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MINIMIZATION OF CONSTRAINED QUADRATIC FORMS IN HILBERT SPACES

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ABSTRACT. A common optimization problem is the minimization of a symmetric positive definite quadratic form $\langle x, Tx \rangle$ under linear constraints. The solution to this problem may be given using the Moore–Penrose inverse matrix. In this work at first we extend this result to infinite dimensional complex Hilbert spaces, where a generalization is given for positive operators not necessarily invertible, considering as constraint a singular operator. A new approach is proposed when T is positive semidefinite, where the minimization is considered for all vectors belonging to $\mathcal{N}(T)^{\perp}$.

1. INTRODUCTION

Quadratic forms have played a central role in the history of mathematics, in both the finite and the infinite dimensional case. Many authors have studied problems on minimizing (or maximizing) quadratic forms under various constraints, such as vectors constrained to lie within the unit simplex (Broom [5]). A similar result is the minimization of a more general case of a quadratic form defined in a finite-dimensional real Euclidean space under linear constraints (see e.g. La Cruz [13], Manherz and Hakimi [15]), with many applications in network analysis and control theory. The same problem is encountered in Rostamian [18], this time directed towards the study of some boundary value problems in the theory of linear elasticity.

In a classical book of Optimization Theory by Luenberger [14], various similar optimization problems are presented, for both finite and infinite dimensions.

In the field of applied mathematics, a strong interest is shown in applications

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of the generalized inverse of matrices or operators. Various types of generalized inverses are used whenever a matrix/operator is singular, in many fields of both computational and also theoretical aspects. An application of the Moore–Penrose inverse in the finite dimensional case, is the minimization of a symmetric positive definite quadratic form under linear constraints. This application can be used in many optimization problems, such as electrical networks (Ben–Israel [2]), finance (Markowitz [16, 17]) etc. A similar result for positive semidefinite quadratic forms with many applications in Signal Processing is presented by Stoica et al [19], Gorkhov and Stoica [10].
In this work at first we extend the result of Ben–Israel [2] for positive operators acting on infinite dimensional complex Hilbert spaces. We will consider the quadratic form as a diagonalizable, diagonal or a positive operator in general, not necessarily invertible. In the sequel we consider the case of a positive semidefinite quadratic form, and the new approach proposed for this problem is the constrained minimization to take place only for the vectors perpendicular to its kernel. This can be achieved using an appropriate decomposition of the Hilbert space.
Another possible candidate for this work would be the class of compact self adjoint operators, making use of the spectral theorem. Unfortunately, compact operators do not have closed range, therefore their generalized inverse is not a bounded operator.

2. Preliminaries and notation

The notion of the generalized inverse of a (square or rectangular) matrix was first introduced by H. Moore in 1920, and again by R. Penrose in 1955. These two definitions are equivalent, and the generalized inverse of an operator or matrix is also called the Moore–Penrose inverse. It is known that when T is singular, then its unique generalized inverse $T^\dagger$ is defined. In the case when T is a real $r \times m$ matrix, Penrose showed that there is a unique matrix satisfying the four Penrose equations, called the generalized inverse of T, noted by $T^\dagger$.
In what follows, we consider $\mathcal{H}$ a separable infinite dimensional Hilbert space and all operators mentioned are supposed to have closed range.
The generalized inverse of an operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, is the unique operator satisfying the following four conditions:

$$TT^\dagger = (TT^\dagger)^*, \qquad T^\dagger T = (T^\dagger T)^*, \qquad TT^\dagger T = T, \qquad T^\dagger T T^\dagger = T^\dagger \tag{2.1}$$

where T^* denotes the adjoint operator of T.
It is easy to see that $TT^\dagger$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$, denoted by P_T, and that $T^\dagger T$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T^*)$ noted by P_{T^*}.
It is well known that $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$, and that $T^\dagger = (T^*T)^{-1}|_{\mathcal{R}(T^*)}T^*$.
It is also known that $T^\dagger$ is bounded if and only if T has a closed range.
If T has a closed range and commutes with $T^\dagger$, then T is called an EP operator. EP operators constitute a wide class of operators which includes the self adjoint operators, the normal operators and the invertible operators.
Let us consider the equation $Tx = b, T \in B(\mathcal{H})$, where T is singular. If $b \notin R(T)$, then the equation has no solution. Therefore, instead of trying to solve the

equation $\|Tx - b\| = 0$, we may look for a vector u that minimizes the norm $\|Tx - b\|$. Note that the vector u is unique. In this case we consider the equation $Tx = P_{R(T)}b$, where $P_{R(T)}$ is the orthogonal projection on $\mathcal{R}(T)$.
The following two propositions can be found in Groetsch [9] and hold for operators and matrices:

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $b \in \mathcal{H}$. Then, for $u \in \mathcal{H}$, the following are equivalent:*

(i) $Tu = P_{R(T)}b$
(ii) $\|Tu - b\| \leq \| Tx - b\|, \forall x \in \mathcal{H}$
(iii) $T^*Tu = T^*b$

Let $\mathbb{B} = \{u \in \mathcal{H} | T^*Tu = T^*b\}$. This set of solutions is closed and convex, therefore, it has a unique vector with minimal norm. In the literature, Groetsch[9], $\mathbb{B}$ is known as the set of the generalized solutions.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H}), b \in \mathcal{H}$, and the equation $Tx = b$. Then, if $T^\dagger$ is the generalized inverse of T, we have that $T^\dagger b = u$, where u is the minimal norm solution defined above.*

This property has an application in the problem of minimizing a symmetric positive definite quadratic form $\langle x, Qx\rangle$ subject to linear constraints, assumed consistent (see Theorem 2.3).
We will denote by $\operatorname{Lat} T$ the set of all closed subspaces of the underlying Hilbert space $\mathcal{H}$ invariant under T.
A self adjoint operator $T \in B(\mathcal{H})$ is positive when $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Let T be an invertible positive operator which is diagonalizable. Then, $T = U^*T_kU$ where U is unitary and T_k is diagonal, of the form

$$T_k(x_1, x_2, \dots) = (k_1x_1, k_2x_2, \dots)$$

where $(k_n)_n$ is a bounded sequence of real numbers, and its terms are the eigenvalues of T_k, assumed positive. Its inverse T_k^{-1} is also a diagonal operator, with corresponding sequence $k'_i = \frac{1}{k_i}$.
When T_k is singular, at least one of the k_i's is equal to zero. Then, its Moore–Penrose inverse has a corresponding sequence of diagonal elements k'_i defined as follows:

$$k'_i = \begin{cases} \frac{1}{k_i}, & k_i \neq 0 \\ 0, & k_i = 0 \end{cases}$$

Since all the diagonal elements are nonnegative, in both cases T_k has a unique square root T_m, which is also a diagonal operator with corresponding sequence $m_n = \sqrt{k_n}$. Similar results concerning diagonalizable and diagonal operators can be found in Conway [7].
As mentioned before, EP operators include normal and self adjoint operators, therefore the operator T in the quadratic form studied in this work is EP. An operator T with closed range is called EP if $\mathcal{N}(T) = \mathcal{N}(T^*)$. It is easy to see that

$$T \text{ EP} \Leftrightarrow \mathcal{R}(T) = \mathcal{R}(T^*) \Leftrightarrow \mathcal{R}(T) \overset{\perp}{\oplus} \mathcal{N}(T) = \mathcal{H} \Leftrightarrow TT^\dagger = T^\dagger T. \tag{2.2}$$

We take advantage of the fact that EP operators have a simple canonical form according to the decomposition $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Indeed an EP operator T has the following simple matrix form:

$$T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

where the operator $A : \mathcal{R}(T) \to \mathcal{R}(T)$ is invertible, and its generalized inverse $T^\dagger$ has the form

$$T^\dagger = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

(see Campbell and Meyer [6], Drivaliaris et al [8]).
Another result used in our work, wherever a square root of a positive operator is used, is the fact that EP operators have index equal to 1 and so, $\mathcal{R}(T) = \mathcal{R}(T^2)$. (see Ben Israel [1], pages 156–157)
As mentioned above, a necessary condition for the existance of a bounded generalized inverse is that the operator has closed range. Nevertheless, the range of the product of two operators with closed range is not always closed.
In Bouldin [4] an equivalent condition is given:

Theorem 2.3. *Let A and B be operators with closed range, and let*

$$H_i = \mathcal{N}(A) \cap (\mathcal{N}(A) \cap \mathcal{R}(B))^\perp = \mathcal{N}(A) \ominus \mathcal{R}(B)$$

The angle between H_i and $\mathcal{R}(B)$ is positive if and only if AB has closed range.

