SOME RESULTS ON $\sigma$-DERIVATIONS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and let $\mathcal{M}$ be a Banach $\mathcal{B}$-bimodule. Suppose that $\sigma: \mathcal{A} \to \mathcal{B}$ is a linear mapping and $d: \mathcal{A} \to \mathcal{M}$ is a $\sigma$-derivation. We prove several results about automatic continuity of $\sigma$-derivations on Banach algebras. In addition, we define a notion for $m$-weakly continuous linear mapping and show that, under certain conditions, $d$ and $\sigma$ are $m$-weakly continuous. Moreover, we prove that if $\mathcal{A}$ is commutative and $\sigma: \mathcal{A} \to \mathcal{A}$ is a continuous homomorphism such that $\sigma^2 = \sigma$ then $\sigma d \sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq \text{rad}(\mathcal{A})$.

1. Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. Suppose that $\sigma: \mathcal{A} \to \mathcal{B}$ is a linear mapping. A linear mapping $d: \mathcal{A} \to \mathcal{M}$ is called a $\sigma$-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. Clearly if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma = \text{id}$, the identity mapping on $\mathcal{A}$, then a $\sigma$-derivation is an ordinary derivation. On the other hand, each homomorphism $\theta: \mathcal{A} \to \mathcal{B}$ is a $\frac{\theta}{2}$-derivation. Mirzavaziri and Moslehian [5] have presented several important results of $\sigma$-derivations. Hosseini et al [8] defined generalized $\sigma$-derivation on Banach algebras and presented some results about automatic continuity of generalized $\sigma$-derivations and $\sigma$-derivations on Banach algebras. So far, numerous derivations have been defined such as $\sigma$-derivation, generalized $\sigma$-derivation, $(\sigma, \tau)$-derivation and so on. In 2009, Mirzavaziri and Omidvar Tehrani [8] defined $(\delta, \varepsilon)$-double derivation and also the automatic continuity of the former derivation on $C^*$-algebras was considered. Next, Hejazian et al [4] studied the automatic

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continuity of \((\delta, \varepsilon)\)-double derivations on Banach algebras. The investigation of automatic continuity of \((\delta, \varepsilon)\)-double derivations and generalized \(\sigma\)-derivations in detail, will result in some theorems about automatic continuity of derivations and \(\sigma\)-derivations. Moreover, Mirzavaziri and Moslehian ([6] and [7]) acquired some results about automatic continuity of \(\sigma\)-derivations. In this article the \( \mathfrak{m}\)-weakly continuity of a linear mapping is defined as follows:

The linear mapping \( T : \mathcal{B} \to \mathcal{A} \) is called \( \mathfrak{m}\)-weakly continuous if the linear mapping \( \varphi T : \mathcal{B} \to \mathbb{C} \) is continuous for all multiplicative linear functional \( \varphi \) from \( \mathcal{A} \) in to \( \mathbb{C} \). Suppose that \( \mathcal{A} \) is unital and \( d : \mathcal{A} \to \mathcal{B} \) is a \( \sigma\)-derivation such that \( \varphi d(1) \neq 0 \) for all \( \varphi \in \Phi_B \), the set of all non-zero multiplicative linear functionals from \( \mathcal{B} \) in to \( \mathbb{C} \). If for all \( \varphi \in \Phi_B \) there exists an element \( a_{\varphi} \in \mathcal{A} \) such that \( a_{\varphi} \notin \ker(\varphi d) \) and \( \varphi d(a_{\varphi}^2) = (\varphi d(a_{\varphi}))^2 \) then \( \varphi d \) is a homomorphism. Moreover, \( d \) and \( \sigma \) are \( \mathfrak{m}\)-weakly continuous. In particular, if \( \mathcal{A} \) is semi-simple and commutative then \( d \) and \( \sigma \) are continuous.

