Ann. Funct. Anal. 2 (2011), no. 1, 206–219
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

IDEAL-TRIANGULARIZABILITY OF UPWARD DIRECTED SETS OF POSITIVE OPERATORS

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Communicated by G. Androulakis

ABSTRACT. In this paper we consider the question when an upward directed set of positive ideal-triangularizable operators on a Banach lattice is (simultaneously) ideal-triangularizable. We prove that a majorized upward directed set of ideal-triangularizable positive operators, which are compact or abstract integral operators is ideal-triangularizable. We also prove that a finite subset of an additive semigroup of positive power compact quasinilpotent operators is ideal-triangularizable. Moreover, we prove that an additive semigroup of positive power compact quasinilpotent operators of bounded compactness index is ideal-triangularizable.

1. INTRODUCTION AND PRELIMINARIES

Suppose $\mathcal{P}$ is a class of operators on a Banach lattice and let $\{T_\alpha\}_\alpha \subseteq \mathcal{P}$ be an upward directed set of positive operators such that $\sup_\alpha T_\alpha$ exists. It is very natural to consider under, which conditions the operator $\sup_\alpha T_\alpha$ is also an element of $\mathcal{P}$. In some cases, the answer is affirmative. One of the most trivial examples is the whole class of positive operators. A negative example is provided by the set of positive compact operators on a Banach lattice l^∞. Indeed, for every positive integer n, the operator $T_n : l^\infty \to l^\infty$ defined by

$$T_n(x_1, x_2, \ldots,) = (x_1, \ldots, x_n, 0, \ldots)$$

is a compact operator, however the operator $\sup_\alpha T_\alpha$ is the identity operator on l^∞, which isn't compact. In this paper we will consider conditions under, which the supremum of upward directed nets of positive ideal-triangularizable operators is ideal-triangularizable. This paper is organised as follows. In Section

Date: Received: 28 February 2011; Accepted: 16 May 2011.
2010 *Mathematics Subject Classification.* Primary 47A15; Secondary 47B65, 16N40.
Key words and phrases. Positive operators, ideal-triangularizability, upward directed sets, power compact operators, nilpotent algebras.

2 we prove (Proposition 2.3) that ideal-triangularization of families of operators is inherited by ideal-quotients. Similar results are proved in [11] for semigroups of compact operators on Banach spaces. We will apply Proposition 2.3 in the proof of Theorem 3.3, which is the main theorem of Section 3. The main result (Theorem 4.1) in the last section states that every additive semigroup of positive quasinilpotent power compact operators of bounded compactness index is ideal-triangularizable, which extends [8, Theorem 2.2].

We first recall some necessary definitions and results. Let E be a Banach lattice. By an *operator* on E we mean a continuous linear transformation from E into itself. An operator T is called *positive* whenever $Tx \in E^+$ for all $x \in E^+$. An operator is *order bounded* (resp. *regular*) if it maps order intervals into order intervals (resp. if it can be written as a difference of two positive operators). For the terminology not explained in the text we refer to the books [9] and [12].

A family $\mathcal{F}$ of operators on E is said to be *reducible* if there exists a nontrivial closed subspace of E that is invariant under every member of $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is *irreducible*. If there exists a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of $\mathcal{F}$, then $\mathcal{F}$ is said to be *triangularizable*. By [11, Theorem 7.1.9], the chain $\mathcal{C}$ of closed subspaces of a Banach space X is maximal if and only if the following conditions are satisfied:

(a) $\{0\}$ and X are in $\mathcal{C}$.
(b) $\mathcal{C}$ is complete in the sense that it is closed under arbitrary intersections and closed spans.
(c) If $\mathcal{M}$ is in the chain and if
$$\mathcal{M}_- = \bigvee\{\mathcal{N} \in \mathcal{C} :\ \mathcal{N} \subseteq \mathcal{M}, \mathcal{N} \neq \mathcal{M}\},$$
then the dimension of $\mathcal{M}/\mathcal{M}_-$ is at most one.

If the chain $\mathcal{C}$ satisfies (a) and (b), then the chain $\mathcal{C}$ is said to be *complete*.

A family $\mathcal{F} \subseteq \mathcal{L}(E)$ is said to be *ideal-reducible* (resp. *band-reducible*) if there exists a nontrivial closed ideal (resp. band) of E, which is invariant under every operator in $\mathcal{F}$. A family $\mathcal{F}$ is *ideal-triangularizable* if there is a chain $\mathcal{C}$ that is maximal as a chain of closed ideals of E and that has the property that every ideal in $\mathcal{C}$ is invariant under all the operators in $\mathcal{F}$. Any such chain of closed ideals is said to be an *ideal-triangularizing* chain for the family $\mathcal{F}$. By [4, Proposition 1.2], every maximal chain of closed ideals of a Banach lattice is also maximal as a chain of closed subspaces of a Banach space.

A subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be a *semigroup ideal* if ST and TS belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$.

The following proposition, which was proved in [5], is very useful in determining if a given semigroup of positive operators is ideal-reducible.

Proposition 1.1. *Let E be a normed Riesz space, and let $\mathcal{S}$ be a nonzero semigroup of positive operators on E. The following statements are equivalent:*

(a) *$\mathcal{S}$ is ideal-reducible;*
(b) *some nonzero semigroup ideal of $\mathcal{S}$ is ideal-reducible.*

A Banach lattice E is *Dedekind complete* if every order bounded subset of E has a supremum and infimum. In this case, the set $\mathcal{L}_b(E)$ of all order bounded operators is a Dedekind complete Riesz space, which coincides with the set $\mathcal{L}_r(E)$ of all regular operators. If $\mathcal{F}$ is a majorized upward directed set of positive operators on a Dedekind complete Banach lattice, then the supremum $\sup \mathcal{F}$ exists and for every $x \geq 0$ it is given by the formula

$$\sup \mathcal{F}(x) = \sup\{Sx : \ S \in \mathcal{F}\}.$$

The following lemma will be used in the proof of Theorem 3.3.