A similar result can be found in Izumino [11], this time using orthogonal projections :

Proposition 2.4. *Let A and B be operators with closed range. Then, AB has closed range if and only if $A^\dagger ABB^\dagger$ has closed range.*

We will use the above two results to prove the existence of the Moore–Penrose inverse of appropriate operators which will be used in our work.
Another tool used in this work, is the reverse order law for the Moore–Penrose inverses. In general, the reverse order law does not hold. Conditions under which the reverse order law holds, are described in the following proposition which is a restatement of a part of R. Bouldin's theorem [3] that holds for operators and matrices.

Proposition 2.5. *Let A, B be bounded operators on $\mathcal{H}$ with closed range. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if the following three conditions hold:*
i) The range of AB is closed,
ii) $A^\dagger A$ commutes with BB^*,
iii) $BB^\dagger$ commutes with A^*A.

A corollary of the above theorem is the following proposition that can be found in Karanasios–Pappas [12] and we will use it in our case.

Proposition 2.6. *Let $A, T \in \mathcal{B}(\mathcal{H})$ be two operators such that A is invertible and T has closed range. Then*

$$(TA)^\dagger = A^{-1}T^\dagger \quad \textit{if and only if} \quad \mathcal{R}(T) \in Lat(AA^*).$$

3. The Generalized inverse and minimization of Quadratic forms

Let Q be a symmetric positive definite matrix. Then, Q can be written as $Q = UDU^*$, where U is unitary and D is diagonal.
Let $D^{\frac{1}{2}}$ denote the positive solution of $X^2 = D$, and let $D^{-\frac{1}{2}}$ denote $(D^{\frac{1}{2}})^{-1}$, which exists since Q is positive definite.
The following theorem can be found in Ben Israel [2].

Theorem 3.1. *Consider the equation $Ax = b$.*
If the set $S = \{x : Ax = b\}$ is not empty, then the problem :

$$minimize\langle x, Qx\rangle, x \in S$$

has the unique solution

$$x = UD^{-\frac{1}{2}}(AUD^{-\frac{1}{2}})^{\dagger}b.$$

3.1. Positive Diagonizable Quadratic Forms.

A generalization of the above theorem in infinite dimensional Hilbert spaces, is by replacing Q with an invertible positive operator T which is diagonalizable. The operator A must be singular, otherwise this problem is trivial. Since T is diagonalizable, we have that $T = U^*T_kU$.
We need first the following Lemma. Note that in the infinite dimensional case the Moore–Penrose inverse of an operator is bounded if and only if the operator has closed range.

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible positive operator which is diagonalizable and $A \in \mathcal{B}(\mathcal{H})$ singular with closed range.*
Then, the range of AU^*X^{-1} is closed, where X is the unique solution of the equation $X^2 = T_k$.

Proof. We have that the range of U^*X^{-1} is closed since both operators are invertible, and invertible operators have closed range. Hence, the range of AU^*X^{-1} is closed because since U^*X^{-1} is invertible, $\mathcal{R}(AU^*X^{-1}) = \mathcal{R}(A)$ which is closed. □

We are now in condition to prove Theorem 3.3.

Theorem 3.3. *Consider the equation $Ax = b$, with $A \in \mathcal{B}(\mathcal{H})$ singular with closed range and $b \in \mathcal{H}$.*
If the set $S = \{x : Ax = b\}$ is not empty, then the problem :

$$minimize\langle x, Tx\rangle, x \in S$$

with $T \in \mathcal{B}(\mathcal{H})$ an invertible positive diagonalizable operator with closed range has the unique solution

$$\hat{x} = U^*X^{-1}(AU^*X^{-1})^{\dagger}b$$

where X is the unique solution of the equation $X^2 = T_k$.