Singer and Wermer (see Corollary 2.7.20 of [2]) proved that, when \( \mathcal{A} \) is a commutative Banach algebra and \( D : \mathcal{A} \to \mathcal{A} \) is a continuous derivation, \( D(\mathcal{A}) \subseteq \text{rad}(\mathcal{A}) \), where \( \text{rad}(\mathcal{A}) \) is the Jacobson radical of \( \mathcal{A} \). They conjectured that \( D(\mathcal{A}) \subseteq \text{rad}(\mathcal{A}) \) for each (possibly discontinuous) derivation \( D \) on \( \mathcal{A} \). In 1988, Thomas [9] proved this conjecture. We prove that if \( d : \mathcal{A} \to \mathcal{A} \) is a \( \sigma\)-derivation on a commutative Banach algebra \( \mathcal{A} \) such that \( \sigma \) is a continuous homomorphism and \( \sigma^2 = \sigma \) then \( \sigma d \sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq \text{rad}(\mathcal{A}) \). In particular if \( d(\mathcal{A}) \subseteq \sigma d \sigma(\mathcal{A}) \) then \( d(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq \text{rad}(\mathcal{A}) \), where \( Q(\mathcal{A}) \) is the set of all quasi-nilpotent elements of \( \mathcal{A} \).

2. Main Results

Throughout this paper \( \mathcal{A} \) and \( \mathcal{B} \) denote two Banach algebras. Moreover, \( \mathcal{M} \) denotes a Banach \( \mathcal{B}\)-bimodule. Furthermore, if an algebra is unital then \( 1 \) will show its unit element. Recall that if \( E \) is a subset of an algebra \( \mathcal{B} \), the right annihilator \( \text{ran}(E) \) of \( E \) (resp. the left annihilator \( \text{lan}(E) \) of \( E \) ) is defined to be \( \{ b \in \mathcal{B} : Eb = \{0\} \} \) (resp. \( \{ b \in \mathcal{B} : bE = \{0\} \} \)). The set \( \text{ann}(E) := \text{ran}(E) \cap \text{lan}(E) \) is called the annihilator of \( E \). Suppose \( S \subseteq \mathcal{M} \). The right annihilator \( \text{ran}(S) \) of \( S \) is defined to be \( \{ b \in \mathcal{B} : Sb = \{0\} \} \). The left annihilator of \( S \) is defined, similarly. Also, recall that if \( Y \) and \( Z \) are Banach spaces and \( T : Y \to Z \) is a linear mapping, then the set \( \{ z \in Z : \exists \{ y_n \} \subseteq Y \text{ s.t. } y_n \to 0, T(y_n) \to z \} \) is called the separating space \( S(T) \) of \( T \). By the closed graph Theorem, \( T \) is continuous if and only if \( S(T) = \{0\} \). The reader is referred to [2] for more about separating spaces.

**Definition 2.1.** Suppose \( \sigma : \mathcal{A} \to \mathcal{B} \) is a linear mapping. A linear mapping \( d : \mathcal{A} \to \mathcal{M} \) is called a \( \sigma\)-derivation if \( d(ab) = d(a)\sigma(b) + \sigma(a)d(b) \) for all \( a, b \in \mathcal{A} \).

It is clear that if \( \mathcal{A} \) is a subalgebra of \( \mathcal{B} \) and \( \sigma = \text{id} \), the identity mapping on \( \mathcal{A} \), then a \( \sigma\)-derivation is an ordinary derivation.

**Theorem 2.2.** Suppose that \( d : \mathcal{A} \to \mathcal{B} \) is a linear mapping. We define \( d_1 : \mathcal{A}_1 \to \mathcal{B}_1 \) by \( d_1(a + \alpha) = d(a) + \alpha \) for all \( a + \alpha \in \mathcal{A}_1 \), whenever \( \mathcal{A}_1 = \mathcal{A} \bigoplus \mathbb{C} \).
and $B_1 = B \oplus \mathbb{C}$ are the unitization of $A$ and $B$, respectively. Then $d_1$ is a $\sigma$-derivation if and only if $d$ is a homomorphism.