Lemma 1.2. *Let T be a positive operator on a Banach lattice E and let $\mathcal{F}$ be a majorized upward directed set of positive operators on E. Then the following statements hold.*

(a) *If T is ideal-triangularizable, then every positive operator S on E, which satisfies $0 \leq S \leq T$ is also ideal-triangularizable having the same ideal-triangularizing chain as the operator T.*
(b) *If E is Dedekind complete, then every band invariant under $\mathcal{F}$ is also invariant under $\sup \mathcal{F}$.*
(c) *If E has an order continuous norm and if $\mathcal{F}$ is ideal-triangularizable, then $\sup \mathcal{F}$ is also ideal-triangularizable having the same ideal-triangularizing chain as $\mathcal{F}$.*

Note that in Lemma 1.2(a) and Lemma 1.2(c) the operators S and $\sup \mathcal{F}$ respectively may have other ideal-triangularizing chains.

Proof. Since (a) is very easy to see and (c) immediately follows from (b), we will prove only (b). Let $\mathcal{I}$ be a band invariant under every operator from the family $\mathcal{F}$. If $x \in \mathcal{I}$ is an arbitrary positive vector, then $\sup \mathcal{F}x = \sup\{Sx : \ S \in \mathcal{F}\} \in \mathcal{I}$, as $\mathcal{I}$ is a band. Since the cone of $\mathcal{I}^+$ is generating, we see that $\mathcal{I}$ is invariant under $\mathcal{F}$. □

Let $\mathcal{F}$ be a family of operators on E, and let I and J be closed ideals of E satisfying $J \subseteq I$ that are invariant under every member of $\mathcal{F}$. Then $\mathcal{F}$ induces a family $\widehat{\mathcal{F}}$ of operators on the quotient Banach lattice I/J as follows. For each $T \in \mathcal{F}$ the operator $\widehat{T}$ is defined on I/J by

$$\widehat{T}(x + J) = Tx + J \ .$$

Because I and J are invariant under T, the operator $\widehat{T}$ is a well-defined operator on I/J. Any such $\widehat{\mathcal{F}}$ is called a *collection of ideal-quotients* of the family $\mathcal{F}$. A set $\mathcal{P}$ of properties is said to be *inherited by ideal-quotients* if every collection of ideal-quotients of a family of operators satisfying properties in $\mathcal{P}$ also satisfies the same properties.

The proof of the following lemma can be found in [3, Lemma 2.3]

Lemma 1.3 (The Ideal-triangularization Lemma)**.** *Let $\mathcal{P}$ be a set of properties inherited by ideal-quotients. If every family of operators on a Banach lattice of dimension greater than one, which satisfies $\mathcal{P}$ is ideal-reducible, then every such family is ideal-triangularizable.*

2. Inheritence of ideal-triangularization

In this section we will prove some order analogs of results that hold for triangularizable families of compact operators on Banach spaces. The monograph [11] contains a very detailed overview of these results.

If $\{I_\alpha\}_\alpha$ is a family of ideals of a Riesz space E, it is well known that their linear span $\sum_\alpha I_\alpha$ is also an ideal of E. The following result is probably known. We include a proof for the convenience of the reader.

Lemma 2.1. *For ideals $\{I_\alpha\}_\alpha$ and J of a Riesz space E the following formula holds.*

$$\left(\sum_\alpha I_\alpha\right) \cap J = \sum_\alpha (I_\alpha \cap J).$$

Proof. Since the inclusion $\sum_\alpha (I_\alpha \cap J) \subseteq (\sum_\alpha I_\alpha) \cap J$ is obvious, we will prove only the opposite inclusion. Let x be any element of $(\sum_\alpha I_\alpha) \cap J$. Then there exists indices $\alpha_1, \ldots, \alpha_n$ such that $x \in (\sum_{i=1}^n I_{\alpha_i}) \cap J$. By [12, Proposition II.2.1] it follows

$$x \in \left(\sum_{i=1}^n I_{\alpha_i}\right) \cap J = \sum_{i=1}^n (I_{\alpha_i} \cap J) \subseteq \sum_\alpha (I_\alpha \cap J).$$

□

Lemma 2.2. *Let E be a Banach lattice and let $J \subseteq E$ be a closed ideal. Let $\mathcal{C} = \{I_\alpha\}_\alpha$ be a chain of closed ideals of E. Then the following set equalities hold*

$$\bigcap_\alpha (I_\alpha \cap J) = \left(\bigcap_\alpha I_\alpha\right) \cap J, \tag{2.1}$$

$$\bigvee_\alpha (I_\alpha \cap J) = \left(\bigvee_\alpha I_\alpha\right) \cap J, \tag{2.2}$$

$$\bigcap_\alpha ((I_\alpha + J)/J) = \left(\bigcap_\alpha I_\alpha + J\right)/J \tag{2.3}$$

and

$$\bigvee_\alpha ((I_\alpha + J)/J) = \left(\bigvee_\alpha I_\alpha + J\right)/J. \tag{2.4}$$

Moreover, if the chain $\mathcal{C}$ is complete, then $\{I_\alpha \cap J\}_\alpha$ and $\{(I_\alpha + J)/J\}_\alpha$ are complete chains of closed ideals of J and E/J respectively.

