Hilbert space, and let $T \in \mathcal{B}(H)$ be such that $m_e(T) = m_e(T^*) = 0$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the theorem in the following three cases separately.

Case(I) dim $N(T) = 0$: Since, $m_e(T) = 0$, we have $m(T) = 0$ and T is injective. Let $T = V|T|$ be the Polar decomposition of T. Now, T is injective implies V is an isometry. We have $|T| \in \mathcal{B}^+(H)$ and $m(|T|) = m(T) = 0$. By Case(II) of Theorem 4.4, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$ for all $n \in \mathbb{N}$ and $S_n \to |T|$ in norm as $n \to \infty$. Let us set $T_n := VS_n$ for all $n \in \mathbb{N}$. We see that $|T_n| = (S_n V^* V S_n)^{\frac{1}{2}} = S_n$ implies that $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$.

Case(II) $0 < \dim N(T) < \infty$: We have $N(T) \neq 0$. Since $m_e(T) = 0$, $R(|T|)$ is not closed. From Lemma 4.13, there exists a sequence of closed range operators $\{T_n\}_{n\geq 1} \subseteq \mathcal{B}(H)$ such that $N(T_n) = N(T) \neq 0$ and $T_n \to T$ in norm as $n \to \infty$. Now, $R(T_n)$ is closed implies that $R(|T_n|)$ is closed and dim $N(|T_n|)$= dim $N(T_n) < \infty$. Therefore, $|T_n|$ is a Fredholm operator of index '0'. By Remark 2.8, $0 \notin \sigma_{ess}(|T_n|)$. But $0 \in \sigma_p(|T_n|) \subseteq \sigma(|T_n|)$. So, $0 \in \sigma_{disc}(|T_n|)$. Since $m(|T_n|) = 0$, we have $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$.

Case(III) dim $N(T) = \infty$: Suppose that dim $N(T^*) = \infty$. Then, ind T = dim$N(T)-$dim$N(T^*) = 0$. From Theorem 2.16(i), we have $\rho_1(T) = \inf\{\|T-S\| : S \in \mathcal{B}(H) \text{ and } \dim N(S) = 1\} = 0$. Then for every $n \in \mathbb{N}$, we can find a S_n with dim $N(S_n) = 1$ such that $\|T - S_n\| \leq \frac{1}{n}$ and using Lemma 4.13, there exists a T_n such that $\|S_n - T_n\| \leq \frac{1}{n}$. We have, dim$N(|T_n|)$ = dim $N(S_n) = 1$ and $R(|T_n|)$ is closed. Therefore, $|T_n|$ is a Fredholm operator of index '0'. By Remark 2.8,

$0 \notin \sigma_{ess}(|T_n|)$. But $0 \in \sigma_p(|T_n|) \subseteq \sigma(|T_n|)$. So, $0 \in \sigma_{disc}(|T_n|)$. Since $m(|T_n|) = 0$, we have $T_n \in \mathcal{M}_d(H)$ for all $n \in \mathbb{N}$. Obviously, $T_n \to T$ in norm as $n \to \infty$.

In the case, if dim $N(T^*) < \infty$, then ind $T = \dim N(T) - \dim N(T^*) = \infty$. From Theorem 2.16(ii), we have $\rho_1(T) = \inf\{\|T - S\| : S \in \mathcal{B}(H) \text{ and } \dim N(S) = 1\} = 0$. Rest of the proof is same as above, when dim $N(T^*) = \infty$. □

Now, we observe that the set $\mathcal{M}_d(H)$ is not dense in $\mathcal{B}(H)$, for the case of separable infinite dimensional complex Hilbert space H.

Remark 4.15. Let H be a separable infinite dimensional complex Hilbert space, and let $T \in \mathcal{B}(H)$ be such that $m_e(T) = 0$ and $m_e(T^*) > 0$. Then, $R(T)$ is closed and dim $N(T) = \infty$. Denote put $r := m(T^*)$. From Theorem 2.16(ii), we have dim $N(S) = \infty$ for all $S \in B(T, r)$ and $\mathcal{M}_d(H) \cap B(T, r) = \emptyset$. Therefore, Theorem 4.12 is not valid in this case.

We pose the following question.

Question 4.16. Are Theorem 4.14 and Remark 4.15 still valid, if H is a non separable complex Hilbert space?

For a fixed $T \in \mathcal{B}(H)$, set

$$B_T^d := \{S \in \mathcal{M}_d(H) \colon S + T \in \mathcal{M}(H)\}.$$

Next theorem measures the size of the set B_T^d in $\mathcal{M}_d(H)$.