Proof. We denote the unit element of $A_1$ and $B_1$ by 1. Clearly $d_1(1) = 1$. Suppose that $d_1$ is a $\sigma$-derivation. We have $1 = d_1(1) = d_1(1)\sigma(1) + \sigma(1)d_1(1)$. Therefore $\sigma(1) = \frac{1}{2}$ and $d_1((a + \alpha)1) = d_1(a + \alpha)\sigma(1) + \sigma(a + \alpha)d_1(1) = \frac{d_1(a + \alpha)}{2} + \sigma(a + \alpha)$. Hence $\sigma(a + \alpha) = \frac{d_1(a + \alpha)}{2}$ for all $a + \alpha \in A_1$. Moreover, we have

$$d_1((a + \alpha)(b + \beta)) = d_1(a + \alpha)\sigma(b + \beta) + \sigma(a + \alpha)d_1(b + \beta)$$

$$= d_1(a + \alpha)\frac{d_1(b + \beta)}{2} + \frac{d_1(a + \alpha)}{2}d_1(b + \beta)$$

$$= d_1(a + \alpha)d_1(b + \beta).$$

It means that $d_1$ is a homomorphism. Hence $d$ is a homomorphism. Conversely, assume that $d$ is a homomorphism, i.e. $d(ab) = d(a)d(b)$ for all $a,b \in A$. We have $d(ab) + \beta d(a) + \alpha d(b) + \alpha \beta = d(a)d(b) + \beta d(a) + \alpha d(b) + \alpha \beta$ for all $a + \alpha, b + \beta \in A_1$. It means that $d_1$ is a homomorphism. Put $\sigma = \frac{d_1}{2}$. Then

$$d_1((a + \alpha)(b + \beta)) = d_1(a + \alpha)d_1(b + \beta)$$

$$= d_1(a + \alpha)\frac{d_1(b + \beta)}{2} + \frac{d_1(a + \alpha)}{2}d_1(b + \beta)$$

$$= d_1(a + \alpha)\sigma(b + \beta) + \sigma(a + \alpha)d_1(b + \beta).$$

Hence $d_1$ is a $\sigma$-derivation. \qed

Corollary 2.3. Suppose $B$ is commutative and semisimple and let $d : A \to B$ be a linear mapping. If $d_1 : A_1 \to B_1$, defined by $d_1(a + \alpha) = d(a) + \alpha$, is a $\sigma$-derivation then $d$ and $d_1$ are continuous operators.

Proof. According to Theorem 2.2, $d$ is a homomorphism. By Theorem 2.3.3 of [2], $d$ is continuous and so $d_1$ is continuous. \qed

Theorem 2.4. Suppose that $A$ is unital and $d : A \to M$ is a $\sigma$-derivation. If $\sigma$ is continuous and $\|\sigma(1)\| < 1$ then $d$ is continuous.

Proof. Suppose $d(1) = 0$. Then for each $a \in A$, $\|d(a)\| = \|d(a)\sigma(1)\| \leq \|d(a)\|\|\sigma(1)\|$. Thus $\|d(a)\|(1 - \|\sigma(1)\|) \leq 0$. It follows that $d(a) = 0$. Since $a$ was arbitrary, $d$ is identically zero and hence $d$ is continuous. Now assume that $d(1) \neq 0$ and $a$ is an arbitrary element of $A$ such that $d(a) \neq 0$. We have

$$\|d(a)\| = \|d(1)\sigma(a) + \sigma(1)d(a)\|$$

$$\leq \|d(1)\sigma(a)\| + \|\sigma(1)d(a)\|$$

$$\leq \|d(1)\|\|\sigma\||\|a\| + \|\sigma(1)\|\|d(a)\|. $$

Hence $(1 - \|\sigma(1)\|)\|d(a)\| \leq \|d(1)\|\|\sigma\||\|a\|$. This implies that $d$ is continuous. \qed

Recall that an element $a$ in a normed algebra $A$ is called quasi-nilpotent if $\lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = 0$. The set of all quasi-nilpotent elements of $A$ is denoted by $Q(A)$. 