Proof. The formula (2.1) is obvious. To show the formula (2.2), recall first that Lemma 2.1 implies

$$\sum_\alpha (I_\alpha \cap J) = \left(\sum_\alpha I_\alpha\right) \cap J. \tag{2.5}$$

Taking closures on both sides of the formula (2.5) and applying [12, Proposition II.1.1] we get

$$\bigvee_\alpha (I_\alpha \cap J) = \overline{\sum_\alpha I_\alpha \cap J} = \overline{\sum_\alpha I_\alpha} \cap J = \left(\bigvee_\alpha I_\alpha\right) \cap J.$$

Let us denote by J_α the sum $I_\alpha + J$. By [15, Theorem 15.18], the set J_α is a closed ideal in E that contains J. Since the inclusions $\left(\bigcap_\alpha J_\alpha\right)/J \subseteq \bigcap_\alpha (J_\alpha/J)$ and $\bigvee_\alpha (J_\alpha/J) \subseteq \left(\bigvee_\alpha J_\alpha\right)/J$ are obvious, we will prove the opposite inclusions. If $x + J \in \bigcap_\alpha (J_\alpha/J)$, then $x \in J_\alpha$ for each α, as $J \subseteq J_\alpha$. Thus, $x \in \bigcap_\alpha J_\alpha$, which proves the formula (2.3).

To finish the proof of the formula (2.4), observe first that we have

$$\sum_\alpha (J_\alpha/J) = \left(\sum_\alpha J_\alpha\right)/J. \tag{2.6}$$

Let us denote by $q : E \to E/J$ the quotient projection. Taking closures on both sides in the formula (2.6) we obtain

$$\bigvee_\alpha (J_\alpha/J) = \overline{\left(\sum_\alpha J_\alpha\right)/J} = \overline{q\left(\sum_\alpha J_\alpha\right)}.$$

Since q is continuous, we have

$$\left(\bigvee_\alpha J_\alpha\right)/J = q\left(\overline{\sum_\alpha J_\alpha}\right) \subseteq \overline{q\left(\sum_\alpha J_\alpha\right)} = \bigvee_\alpha (J_\alpha/J),$$

which completes the proof of the formula (2.4).

If the chain $\mathcal{C}$ is complete, applying formulas (2.1), (2.2), (2.3) and (2.4) we see that the chains $\{I_\alpha \cap J\}_\alpha$ and $\{(I_\alpha + J)/J\}_\alpha$ are also complete, which finishes the proof. □

The following proposition states that ideal-triangularizability of operators is inherited by ideal-quotients.

Proposition 2.3. *Let E be a Banach lattice and let $\mathcal{F}$ be a family of operators on E. Let J be a closed ideal in E, which is invariant under every operator from $\mathcal{F}$. Then $\mathcal{F}$ is ideal-triangularizable if and only if the families $\{S|_J : J \to J,\ S \in \mathcal{F}\}$ and $\{\widehat{S} : E/J \to E/J,\ S \in \mathcal{F}\}$ are ideal-triangularizable.*

Proof. Suppose first that the family $\mathcal{F}$ is ideal-triangularizable. Then there exists a maximal chain $\mathcal{C}$ of closed ideals invariant under every operator $S \in \mathcal{F}$. We claim that the chain $\mathcal{C}' = \{I \cap J\}_{I \in \mathcal{C}}$ is a maximal chain of closed ideals in J. It is obvious that each closed ideal from $\mathcal{C}'$ is invariant under every operator $S \in \mathcal{F}$. Since $\{0\}$ and E are in $\mathcal{C}$, $\{0\}$ and J are in $\mathcal{C}'$. Since $\mathcal{C}$ is a complete chain, Lemma 2.2 implies that the chain $\mathcal{C}'$ is also complete. Let $I_\alpha \cap J$ be any closed ideal in the chain $\mathcal{C}'$. By the formula (2.2) we have $(I_\alpha \cap J)_- = (I_\alpha)_- \cap J$ and so the inequality

$$\dim((I_\alpha \cap J)/((I_\alpha)_- \cap J)) \leq \dim(I_\alpha/(I_\alpha)_-) \leq 1$$

implies that the chain $\mathcal{C}'$ is an ideal-triangularizing chain for the family $\{S|_J : J \to J,\ S \in \mathcal{F}\}$.

It is obvious that every closed ideal in the chain $\mathcal{C}'' = \{(I_\alpha + J)/J\}_\alpha$ is invariant under every operator from the family $\{\widehat{S} : E/J \to E/J,\ S \in \mathcal{F}\}$. We claim that $\mathcal{C}''$ is in fact an ideal-triangularizing chain for the family $\{\widehat{S} : E/J \to E/J,\ S \in \mathcal{F}\}$. Since $\{0\}, E \in \mathcal{C}$, by the definition of $\mathcal{C}''$, closed ideals 0 and E/J are also elements of $\mathcal{C}''$. By Lemma 2.2, the chain $\mathcal{C}''$ is also complete. Let $(I_\alpha + J)/J$ be any element of the chain $\mathcal{C}''$. Then the formula (2.4) implies

$$((I_\alpha + J)/J)_- = ((I_\alpha)_- + J)/J.$$

Since the lattice of closed ideals of a Banach lattice is distributive, we apply [14, Theorem 3.7] and [14, Theorem 3.8] to get the following chain of vector space isomorphisms

$$((I_\alpha + J)/J) \,/\, (((I_\alpha)_- + J)/J) \cong (I_\alpha + J) \,/\, ((I_\alpha)_- + J) =$$
$$= (I_\alpha + (I_\alpha)_- + J) \,/\, ((I_\alpha)_- + J) \cong I_\alpha / \,(I_\alpha \cap ((I_\alpha)_- + J)) = I_\alpha/((I_\alpha)_- + (I_\alpha \cap J)).$$

Since $\mathcal{C}$ is a triangularizing chain for every operator from the family $\mathcal{F}$, the dimension $\dim(I_\alpha/((I_\alpha)_- + (I_\alpha \cap J)))$ is at most 1. Therefore, $\mathcal{C}''$ is a maximal chain of closed ideals of E/J and the family $\{\widehat{S} : E/J \to E/J,\ S \in \mathcal{F}\}$ is ideal-triangularizable.