Proof. The idea of the proof is similar to Ben Israel [2], but the existence of a bounded Moore–Penrose inverse is not trivial like in the finite dimensional case. It is easy to see that since $T = U^*T_kU$ is positive, T_k is also positive. Then,

$$\langle x, Tx\rangle = \langle x, U^*T_kUx\rangle = \langle Ux, T_kUx\rangle = \langle Ux, X^2Ux\rangle =$$

$$\langle XUx, XUx \rangle = \langle y, y \rangle$$

So, the problem of minimizing $\langle x, Tx \rangle$ is equivalent of minimizing $\langle y, y \rangle = \| y \|^2$ where $y = XUx$. We also have that $y = XUx \Leftrightarrow x = U^* X^{-1} y$.
From proposition 2.2 we know that the minimal norm solution of the equation $Ax = b$ is the vector $\hat{x} = A^\dagger b$, so by substitution, $AU^* X^{-1} y = b$ and the minimal norm vector $\hat{y}$ is equal to $\hat{y} = [AU^* X^{-1}]^\dagger b$, since $AU^* X^{-1}$ is a singular operator. Therefore, $XU\hat{x} = [AU^* X^{-1}]^\dagger b \Leftrightarrow \hat{x} = U^* X^{-1} (AU^* X^{-1})^\dagger b$

□

We can verify that the solution $\hat{x} = U^* X^{-1} (AU^* X^{-1})^\dagger b$ satisfies the constraint $Ax = b$:
We have that $A\hat{x} = AU^* X^{-1} (AU^* X^{-1})^\dagger b = SS^\dagger b = P_S b$, where $S = AU^* X^{-1}$ and we can see that $\mathcal{R}(S) = \mathcal{R}(AU^* X^{-1}) = \mathcal{R}(A)$ since X and U are invertible. Therefore, $P_S = P_A$ and $P_S b = P_A b = b$ since $b \in \mathcal{R}(A)$.
We can also compute the value of the minimum $\langle x, Tx \rangle, x \in S$:

$$\langle \hat{x}, T\hat{x} \rangle = \langle U^* X^{-1} (AU^* X^{-1})^\dagger b, U^* T_k U U^* X^{-1} (AU^* X^{-1})^\dagger b \rangle =$$

$$\langle (AU^* X^{-1})^\dagger b, (AU^* X^{-1})^\dagger b \rangle = \| (AU^* X^{-1})^\dagger b \|^2$$

3.2. **Positive Definite Quadratic Forms.** In this section we extend the results presented above, in the general case when T is a positive operator following the same point of view.
Let T be a positive operator, having a unique square root R.

Theorem 3.4. *Consider the equation $Ax = b$, with $A \in \mathcal{B}(\mathcal{H})$ singular with closed range and $b \in \mathcal{H}$. If the set $S = \{x : Ax = b\}$ is not empty, then the problem :*

$$minimize \langle x, Tx \rangle, x \in S$$

with $T \in \mathcal{B}(\mathcal{H})$ an invertible positive operator with closed range has the unique solution

$$\hat{x} = R^{-1} (AR^{-1})^\dagger b$$

Proof. We have that $\langle x, Tx \rangle = \langle Rx, Rx \rangle = \| Rx \|^2 = \| y \|^2$. So,

$$Ax = b \Leftrightarrow \hat{y} = (AR^{-1})^\dagger b \Leftrightarrow \hat{x} = R^{-1} (AR^{-1})^\dagger b$$

The range of the operator AR^{-1} is closed, as discussed in Lemma 3.2. □

We can see by easy computations, that the minimum $\langle x, Tx \rangle, x \in S$ is then equal to

$$\langle \hat{x}, T\hat{x} \rangle = \langle R^{-1} (AR^{-1})^\dagger b, R^2 R^{-1} (AR^{-1})^\dagger b \rangle =$$

$$\langle (AR^{-1})^\dagger b, (AR^{-1})^\dagger b \rangle = \| (AR^{-1})^\dagger b \|^2$$

Remark 3.5. At this point we can say that Theorem 3.3 is now a Corollary of Theorem 3.4 since when T is an invertible positive diagonalizable operator we have that $R^{-1} (AR^{-1})^\dagger = U^* X^{-1} U (AU^* X^{-1} U)^\dagger = U^* X^{-1} (AU^* X^{-1})^\dagger$.