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Theorem 2.5. Suppose that $A$ and $B$ are unital and $B$ has no zero divisors and assume that $d : A \to B$ is a $\sigma$-derivation such that $d(1) \neq 0$. If there exists a sequence $\{a_n\} \subseteq A$ such that $d(a_n) \to a_0$ and $\sigma(a_n) \to a_0$, where $a_0 \neq 0$, then $d = \sigma$. Moreover, if $d$ is continuous then $d(Q(A)) \subseteq Q(B)$.

Proof. We have $d(a_n) = d(a_n)\sigma(1) + \sigma(a_n)d(1)$. Thus $a_0(\sigma(1) + d(1) - 1) = 0$. Since $B$ has no zero divisors and $a_0 \neq 0$, $d(1) + \sigma(1) = 1$. We have $d(1) \neq 1$, since if $d(1) = 1$ then $\sigma(1) = 0$. Thus $d(1) = d(1)\sigma(1) + \sigma(1)d(1) = 0$, which is a contradiction. We have $d(1) = (1 - \sigma(1))\sigma(1) + \sigma(1)(1 - \sigma(1))$. Therefore $(1 - 2\sigma(1))d(1) = 0$. Since $d(1) \neq 0$ and $B$ has no zero divisors, $\sigma(1) = \frac{1}{2}$. It follows that $d(1) = \frac{1}{2}$. Let $a$ be an arbitrary element of $A$. We have

$$d(a) = d(a)\sigma(1) + \sigma(a)d(1) = \frac{d(a)}{2} + \frac{\sigma(a)}{2},$$

and hence $d = \sigma$. By induction on $n$, we obtain

$$d(a^n) = 2^{n-1}(d(a))^n$$

therefore $(d(a))^n = \frac{d(a^n)}{2^{n-1}}$. Assume that $d$ is continuous and $a \in Q(A)$. Then

$$\|d(a^n)\|^{\frac{1}{n}} = \|\frac{d(a^n)}{2^{n-1}}\|^{\frac{1}{n}} \leq \left(\frac{1}{2^{n-1}}\right)^{\frac{1}{n}} \|d\|^{\frac{1}{n}} \|a^n\|^{\frac{1}{n}} \to 0.$$

It means that $d(a) \in Q(B)$. \hfill $\square$

Remark 2.6. Suppose that $\sigma : A \to B$ is a continuous linear mapping and $\{\sigma(ab) - \sigma(a)\sigma(b) \mid a, b \in A\} \subseteq \text{ann}(M)$. Then $U_\sigma = A \oplus M$ is an algebra by the following action: $(a, x) \bullet (b, y) = (ab, \sigma(a)y + x\sigma(b))$ for all $a, b \in A$ and $x, y \in M$. Put $m = \max\{1, \|\sigma\|\}$. We define $\|a\| = m\|a\|$ (a $\in A$), which is clearly a complete norm on $A$. Then $\|ab\| = m\|ab\| \leq m^2\|a\|\|b\| = m\|a\|m\|b\| = \|\sigma(a)||\|b\||$. Let $d : A \to M$ be a $\sigma$-derivation. Define two norms $\|\|_1$ and $\|\|_2$ on $U_\sigma$ by $\|(a, x)\|_1 = \|\|a\| + \|x\|$, $\|(a, x)\|_2 = \|\|a\| + \|d(a) - x\|$.

Theorem 2.7. Suppose that $U_\sigma, \|\|_1$ and $\|\|_2$ are as in the Remark 2.6. Then $U_\sigma$ is a Banach algebra with respect to $\|\|_1$ and $\|\|_2$. Furthermore, these two norms are equivalent if and only if $d$ is continuous.