Now suppose that the families $\{S|_J : J \to J,\ S \in \mathcal{F}\}$ and $\{\widehat{S} : E/J \to E/J,\ S \in \mathcal{F}\}$ are ideal-triangularizable with ideal-triangularizing chains $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively and let $q : E \to E/J$ denotes the quotient projection. Then it is obvious that the chain $\mathcal{C}_1 \cup \{q^{-1}(I)\}_{I \in \mathcal{C}_2}$ is a maximal chain of closed ideals in E whose elements are invariant under every operator $S \in \mathcal{F}$. $\square$

The following results are order analogs of [11, Theorem 7.3.9] and [11, Lemma 8.2.15], respectively.

Corollary 2.4. *Let $\mathcal{F}$ be an ideal-triangularizable family of operators on a Banach lattice E. Then every chain of closed ideals invariant under $\mathcal{F}$ is contained in an ideal-triangularizing chain.*

Proof. Let $\mathcal{C}$ be a chain of closed ideals invariant under $\mathcal{F}$. A simple Zorn lemma argument shows the existence of a maximal chain $\mathcal{C}'$ of closed ideals invariant under $\mathcal{S}$, which contains $\mathcal{C}$. Since the chain is maximal, it is a complete chain. We claim that this chain is also maximal as a subspace chain of E. Suppose that there exists $J \in \mathcal{C}'$ such that $\dim(J/J_-) \geq 2$ and let us denote by $q : J \to J/J_-$ the quotient projection. By Proposition 2.3, the induced family $\widehat{\mathcal{F}}$ on J/J_- is ideal-reducible. If $\mathcal{K}$ is any nontrivial closed ideal in J/J_- invariant under $\widehat{\mathcal{F}}$, then $q^{-1}(\mathcal{K})$ is a closed ideal in E that satisfies $J_- \subsetneq q^{-1}(\mathcal{K}) \subsetneq J$ and is invariant under $\mathcal{F}$. The chain $\mathcal{C}' \cup \{q^{-1}(\mathcal{K})\}$ is larger than a chain $\mathcal{C}'$, which is in contradiction with the maximality of $\mathcal{C}'$. $\square$

Corollary 2.5. *Let $\mathcal{S}$ be a semigroup of positive operators on a Banach lattice E. If a semigroup ideal $\mathcal{J}$ of $\mathcal{S}$ has a unique ideal-triangularizing chain of closed ideals, then $\mathcal{S}$ is ideal-triangularizable and has a unique ideal-triangularizing chain.*

Proof. Since the ideal $\mathcal{J}$ is ideal-reducible, Proposition 1.1 implies that $\mathcal{S}$ is ideal-reducible. To apply the Ideal-triangularization Lemma, we have to show that the property of having a unique ideal-triangularizing chain is inherited by ideal-quotients. Let $\mathcal{L}$ and $\mathcal{K}$ be closed ideals invariant under $\mathcal{J}$ satisfying $\mathcal{L} \subsetneq \mathcal{K}$. By [12, Proposition III.1.3], chains of closed ideals of $\mathcal{K}$ invariant under $\mathcal{J}$, which start with $\mathcal{L}$ are in a bijective correspondence with chains of closed ideals invariant under the semigroup ideal $\widehat{\mathcal{J}}$ of ideal-quotients by an ideal $\mathcal{J}$ on a quotient Banach lattice $\mathcal{K}/\mathcal{L}$. Every chain of closed ideals invariant under $\mathcal{J}$ is contained in an ideal-triangularizing chain by Theorem 2.4. Since $\mathcal{J}$ has a unique ideal-triangularizing chain, it follows that also $\widehat{\mathcal{J}}$ has a unique ideal-triangularizing chain. □

For a family $\mathcal{F}$ of operators on a Banach lattice E we define $\mathcal{F}^n$ as the set of all products of at least n operators from $\mathcal{F}$, and by $\mathrm{sg}(\mathcal{F})$ we denote the semigroup generated by $\mathcal{F}$. It is obvious that for each $n \in \mathbb{N}$ the set $\mathcal{F}^n$ is a semigroup ideal in $\mathrm{sg}(\mathcal{F}) = \mathcal{F}^1$. The following proposition can be seen as an order analog of [11, Lemma 8.2.14].