A natural question to ask, is what happens if the reverse order law for generalized inverses holds. In this case, the solution given by Theorem 3.4, using Proposition 2.4 will be as follows:

Corollary 3.6. *Considering all the assumptions of Theorem 3.4, let* $\mathcal{R}(A) \in Lat(T)$. *Then,* $\hat{x} = A^{\dagger}b$.

The proof is obvious, since in this case, $Lat(RR^*) = Lat(R^2) = Lat(T)$, and $R^{-1}(AR^{-1})^{\dagger} = R^{-1}RA^{\dagger}$
We can also see that in this case, the minimum value of $\langle x, Tx\rangle, x \in S$ is equal to $\langle A^{\dagger}b, R^2A^{\dagger}b\rangle = \parallel RA^{\dagger}b \parallel^2$
We will present an example for Theorem 3.4 and Corollary 3.6.

Example 3.7. Let $T : l_2 \to l_2 : T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, x_3, 2x_4 \dots)$ which is a bounded diagonal linear operator.
Let $L : l_2 \to l_2 : L(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$, the well known left shift operator which is singular and $S = \{x : Lx = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)\}$. It is also well known that $L^{\dagger} = R$, the right shift operator. Since T is positive and invertible, the problem of minimizing $\langle x, Tx\rangle$, $x \in S$ following Corollary 3.6 has the unique solution $\hat{x} = (0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$.
Indeed, since $\mathcal{R}(L) = \mathcal{H} = l_2$ which is invariant under T, we have that $\mathcal{R}(L) \in Lat(T)$ and so

$$\hat{x} = (L)^{\dagger}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$$

Using this vector, we have that the problem of minimizing $\langle x, Tx\rangle, x \in S$ has a minimum value as shown in what follows:

$$min\langle x, Tx\rangle = \langle \hat{x}, T\hat{x}\rangle = \langle (0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots), T(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)\rangle =$$

$$\langle (0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots), (0, 2, \frac{1}{2}, 2\times\frac{1}{3}, \frac{1}{4}, 2\times\frac{1}{5} \dots)\rangle = 0+2\times1+\frac{1}{2^2}+2\times\frac{1}{3^2}+\frac{1}{4}+2\times\frac{1}{5^2}\dots$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{6} + \frac{\pi^2}{8} = \frac{7\pi^2}{24}$$

This value is equal to $\parallel RA^{\dagger}b \parallel^2$ as presented in the above corollary, since in this case $R(x_1, x_2, x_3, \dots) = (x_1, \sqrt{2}x_2, x_3, \sqrt{2}x_4 \dots)$ and

$$\parallel RA^{\dagger}b \parallel^2 = \parallel RL^{\dagger}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) \parallel^2 = 0 + 2\times 1 + \frac{1}{2^2} + 2\times\frac{1}{3^2} + \frac{1}{4} + \dots = \frac{7\pi^2}{24}$$

and this verifies Corollary 3.6.
We can see that the minimizing vector found by Theorem 3.4 has the minimum norm among all possible solutions, of the form $(c, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots), c \in \mathcal{C}$, as expected.

3.3. **Positive Semidefinite Quadratic Forms.** We can also consider the case when the positive operator T is singular, that is, T is positive semidefinite. In this case, since $\mathcal{N}(T) \neq \emptyset$, we have that $\langle x, Tx \rangle = 0$ for all $x \in \mathcal{N}(T)$ and so, the problem :

$$minimize \langle x, Tx \rangle, x \in S$$

has many solutions when $\mathcal{N}(T) \cap S \neq \emptyset$.
In Stoika et al [19] a method is presented for the minimization of a positive semidefinite quadratic form under linear constraints, with many applications in the finite dimensional case. In fact, since this problem has an entire set of solutions, the minimum norm solution is given explicitly.
A different approach to this problem in both the finite and infinite dimensional case would be to look among the vectors $x \in \mathcal{N}(T)^{\perp} = \mathcal{R}(T^*) = \mathcal{R}(T)$ for a minimizing vector for $\langle x, Tx \rangle$. In other words, we will look for the minimum of $\langle x, Tx \rangle$ under the constraints $Ax = b, x \in \mathcal{R}(T)$.
Using the fact that T is an EP operator, we will make use of the first two conditions in the following proposition that can be found in Drivaliaris et al [8]:

Proposition 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then the following are equivalent:*
i) T is EP.
ii) There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $U_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ unitary and $A_1 \in \mathcal{B}(\mathcal{K}_1)$ isomorphism such that

$$T = U_1(A_1 \oplus 0)U_1^*.$$

iii) There exist Hilbert spaces $\mathcal{K}_2$ and $\mathcal{L}_2$, $U_2 \in \mathcal{B}(\mathcal{K}_2 \oplus \mathcal{L}_2, \mathcal{H})$ isomorphism and $A_2 \in \mathcal{B}(\mathcal{K}_2)$ isomorphism such that

$$T = U_2(A_2 \oplus 0)U_2^*.$$

iv) There exist Hilbert spaces $\mathcal{K}_3$ and $\mathcal{L}_3$, $U_3 \in \mathcal{B}(\mathcal{K}_3 \oplus \mathcal{L}_3, \mathcal{H})$ injective and $A_3 \in \mathcal{B}(\mathcal{K}_3)$ isomorphism such that

$$T = U_3(A_3 \oplus 0)U_3^*.$$

We present a sketch of the proof for (1)⇒(2):

Proof. Let $\mathcal{K}_1 = \mathcal{R}(T)$, $\mathcal{L}_1 = \mathcal{N}(T)$, $U_1 : \mathcal{K}_1 \oplus \mathcal{L}_1 \to \mathcal{H}$ with

$$U_1(x_1, x_2) = x_1 + x_2,$$

for all $x_1 \in \mathcal{R}(T)$ and $x_2 \in \mathcal{N}(T)$, and $A_1 = T|_{\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(T)$. Since T is EP, $\mathcal{R}(T) \oplus^{\perp} \mathcal{N}(T) = \mathcal{H}$ and thus U_1 is unitary. Moreover it is easy to see that $U_1^* x = (P_T x, P_{\mathcal{N}(T)} x)$, for all $x \in \mathcal{H}$. It is obvious that A_1 is an isomorphism. A simple calculation shows that

$$T = U_1(A_1 \oplus 0)U_1^*.$$

□

It is easy to see that when $T = U_1(A_1 \oplus 0)U_1^*$ and T is positive, so is A_1, since $\langle x, Tx \rangle = \langle x_1, A_1 x_1 \rangle, x_1 \in \mathcal{R}(T)$.
In what follows, T will denote a singular positive operator with a canonical form

$T = U_1(A_1 \oplus 0)U_1^*$, R is the unique solution of the equation $R^2 = A_1$ and we can define

$$V = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{therefore } V^\dagger = \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

As in the previous cases, since the two operators A and R are arbitrary, one does not expect that the range of their product will always be closed.
Using Proposition 3.8, we have the following theorem:

Theorem 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be an singular positive operator, and the equation $Ax = b$, with $A \in \mathcal{B}(\mathcal{H})$ singular with closed range and $b \in \mathcal{H}$. If the set $S = \{x \in \mathcal{N}(T)^\perp : Ax = b\}$ is not empty, then the problem :*

$$minimize \langle x, Tx \rangle, x \in S$$

has the unique solution

$$\hat{x} = U_1 V^\dagger (AU_1V^\dagger)^\dagger b$$

assuming that $P_{A^*}P_T$ has closed range.

Proof. We have that

$$\langle x, Tx \rangle = \langle x, U_1(A_1 \oplus 0)U_1^* x \rangle = \langle U_1^* x, (A_1 \oplus 0)U_1^* x \rangle = \langle U_1^* x, (R^2 \oplus 0)U_1^* x \rangle$$