Proof. Clearly $(U_\sigma, \|\|_1)$ is a Banach algebra and $\|\|_2$ is a norm on $U_\sigma$. We prove that $\|\|_2$ is a complete algebra norm on $U_\sigma$. Suppose $\{(a_n, x_n)\}$ is a Cauchy sequence in $(U_\sigma, \|\|_2)$. Then $\{a_n\}$ and $\{d(a_n) - x_n\}$ are Cauchy sequences in $A$ and $\mathcal{M}$, respectively. Since $A$ and $\mathcal{M}$ are Banach spaces, there exist $a \in A$ and $x \in \mathcal{M}$ such that $a_n \to a$ in $A$ and $d(a_n) - x_n \to x$ in $\mathcal{M}$. Therefore $(a_n, x_n) \to (a, d(a) - x)$ in $\|\|_2$. Thus $(U_\sigma, \|\|_2)$ is a Banach space. Assume that
(a, x) and (b, y) are two arbitrary elements of \( U_\sigma \). We have
\[
\|(a, x) \bullet (b, y)\|_2 = \|(ab, \sigma(a)y + x\sigma(b))\|_2 \\
= \|ab\| + \|d(ab) - \sigma(a)y - x\sigma(b)\| \\
= \|ab\| + \|d(ab)\sigma(b) + \sigma(a)d(b) - \sigma(a)y - x\sigma(b)\| \\
\leq \|a\| \|b\| + \|d(a) - x\| \|\sigma\| \|b\| + \|\sigma\| \|a\| \|d(b) - y\| \\
\leq (\|a\| \|b\| + \|d(a) - x\| \|\sigma\| \|b\| + \|\sigma\| \|a\| \|d(b) - y\|) \\
= \|(a, x)\|_2 \|(b, y)\|_2.
\]
Therefore \((U_\sigma, \|\cdot\|_2)\) is a Banach algebra. Suppose \( d \) is continuous. We have
\[
\|(a, x)\|_2 = \|a\| + \|d(a) - x\| \\
\leq \|a\| + \|d(a)\| + \|x\| \\
\leq \|a\| + \|d\| \|a\| + \|x\| \\
\leq \|a\| + \|d\| \ m \|a\| + \|x\| \\
= \|a\| + \|d\| \|a\| + \|x\| \\
\leq (1 + \|d\|)(\|a\| + \|x\|) \\
= (1 + \|d\|)\|(a, x)\|_1
\]
for all \((a, x)\in U_\sigma\). Applying the open mapping Theorem, we obtain that \(\|\cdot\|_1\) and \(\|\cdot\|_2\) are equivalent. Conversely, suppose that \(\|\cdot\|_1\) and \(\|\cdot\|_2\) are equivalent. Then there exists a positive number \( c \) such that \(\|(a, x)\|_2 \leq c\|(a, x)\|_1 \) \( (a, x) \in U_\sigma \). Thus \(\|d(a)\| \leq \|(a, 0)\|_2 \leq c\|(a, 0)\|_1 = c\|a\|\). It means that \( d \) is continuous. \(\Box\)

Suppose that \( d : \mathcal{A} \to \mathcal{M} \) is a linear mapping. We define a linear mapping \( \Theta : U_\sigma \to U_\sigma \) by \( \Theta(a, x) = (a, d(a) - x) \) \( (a \in \mathcal{A}, x \in \mathcal{M}) \). It is clear that \( \Theta \) is an endomorphism if and only if \( d \) is a \( \sigma \)-derivation.