Proposition 2.6. *Let E be a Banach lattice and $\mathcal{F}$ a family of positive operators on E. If there exists $n \in \mathbb{N}$ such that $\mathcal{F}^n$ is ideal-triangularizable, then $\mathcal{F}$ is ideal-triangularizable.*

Proof. Since ideal-triangularizability of families of operators is inherited by ideal-quotients by Proposition 2.3, it suffices to show that the family $\mathcal{F}$ is ideal-reducible. If there exists $n \in \mathbb{N}$ such that $\mathcal{F}^n = \{0\}$, then the same argument applied in the proof of [8, Proposition 2.1] shows that $\mathcal{F}$ is ideal-reducible, otherwise ideal-reducibility of $\mathcal{F}^n$ and Proposition 1.1 imply ideal-reducibility of $\mathrm{sg}(\mathcal{F})$. □

A Banach lattice E is an AL- (resp. AM-) space if for each pair of disjoint positive vectors $x, y \in E$ it holds that $\|x+y\| = \|x\| + \|y\|$ (resp. $\|x+y\| = \max\{\|x\|, \|y\|\}$). A Banach space X has the *Dunford-Pettis property* whenever for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging weakly to zero and each sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in X^* converging weakly to 0, the sequence $\{\varphi_n(x_n)\}_{n\in\mathbb{N}}$ converges to zero. By Grothendieck's Theorem [1, Theorem 5.85], AL- and AM-spaces have the Dunford-Pettis property. The following result, which is an easy application of Proposition 2.6 extends [5, Theorem 7.2] asserting that every semigroup of positive weakly compact quasinilpotent operators on an AL-space or an AM-space is ideal-triangularizable.

Theorem 2.7. *Every semigroup $\mathcal{S}$ of positive quasinilpotent weakly compact operators on Banach lattice with the Dunford-Pettis property is ideal-triangularizable.*

Proof. Since E has the Dunford-Pettis property, [1, Theorem 19.8] implies that every product of two operators from $\mathcal{S}$ is a compact operator. By [3, Theorem 4.5], the semigroup ideal $\mathcal{S}^2$ is ideal-triangularizable and so Proposition 2.6 implies ideal-triangularizability of $\mathcal{S}$. □

A nonzero positive vector a of a Banach lattice E is said to be an *atom* whenever for each vector $0 \leq x \leq a$ there exists a scalar $\lambda \geq 0$ such that $x = \lambda a$. If E does not have any atoms, it is said to be *atomless*. A Banach lattice E is said to be *atomic* if the band generated by the set of all atoms of E is E. We proceed with an extension of [5, Theorem 7.3]. Since the proof is very similar to the proof of Theorem 2.7 and [5, Theorem 7.3], we omit it.

Theorem 2.8. *Every semigroup of positive weakly compact ideal-triangularizable operators on an atomless Banach lattice with order continuous norm and the Dunford-Pettis property is ideal-triangularizable.*

In Section 4, we will extend Theorem 2.7 to the case of upward directed sets of positive quasinilpotent weakly compact operators. In general, a semigroup of quasinilpotent power compact positive operators is not ideal-triangularizable. For the construction of such semigroups see [7] or [8].

3. Ideal-triangularizability of upward directed sets of positive operators

Let E be a Banach lattice and $\mathcal{F}$ a family of positive ideal-triangularizable operators on E. In this section we consider the question when the operator $\sup \mathcal{F}$ (if it exists) is also ideal-triangularizable. The following example shows that even if $\mathcal{F}$ is a semigroup of positive operators on a finite dimensional Banach lattice, $\sup \mathcal{F}$ need not be ideal-triangularizable.

Example 3.1. Let $n \geq 2$ be a positive integer. For all $1 \leq i, j \leq n$ we denote by E_{ij} the positive matrix unit whose (i,j)-the entry is 1 and others entries are zero. The set $\mathcal{F} = \{E_{ij} : \ 1 \leq i, j \leq n\} \cup \{0\}$ is a multiplicative semigroup of ideal-triangularizable positive matrices, however

$$\sup \mathcal{F} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix},$$

is obviously ideal-irreducible positive matrix.

Before we proceed to the main theorem of this section, we need to recall some definitions. If a is an atom of a Banach lattice E, then [9, Theorem 26.4] implies that B_a is a projection band and so we have $E = B_a \oplus B_a^d$. Therefore, for every $f \in E$ there exist $\lambda \in \mathbb{R}$ and $g \in B_a^d$ such that $f = \lambda a + g$. Note that λ and g are uniquely determined. Let φ_a be the positive linear functional on E defined by $\varphi_a(f) = \lambda$. It follows from $|f| = |\lambda| a + |g| \geq |\lambda| a$ that the functional φ_a is bounded. To each positive operator T on E we associate the *zero-set* $Z(T)$ of all atoms $a \in E$ such that $\varphi_a(Ta) = 0$, or equivalently, $Ta \wedge a = 0$. An atom $a \in Z(T)$ is called *zero* of an operator T. The notion of a zero set of positive operators on L^p spaces was first introduced in [10], however the Banach lattice setting was first considered in [5].

If $\mathcal{F}$ is assumed to be an upward directed set of positive operators, there are some positive results.

Proposition 3.2. *Let E be a Banach lattice with order continuous norm and let $\{T_\alpha\}_\alpha$ be an upward directed set of positive ideal-triangularizable operators on E.*

(a) *If E contains atoms, then $\{T_\alpha\}_\alpha$ is ideal-reducible. If $\{T_\alpha\}_\alpha$ is majorized, then $\sup_\alpha T_\alpha$ is ideal-reducible.*
(b) *If E is atomic, then $\{T_\alpha\}_\alpha$ is ideal-triangularizable. If $\{T_\alpha\}_\alpha$ majorized, then $\sup_\alpha T_\alpha$ is ideal-triangularizable.*

Proof. To see (a), assume that E contains atoms. Let $\mathcal{S}$ be the set

$$\{T \geq 0 : \text{ there exist } \alpha, \text{ a scalar } t \geq 0 \text{ and } n \in \mathbb{N} \text{ such that } 0 \leq T \leq tT_\alpha^n\}.$$