We have that $U_1^* x = (x_1, x_2)$ and $\langle U_1^* x, (A_1 \oplus 0)U_1^* x \rangle = \langle x_1, A_1 x_1 \rangle, x_1 \in \mathcal{R}(T)$.
Therefore $\langle x, Tx \rangle = \langle (R \oplus 0)U_1^* x, (R \oplus 0)U_1^* x \rangle = \langle Rx_1, Rx_1 \rangle = \langle y, y \rangle$, where $y = Rx_1$, with $x_1 \in \mathcal{N}(T)^\perp$.
The problem of minimizing $\langle x, Tx \rangle$ is equivalent of minimizing $\| y \|^2$ where $y = Rx_1 = (R \oplus 0)U_1^* x \Leftrightarrow x = U_1(R^{-1} \oplus 0)y = U_1 V^\dagger y$.
As before, the minimal norm solution $\hat{y}$ is equal to $\hat{y} = [AU_1V^\dagger]^\dagger b$.
Therefore, $\hat{x_1} = U_1 V^\dagger (AU_1V^\dagger)^\dagger b$, with $\hat{x_1} \in S$.
As in Theorem 3.3 , we still have to prove that $AU_1V^\dagger$ has closed range.
Using Theorem 2.3, the range of $U_1V^\dagger$ is closed since

$$H_i = \mathcal{N}(U_1^*) \cap (\mathcal{N}(U_1^*) \cap \mathcal{R}(V^\dagger))^\perp = 0$$

and so the angle between U_1^* and $V^\dagger$ is equal to $\frac{\pi}{2}$.
From Proposition 2.4 the operator $P_{A^*}P_T$ must have closed range because

$$A^\dagger A U_1 V^\dagger (U_1 V^\dagger)^\dagger = P_{A^*} U_1 P_R U_1^* = P_{A^*} U_1 P_{A_1} U_1^* = P_{A^*} P_T$$

making use of Proposition 2.6 and the fact that $\mathcal{R}(R) = \mathcal{R}(A_1) = \mathcal{R}(T)$.

□

Corollary 3.10. *Under all the assumptions of Theorem 3.9 we have that the minimum value of $f(x) = \langle x, Tx \rangle, x \in S$ is equal to $\| (AU_1V^\dagger)^\dagger b \|^2$*

Proof. We have that

$$f_{min}(x) = \langle \hat{x}, T\hat{x} \rangle = \langle U_1V^\dagger (AU_1V^\dagger)^\dagger b, TU_1V^\dagger (AU_1V^\dagger)^\dagger b \rangle$$

Since $T = U_1(R^2 \oplus 0)U_1^*$ we have that

$$f_{min}(x) = \langle U_1V^\dagger (AU_1V^\dagger)^\dagger b, U_1(R \oplus 0)(AU_1V^\dagger)^\dagger b \rangle =$$

$$\langle P_T (AU_1V^\dagger)^\dagger b, (AU_1V^\dagger)^\dagger b \rangle = \| (AU_1V^\dagger)^\dagger b \|^2$$

since

$$V^\dagger(R \oplus 0) = (I \oplus 0) = P_T$$

and $\mathcal{R}(AU_1V^\dagger)^\dagger = \mathcal{R}(AU_1V^\dagger)^* = \mathcal{R}(RU_1A^*) \subseteq \mathcal{R}(R) = \mathcal{R}(T)$, therefore

$$P_T(AU_1V^\dagger)^\dagger b = (AU_1V^\dagger)^\dagger b$$

□

In the sequel, we present an example which clarifies Theorem 3.9 and Corollary 3.10. In addition, the difference between the proposed minimization ($x \in \mathcal{N}(T)^\perp$) and the minimization for all $x \in \mathcal{H}$ is clearly indicated.

Example 3.11. Let $\mathcal{H} = \mathcal{R}^3$, and the positive semidefinite matrix

$$Q = \begin{bmatrix} 14 & 20 & 28 \\ 20 & 83 & 40 \\ 28 & 40 & 56 \end{bmatrix}$$