**Theorem 2.8.** Suppose that \( \sigma : \mathcal{A} \to \mathcal{B} \) is a continuous linear mapping such that \( \{\sigma(ab) - \sigma(a)\sigma(b) \mid a, b \in \mathcal{A}\} \subseteq \text{ann}(\mathcal{M}) \) and assume that \( d : \mathcal{A} \to \mathcal{M} \) is a \( \sigma \)-derivation. Consider \( U_\sigma \) and \( \|\cdot\|_2 \) as in Remark 2.6. Then \( d \) is continuous if and only if \( \Theta : (U_\sigma, \|\cdot\|_2) \to (U_\sigma, \|\cdot\|_2) \) is continuous.

**Proof.** We have \( \|\Theta(a, x)\|_2 = \|(a, d(a) - x)\|_2 = \|a\| + \|x\| = \|(a, x)\|_1 \). Let \( d \) be continuous. By Theorem 2.7, \(\|\cdot\|_1\) and \(\|\cdot\|_2\) are equivalent. So there exists a positive number \( c \) such that \(\|(a, x)\|_1 \leq c\|(a, x)\|_2 \). On the other hand, \(\|\Theta(a, x)\|_2 = \|(a, x)\|_1 \leq c\|(a, x)\|_2 \). It means that \( \Theta \) is continuous. Now assume that \( \Theta \) is continuous. Then there exists a positive number \( c \) such that \(\|\Theta(a, x)\|_2 \leq c\|(a, x)\|_2 \). This implies that \(\|(a, x)\|_1 \leq c\|(a, x)\|_2 \). It follows from Theorem 2.7 that \( d \) is continuous. \(\Box\)

Suppose that \( \mathcal{A} \) is a Banach algebra. We denote by \( \Phi_\mathcal{A} \), the set of all non-zero multiplicative linear functionals from \( \mathcal{A} \) into \( \mathbb{C} \). We know that each member of \( \Phi_\mathcal{A} \) is continuous. Since the case \( \Phi_\mathcal{A} = \emptyset \) makes every thing trivial, so we will assume that \( \Phi_\mathcal{A} \) is not equal to empty set.
Definition 2.9. Let $\mathcal{B}$ and $\mathcal{A}$ be two Banach algebras and suppose that $T : \mathcal{B} \to \mathcal{A}$ is a linear mapping. $T$ is called $m$-weakly continuous if the linear mapping $\varphi T : \mathcal{B} \to \mathbb{C}$ is continuous for all $\varphi \in \Phi_\mathcal{A}$.

It is clear that if a linear mapping is continuous then it is $m$-weakly continuous but the converse is not true, in general. To see this, suppose that $\mathcal{A}$ is a Banach algebra. Set $\mathcal{B} = \mathbb{C} \bigoplus \mathcal{A}$. Consider $\mathcal{B}$ as a commutative algebra with pointwise addition and scalar multiplication and the product defined by $(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a)$ $(\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A})$. The algebra $\mathcal{B}$ with the norm $\| (a, a) \| = |a| + \| a \|$ is a Banach algebra. Hence $rad(\mathcal{B}) = Q(\mathcal{B}) = \{0\} \bigoplus \mathcal{A}$. On the other hand, $rad(\mathcal{B}) = \bigcap_{\varphi \in \Phi_\mathcal{B}} ker(\varphi)$. Note that $\Phi_\mathcal{B} \neq \emptyset$, since $\mathcal{B}$ is a unital commutative Banach algebra. Assume that $T : \mathcal{A} \to \mathcal{A}$ is a discontinuous linear mapping. Define $D : \mathcal{B} \to \mathcal{B}$ by $D(a, a) = (0, T(a))$. Clearly $D$ is discontinuous and $D(\mathcal{B}) \subseteq \{0\} \bigoplus \mathcal{A} = rad(\mathcal{B}) = \bigcap_{\varphi \in \Phi_\mathcal{B}} ker(\varphi)$. So $\varphi(D(\mathcal{B})) = \{0\}$ for all $\varphi \in \Phi_\mathcal{B}$ and it cause that $\varphi D : \mathcal{B} \to \mathbb{C}$ is continuous for all $\varphi \in \Phi_\mathcal{B}$. Thus $D$ is $m$-weakly continuous but it is not continuous. In fact $D$ is a discontinuous derivation on $\mathcal{B}$. Moreover, every derivation from a commutative Banach algebra $\mathcal{A}$ into $\mathcal{A}$ is $m$-weakly continuous (see Theorem 4.4 of [9]).