We claim that $\mathcal{S}$ is a multiplicative semigroup of positive operators such that every pair of operators from $\mathcal{S}$ is ideal-triangularizable. To prove this, let T_1 and T_2 be elements of $\mathcal{S}$. By the assumption, there exist positive operators $T_{\alpha_1}, T_{\alpha_2}$, positive real numbers t_1, t_2 and positive integers n_1, n_2 such that

$$T_1 \leq t_1 T_{\alpha_1}^{n_1} \qquad \text{and} \qquad T_2 \leq t_2 T_{\alpha_2}^{n_2}.$$

Since $\{T_\alpha\}$ is upward directed, there exists T_β such that $T_{\alpha_i} \leq T_\beta$ for $i = 1, 2$. This implies

$$T_1 \leq t_1 T_\beta^{n_1}, \qquad T_2 \leq t_2 T_\beta^{n_2} \tag{3.1}$$

and

$$T_1 T_2 \leq t_1 t_2 T_{\alpha_1}^{n_1} T_{\alpha_2}^{n_2} \leq (t_1 t_2) T_\beta^{n_1+n_2}. \tag{3.2}$$

The inequality (3.2) implies that $\mathcal{S}$ is a multiplicative semigroup. Lemma 1.2 and inequalities in (3.1) imply that the pair $\{T_1, T_2\}$ is ideal-triangularizable.

If there exists a nonzero operator $S \in \mathcal{S}$ such that the zero set $Z(S) \neq \emptyset$, then [5, Lemma 4.1] implies that every operator in the semigroup ideal $\mathcal{J}$ generated by the operator S in $\mathcal{S}$ has a common zero. [5, Proposition 4.3] implies that $\mathcal{J}$ is ideal-reducible and so by Proposition 1.1 the semigroup $\mathcal{S}$ is ideal-reducible. So we may assume that $Z(S) = \emptyset$ for all $S \in \mathcal{S}$. Since every operator in $\mathcal{S}$ is ideal-triangularizable, by [5, Proposition 4.4] the semigroup $\mathcal{S}$ is ideal-reducible. Lemma 1.2 implies that $\sup_\alpha T_\alpha$ is ideal-reducible, which proves (a).

To see (b) we first recall the fact that every ideal-quotient of an atomic Banach lattice with order continuous norm is also atomic with order continuous norm. An application of Proposition 2.3, Lemma 1.3 and (a) finishes the proof of (b). □

The following theorem is the main theorem of this section.

Theorem 3.3. *Let E be a Banach lattice with order continuous norm and let $\{T_\alpha\}_\alpha$ be an upward directed set of positive ideal-triangularizable operators on E.*

(a) *If for every α the operator T_α is a compact operator or an abstract integral operator, then $\{T_\alpha\}_\alpha$ is ideal-triangularizable. Moreover, if $\{T_\alpha\}$ is majorized, then $\sup_\alpha T_\alpha$ is ideal-triangularizable.*
(b) *If $\{T_\alpha\}_\alpha$ is majorized and if the operator $\sup_\alpha T_\alpha$ is power compact, then $\sup_\alpha T_\alpha$ is also an ideal-triangularizable operator.*

Proof. Let $\mathcal{S}$ be the set

$$\{T \geq 0 : \text{ there exist } \alpha, \text{ a scalar } t \geq 0 \text{ and } n \in \mathbb{N} \text{ such that } 0 \leq T \leq tT_\alpha^n\}.$$

In the proof of Proposition 3.2 we have seen that $\mathcal{S}$ is a multiplicative semigroup with the property that every pair of operators from $\mathcal{S}$ is ideal-triangularizable.

To prove (a), Lemma 1.2 implies that it suffices to prove that $\mathcal{S}$ is ideal-triangularizable. Since ideal-triangularizability of operators is inherited by ideal-quotients by Proposition 2.3, it also suffices to prove that $\mathcal{S}$ is ideal-reducible. If E contains atoms, then $\mathcal{S}$ is ideal-reducible by Proposition 3.2. So we may assume that E is an atomless Banach lattice with order continuous norm.

Obviously, $\mathcal{S} = \{0\}$ if for every α the operator T_α is zero. Assume first that there exists α such that T_α is a compact operator. Let $\mathcal{J}$ be the semigroup ideal in $\mathcal{S}$ generated by the operator T_α. Then $\mathcal{J}$ consists of positive compact operator such that every pair of operators from $\mathcal{J}$ is ideal-triangularizable. By [5, Theorem 6.7], the semigroup ideal $\mathcal{J}$ is ideal-triangularizable and so $\mathcal{S}$ is ideal-reducible by Proposition 1.1. Suppose now that for every α the operator T_α is an abstract integral operator. Since the set of all abstract integral operators forms a band, every operator in the semigroup $\mathcal{S}$ is an ideal-triangularizable abstract integral operator by [5, Lemma 3.1] and so [5, Corollary 6.5] implies that $\mathcal{S}$ is ideal-triangularizable.