We are looking for the minimum of $f(u) = u'Qu, u \in \mathcal{N}(Q)^\perp$ under the constraint $3x + 2y + z = 4$.
Then, all vectors $u \in \mathcal{N}(Q)^\perp$ have the form $u = (x, y, 2x)^T$. The matrices $U, V^\dagger$ are

$$U = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \qquad V^\dagger = \begin{bmatrix} 0.1410 & -0.0436 & 0 \\ -0.0436 & 0.1284 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using theorem 3.9 we see that the minimizing vector of $f(u)$ under $Au = b, u \in \mathcal{N}(Q)^\perp$, where $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ and b = 4, is

$$\hat{u} = U_1V^\dagger(AU_1V^\dagger)^\dagger b = (0.5827, 0.5432, 1.1655)^T$$

The minimum value of $f(u)$ is then equal to 206.7133
We can verify that it is equal to the minimum value found in Corollary 3.10.

$$\| (AU_1V^\dagger)^\dagger b \|^2 = \| (11.7871, 8.2327, 0)^T \|^2 = 206.7133$$

This example can be represented graphically as follows, clearly showing the constrained minimization and the uniqueness of the solution:
In figure 1(a) we show in red the surface of the quadratic form and in blue, the plane $3x + 2y + z = 4$ (the constraint).
In figure 1(b) we can see we can see that among all vectors belonging to $\mathcal{N}(Q)^\perp$ satisfying the constraint $Au = b$, which are having the form

$$u = (x, 2 - 2.5x, 2x)^T, x \in \mathbb{R}$$

the vector $\hat{u} = (0.5827, 0.5432, 1.1655)^T$ found from Theorem 3.9 minimizes the function $f(u)$.

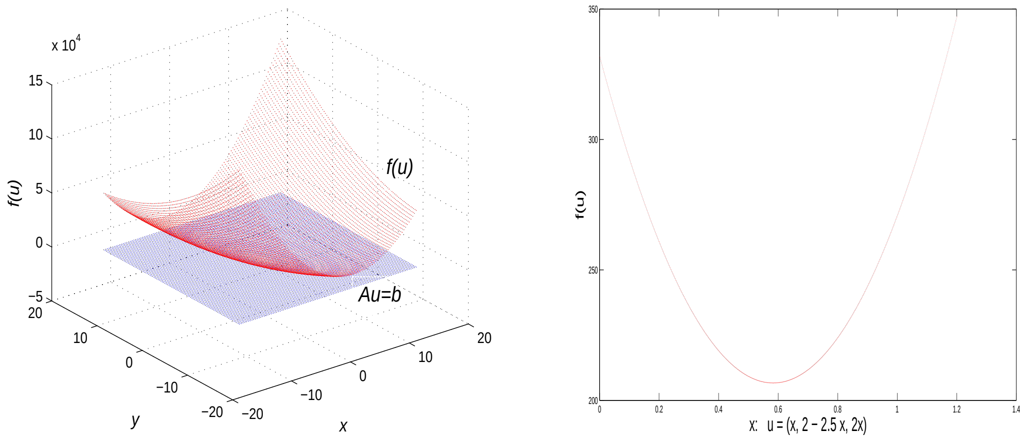

FIGURE 1. Constrained minimization of f(u) = u'Qu, $x \in S = \{x \in \mathcal{N}(Q)^{\perp} : Ax = b\}$

We will also examine the minimization of $f(u)$ in the case when $u \in \mathcal{N}(Q)$, given that $u \in S$, so that the difference between them is clearly indicated:
When the minimization takes place for all vectors $u \in \mathcal{N}(Q)$ the set

$$S' = \{u \in \mathcal{N}(Q) : Au = b\}$$

is nonempty.
The vector $v = (1.6, 0, -0.8)^T$ belongs to $\mathcal{S}'$ and therefore, $f(v) = 0$.
If we consider the case when when u is a random vector in $\mathbb{R}^3$, the answer is still the same, since this vector gives the minimum value $f(v) = 0$ for the constrained minimization. Obviously, the same answer is given using the algorithm proposed by Stoika et al [19].

4. Conclusions

In this work we extend a minimization result concerning non singular quadratic forms using the Moore–Penrose inverse, to infinite dimensional Hilbert spaces. In addition, in the case of a singular quadratic form the minimization takes place for all its non zero values. This proposed Constrained minimization method has the advantage of a unique solution and is easy to implement. Practical importance of this result can be in numerous applications such as filter design, spectral analysis, direction finding etc. In many of these cases the quadratic form may be very close to, or even exactly singular, and therefore the knowledge of the non zero part of the solution may be of importance.

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