Proposition 2.10. Suppose that $\mathcal{A}$ is a Banach algebra. Then $\mathcal{A}$ is commutative and semi-simple if and only if $\bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$.

Proof. Obviously if $\mathcal{A}$ is commutative and semi-simple then $\bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$. Conversely, suppose that $\bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$ and $a, b$ are two arbitrary elements of $\mathcal{A}$. Then $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$ for all $\varphi \in \Phi_\mathcal{A}$. So $\varphi(ab - ba) = 0$. Since $\varphi$ was arbitrary, we have $ab - ba \in \bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$. Hence $\mathcal{A}$ is commutative. Since $\mathcal{A}$ is commutative and $\bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$, $rad(\mathcal{A}) = \{0\}$. Thus $\mathcal{A}$ is semi-simple. □

Theorem 2.11. Suppose that $\mathcal{B}$ and $\mathcal{A}$ are two Banach algebras and assume that $T : \mathcal{B} \to \mathcal{A}$ is an $m$-weakly continuous linear mapping. If $\bigcap_{\varphi \in \Phi_\mathcal{A}} ker(\varphi) = \{0\}$ then $T$ is continuous.

Proof. By part (ii) of Proposition 5.2.2 in [2], we have the result. □

Theorem 2.12. Suppose that $d : \mathcal{A} \to \mathcal{B}$ is a $\sigma$-derivation such that $\sigma$ is $m$-weakly continuous. If $\bigcap_{\varphi \in \Phi_\mathcal{B}} ker(\varphi) = \{0\}$ and $S(\varphi d) \neq \{0\}$ for all $\varphi \in \Phi_\mathcal{B}$ then $\sigma$ is a homomorphism.

Proof. Suppose that $\varphi$ is an arbitrary element of $\Phi_\mathcal{B}$. Put $\varphi d = d_1$ and $\varphi \sigma = \sigma_1$. Obviously $d_1$ is a $\sigma_1$-derivation. Since $\sigma_1$ is continuous, $\{\sigma_1(ab) - \sigma_1(a)\sigma_1(b) \mid a, b \in \mathcal{A}\} \subseteq \text{ann}(S(d_1)) = \{0\}$ (see Lemma 2.3 of [6]). Therefore $\{\sigma(ab) - \sigma(a)\sigma(b) \mid a, b \in \mathcal{A}\} \subseteq \bigcap_{\varphi \in \Phi_\mathcal{B}} ker(\varphi) = \{0\}$. So $\sigma$ is a homomorphism. □