Theorem 4.17. *Let $T \in \mathcal{B}(H)$. Then B_T^d is dense in $\mathcal{M}_d(H)$.*

Proof. Let $S \in \mathcal{M}_d(H)$, and let $S \notin B_T^d$. From Theorem 3.5 of [17], there exists a sequence of bounded operators $\{R_n\}_{n\geq 1}$ such that $T_n := S + T + R_n \in \mathcal{M}(H)$ for all $n \in \mathbb{N}$ and $T_n := S + T + R_n \to S + T$ in norm as $n \to \infty$. Since $S \in \mathcal{M}_d(H)$, by Theorem 4.1, it follows that $S + R_n \in \mathcal{M}_d(H)$ for all large n. Therefore, $S + R_n \in B_T^d$ and B_T^d is dense in $\mathcal{M}_d(H)$. □

We know that the set of minimum attaining operators is dense in $\mathcal{B}(H)$. On the other hand we observe that the set of not minimum attaining operators is very small in $\mathcal{B}(H)$.

Theorem 4.18. *Let H be a separable Hilbert space. Then the set of all non minimum attaining operators is nowhere dense in $\mathcal{B}(H)$.*

Proof. Let $E = \{T \in \mathcal{B}(H) \colon T \notin \mathcal{M}(H)\}$, and let $S := \{T \in \mathcal{B}(H) \colon m(T) \in \sigma_{ess}(|T|)\}$. Let $\{A_n\}$ be a Cauchy sequence in S. Then by the completeness of $\mathcal{B}(H)$, there exists a $A \in \mathcal{B}(H)$ such that $A_n \to A$ in norm. Suppose that $m(A) \notin \sigma_{ess}(|A|)$. Then $m(A) \in \sigma_{disc}(|A|)$. Now, by Theorem 4.1, $m(A_n) \in \sigma_{disc}(|A|)$ for large n. This contradicts $\{A_n\} \subseteq S$. Therefore, $A \in S$ and S is a closed set.

To prove E is nowhere dense, it is enough to prove that $\overline{E}^{\complement}$ is dense in $\mathcal{B}(H)$. If $m(T) = m(|T|) \notin \sigma_{ess}(|T|)$, then $m(T) \in \sigma_{disc}(|T|)$, and Proposition 2.11 implies that $T \in \mathcal{M}(H)$. Consequently, $E \subseteq S$ and so $\overline{E} \subseteq S$. In view of Remark 4.15, we also have $\overline{E} \subseteq \mathcal{B}(H) \setminus \{T \in \mathcal{B}(H) \colon m(T) = 0 \text{ and } m(T^*) > 0\}$. It follows that $\mathcal{M}_d(H) \cup \{T \in \mathcal{B}(H) \colon m(T) = 0 \text{ and } m(T^*) > 0\} \subseteq \overline{E}^{\complement}$. Applying Theorem 4.12

and Theorem 4.14, we can conclude that $\overline{E}^{\complement}$ is dense in $\mathcal{B}(H)$. Hence the result holds. $\square$

Now, we pose the following question.

Question 4.19. Is Theorem 4.18 still valid, if H is a non separable complex Hilbert space?

For a fixed $T \in \mathcal{B}(H)$, put

$$C_T^d := \{S \in \mathcal{M}_d(H) \colon S + T \notin \mathcal{M}(H)\}.$$

The following theorem measures the size of the set C_T^d in $\mathcal{M}_d(H)$.

Theorem 4.20. *Let $T \in \mathcal{B}(H)$. Then C_T^d is nowhere dense in $\mathcal{M}_d(H)$.*

Proof. Let us define the set F by $F := \{S \in \mathcal{B}(H) \colon S + T \notin \mathcal{M}(H)\}$ and the set R by $R := \{S \in \mathcal{B}(H) \colon m(S+T) \in \sigma_{ess}(|S+T|)\}$. Note that we can prove R is closed by the similar arguments as given in Theorem 4.18.

We observe that $F \subseteq R$. Let $m(S+T) = m(|S+T|) \notin \sigma_{ess}(|S+T|)\}$. Then, $m(S+T) \in \sigma_{disc}(|S+T|)\}$ and Proposition 2.11 implies that $S + T \in \mathcal{M}(H)$. Consequently, $F \subseteq R$ and so $\overline{F} \subseteq R$. From Theorem 4.17, we have $R^{\complement} = B_T^d$ is dense in $\mathcal{M}_d(H)$. It follows that R is nowhere dense in $\mathcal{M}_d(H)$. Since $F \subseteq R$, we can conclude that F is also nowhere dense in $\mathcal{M}_d(H)$. $\square$

Finally, we pose the following question.

Question 4.21. Let $T \in \mathcal{B}(H)$. Then, is the set C_T^d porous in $\mathcal{M}_d(H)$?

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