To see (b), we have to prove that $\{T_\alpha\}_\alpha$ is ideal-reducible as every property in (b) is inherited by ideal-quotients and then we can apply Lemma 1.3. We may also assume that the operator $T := \sup_\alpha T_\alpha$ is not quasinilpotent since every positive power compact quasinilpotent operator is ideal-triangularizable by [6, Theorem 1.3]. Therefore, there exists a positive integer n such that T^n is a nonzero compact operator. Since the norm of E is order continuous, every positive operator on E is order continuous and so [1, Exercise 1.4.8] (see also [2]) implies that for every positive integer n we have $0 \le T_\alpha^n \uparrow T^n$. If all operators T_α are nilpotent, a result of H. J. Krieger (see [16, Theorem 135.8] or [13, Theorem 2.4]) implies that T is quasinilpotent, which is a contradiction. Therefore, we may assume that there exists α such that T_α is nonnilpotent power compact operator. A theorem of Aliprantis and Burkinshaw ([1, Theorem 5.14]) implies that the operator T_α^{3n} is also compact. The semigroup ideal $\mathcal{J}$ generated by the positive compact operator T_α^{3n} of the semigroup $\mathcal{S}$ consists of positive compact operators and every pair of operators in $\mathcal{J}$ is ideal-triangularizable. Again, [5, Theorem 6.7] implies that $\mathcal{J}$ is ideal-triangularizable and so $\mathcal{S}$ is ideal-reducible. ☐

Corollary 3.4. *Let $\{T_\alpha\}_\alpha$ be an upward directed set of positive quasinilpotent compact operators on a Banach lattice with an order continuous norm. If the supremum $\sup_\alpha T_\alpha$ exists and is a compact operator, then $\sup_\alpha T_\alpha$ is an ideal-triagularizable positive compact quasinilpotent operator.*

Proof. By [6, Theorem 1.3], every operator T_α is ideal-triangularizable and so the operator $T := \sup_\alpha T_\alpha$ is an ideal-triangularizable positive compact operator by Theorem 3.3. By [16, Theorem 135.8], the operator $\sup_\alpha T_\alpha$ is quasinilpotent, which finishes the proof. ☐

Corollary 3.5. *Let E be an atomless Banach lattice with order continuous norm and let $\{T_\alpha\}_\alpha$ be an upward directed set of positive ideal-triangularizable compact operators on E. If the supremum $\sup_\alpha T_\alpha$ exists and is a compact operator, then $\sup_\alpha T_\alpha$ is an ideal-triagularizable positive compact quasinilpotent operator.*

In general, every ideal-triangularizable set $\mathcal{F}$ of positive operators is contained in an ideal-triangularizable upward directed set of positive operators. Indeed, if $\mathcal{S}$ is the additive semigroup generated by the set $\mathcal{F}$, then it is obviously ideal-triangularizable and upward directed. Ideal-triangularizability of additive semigroups will be considered in the following section.

4. Ideal-triangularizability of power compact operators

In [8], the author considered ideal-triangularizability of algebras generated by positive operators. The author proved that every nil-algebra generated by positive operators and with bounded nilpotency index is necessarily ideal-triangularizable and nilpotent. If T is a power compact operator on a Banach space, then the smallest positive integer n such that T^n is a compact operator is called the *index of compactness* of the operator T. In this section we will extend [8, Theorem 2.2] and partly extend [8, Corollary 2.4] to the case of additive semigroups of positive power compact quasinilpotent operators with bounded compactness index. We also deduce some results concerning upward directed sets of positive power compact quasinilpotent operators.

Theorem 4.1. *Let $\mathcal{S}$ be an additive semigroup of positive quasinilpotent power compact operators on a Banach lattice E. If the index of compactness of operators from $\mathcal{S}$ is bounded, then $\mathcal{S}$ is ideal-triangularizable. Moreover, if every operator in $\mathcal{S}$ is nilpotent, then the algebra generated by $\mathcal{S}$ is a nilpotent algebra.*

Proof. It is enough to prove that $\mathcal{S}$ is ideal-reducible since ideal-triangularizability of $\mathcal{S}$ follows from Lemma 1.3. By the assumption, there exists a positive integer $n \in \mathbb{N}$ such that S^n is a compact operator for every operator from the additive semigroup $\mathcal{S}$. Let $\mathcal{S}_1$ be the multiplicative semigroup generated by $\mathcal{S}$. We claim that in $\mathcal{S}_1$ every operator is a quasinilpotent power compact operator and the compactness index of operators in $\mathcal{S}_1$ is bounded. To see this, let $S_1, \ldots, S_m$ be arbitrary operators in $\mathcal{S}$. Then the operator $S_1 + \cdots + S_m$ is an element of $\mathcal{S}$ and so the operator $(S_1 + \cdots + S_m)^{mn}$ is compact and quasinilpotent. Since

$$0 \leq (S_1 \cdots S_m)^n \leq (S_1 + \cdots + S_m)^{mn},$$

the operator $S_1 \cdots S_m$ is quasinilpotent and Aliprantis-Burkinshaw theorem [1, Theorem 5.14] implies that the operator $(S_1 \cdots S_m)^{3n}$ is compact. If there exists a nonnilpotent operator S in $\mathcal{S}$, then the semigroup ideal generated with S^n in $\mathcal{S}_1$ is ideal-triangularizable by [3, Theorem 4.5] and so $\mathcal{S}_1$ is ideal-reducible by Proposition 1.1.

Suppose now that every operator in $\mathcal{S}$ is nilpotent. Since n is the index of compactness of operators from $\mathcal{S}$, we have $S^n = 0$ for all $S \in \mathcal{S}$. If $S_1, \ldots, S_n$ are arbitrary operators from $\mathcal{S}$, then

$$0 \leq S_1 \cdots S_n \leq (S_1 + \cdots + S_n)^n = 0$$

implies that every product of at least n operators from $\mathcal{S}$ is zero. So, $\mathcal{S}_1^n = \{0\}$ and therefore, the algebra $\mathcal{A}$ generated by $\mathcal{S}$ satisfies $\mathcal{A}^n = \{0\}$, which is obviously nilpotent and ideal-triangularizable by [8, Proposition 2.1]. □

Theorem 4.2. *Let $\mathcal{S}$ be an additive semigroup of positive quasinilpotent power compact operators and let $\mathcal{A}$ be the algebra generated by $\mathcal{S}$. Then the following statements hold.*

(a) *Every finite subset (of not necessarily positive operators) of $\mathcal{A}$ is ideal-triangularizable.*

(b) *If $\mathcal{A}$ is Artinian, then $\mathcal{A}$ is ideal-triangularizable. In particular, every finite dimensional algebra of quasinilpotent power compact operators generated by positive operators is ideal-triangularizable.*

Proof. To prove (a), let $\mathcal{F}$ be a finite subset (of not necessarily positive operators) in $\mathcal{A}$. Then $\mathcal{F}$ is contained in an algebra $\mathcal{A}_1$ generated by positive quasinilpotent power compact operators $S_1, \ldots, S_m \in \mathcal{F}$. Let n be the smallest positive integer such that $(S_1 + \cdots + S_m)^n$ is a compact operator and let $\mathcal{S}_1$ be the additive semigroup generated by operators $S_1, \ldots, S_m$. Obviously we have $\mathcal{S}_1 \subseteq \mathcal{S}$ and for every operator $S \in \mathcal{S}_1$ the operator S^{3n} is a compact operator by Aliprantis-Burkinshaw theorem [1, Theorem 5.14]. Theorem 4.1 implies that $\mathcal{S}_1$ is ideal-triangularizable.

To prove (b), assume first that $\mathcal{S}$ contains a nonnilpotent operator S and let $\mathcal{S}_1$ be a multiplicative semigroup generated by an additive semigroup $\mathcal{S}$. There exists a positive integer n such that S^n is a compact operator. Since every operator in $\mathcal{S}_1$ is quasinilpotent, the semigroup ideal in $\mathcal{S}_1$ generated by the operator S^n is ideal-triangularizable by [3, Theorem 4.5] and so $\mathcal{S}_1$ is ideal-reducible by Proposition 1.1. If every operator in $\mathcal{S}$ is nilpotent, then the algebra $\mathcal{A}$ is a nil-algebra which is ideal-triangularizable by [8, Theorem 2.5]. To finish the proof note that the property of being an Artinian algebra is inherited by ideal-quotients.

□

Corollary 4.3. *Let $\mathcal{S}$ be an additive semigroup of ideal-triangularizable power compact operators on an atomless Banach lattice E with an order continuous norm. Then all the statements of Theorem 4.2 hold for the algebra generated by $\mathcal{S}$.*

Proof. Let T be an arbitrary power compact in $\mathcal{A}$. If T is not nilpotent, then there exists a positive integer n such that T^n is a nonzero compact operator. Similarly as in [5, Proposition 4.5] we can see that T^n is quasinilpotent. Therefore, every operator from $\mathcal{A}$ is quasinilpotent and so we apply Theorem 4.2. □

The following corollary is a companion to Theorem 3.3(b). In Theorem 3.3 we assumed that the upward directed set is dominated by a power compact operator. In that case, the compactness index of operators in the upward directed set was automatically bounded by Aliprantis-Burkinshaw theorem. In the following corollary we assume quasinilpotence of the operators in the upward directed set instead of their ideal-triangularizability.

Corollary 4.4. *An upward directed set of positive power compact quasinilpotent operators of bounded compactness index is ideal-triangularizable.*

Proof. Let $\mathcal{F}$ be an upward directed set satisfying the hypothesis of the corollary, and let $\mathcal{S}$ be an additive semigroup generated by the set $\mathcal{F}$. If $S_1, \ldots, S_n$ are arbitrary operators in $\mathcal{S}$, then

$$\sum_{i=1}^{n} S_i = \sum_{i=1}^{k} T_i$$

for some operators $T_i \in \mathcal{F}$ and $i = 1, \ldots, k$. Since the set $\mathcal{F}$ is upward directed, there exists an operator $T \in \mathcal{F}$, such that $\sum_{i=1}^{k} T_i \leq kT$. If m is the compactness index of $\mathcal{F}$, then T^m is a compact operator and so the inequality $0 \leq (\sum_{i=1}^{n} S_i)^m \leq k^m T^m$ and Theorem [1, Theorem 5.14] imply that the operator $(\sum_{i=1}^{n} S_i)^{3m}$ is compact and quasinilpotent. To finish the proof we apply Theorem 4.1. □

Corollary 4.5. *Every upward directed set of positive compact quasinilpotent operators is ideal-triangularizable.*

The authors in [7] constructed an irreducible set of positive nilpotent operators of unbounded nilpotency index on the space $L^p[0,1)$, which is multiplicative, additive and is closed under multiplication by positive scalars. This example shows that in general we cannot omit the assumption that the index of compactness is bounded in Theorem 4.1 and Corollary 4.4. In the same paper, the authors constructed an ideal-irreducible multiplicative semigroup of square-zero operators which shows that in general we cannot omit the assumption that the semigroup is additive. We finish this paper with the following corollaries.

Corollary 4.6. *Let $\mathcal{S}$ be an additive semigroup of positive weakly compact quasinilpotent operators on a Banach lattice with the Dunford-Pettis property. Then for the algebra $\mathcal{A}$ generated by $\mathcal{S}$ the statements of Theorems 4.1 and 4.2 hold .*

Corollary 4.7. *Every upward directed set of quasinilpotent weakly compact operators on a Banach lattice with the Dunford-Pettis property is ideal-triangularizable.*

Acknowledgement. The author was supported by the Slovenian Research Agency. The author also like to thank professor Roman Drnovšek for useful comments and discussions.

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