Theorem 2.13. Suppose that $\mathcal{A}$ is unital and $d : \mathcal{A} \to \mathcal{B}$ is a $\sigma$-derivation such that $\varphi d(1) \neq 0$ for all $\varphi \in \Phi_\mathcal{B}$. If for all $\varphi \in \Phi_\mathcal{B}$ there exists an element $a_\varphi \in \mathcal{A}$ such that $a_\varphi \notin ker(\varphi d)$ and $\varphi d(a_\varphi^2) = (\varphi d(a_\varphi))^2$ then $\varphi d$ is a homomorphism. Moreover, $d$ and $\sigma$ are $m$-weakly continuous.
Proof. Suppose that \( \varphi \) is an arbitrary element of \( \Phi_B \). Put \( \varphi d = d_1 \) and \( \varphi \sigma = \sigma_1 \). At first we show that \( \ker(d_1) \subseteq \ker(\sigma_1) \). Let \( a \in \ker(d_1) \). We have
\[
0 = d_1(a) \\
= d_1(a)\sigma_1(1) + \sigma_1(a)d_1(1) \\
= \sigma_1(a)d_1(1).
\]
Since \( d_1(1) \neq 0 \), \( \sigma_1(a) = 0 \) and hence \( a \in \ker(\sigma_1) \). It means that \( \ker(d_1) \subseteq \ker(\sigma_1) \). Therefore there exists a complex number \( \lambda_\varphi \) such that \( \sigma_1 = \lambda_\varphi d_1 \). By hypothesis, there exists \( a_\varphi \notin \ker(\varphi d) \) such that \( \varphi d(a_\varphi^2) = (\varphi d(a_\varphi))^2 \). We have
\[
(d_1(a_\varphi))^2 = d_1(a_\varphi^2) \\
= d_1(a_\varphi)\sigma_1(a_\varphi) + \sigma_1(a_\varphi)d_1(a_\varphi) \\
= d_1(a_\varphi)\lambda_\varphi d_1(a_\varphi) + \lambda_\varphi d_1(a_\varphi)d_1(a_\varphi) \\
= 2\lambda_\varphi(d_1(a_\varphi))^2.
\]
Since \( d_1(a_\varphi) \neq 0 \), \( \lambda_\varphi = \frac{1}{2} \). This implies that \( \sigma_1 = \frac{d_1}{2} \). We have
\[
d_1(ab) = d_1(a)\sigma_1(b) + \sigma_1(a)d_1(b) \\
= d_1(a)\frac{d_1(b)}{2} + \frac{d_1(a)}{2}d_1(b) \\
= d_1(a)d_1(b)
\]
for all \( a, b \in \mathcal{A} \). Hence \( d_1 : \mathcal{A} \to \mathbb{C} \) is a complex homomorphism. We know that every complex homomorphism on a Banach algebra is continuous. Clearly \( \sigma_1 \) is also continuous. Since \( \varphi \) was arbitrary, \( d \) and \( \sigma \) are m-weakly continuous. \( \square \)

Suppose that \( a \in \mathcal{A} \) we define \( L_a : \mathcal{A} \to \mathcal{A} \) by \( L_a(b) = ab \) for all \( b \in \mathcal{A} \). Set \( L_\mathcal{A} = \{L_a \mid a \in \mathcal{A} \} \). It is clear that \( L_\mathcal{A} \) is a subalgebra of \( B(\mathcal{A}) \), here \( B(\mathcal{A}) \) denotes the set of all continuous linear mapping from \( \mathcal{A} \) into \( \mathcal{A} \). It is well known that \( a \in Q(\mathcal{A}) \) if and only if \( L_a \in Q(L_\mathcal{A}) \).

**Theorem 2.14.** \( Q(\mathcal{A}) = lan(\mathcal{A}) \) if and only if \( Q(L_\mathcal{A}) = \{0\} \).

**Proof.** Suppose that \( Q(L_\mathcal{A}) = \{0\} \) and \( a \in Q(\mathcal{A}) \). So \( L_a \in Q(L_\mathcal{A}) = \{0\} \) and hence \( a \in lan(\mathcal{A}) \). It means that \( Q(\mathcal{A}) \subseteq lan(\mathcal{A}) \). It is easy to see that \( lan(\mathcal{A}) \subseteq Q(\mathcal{A}) \). Thus \( Q(\mathcal{A}) = lan(\mathcal{A}) \). Conversely, assume that \( Q(\mathcal{A}) = lan(\mathcal{A}) \). Suppose that \( L_a \in Q(L_\mathcal{A}) \). So \( a \in Q(\mathcal{A}) = lan(\mathcal{A}) \). It follows that \( ab = 0 \) for all \( b \in \mathcal{A} \). It means that \( L_a = 0 \). Hence \( Q(L_\mathcal{A}) = \{0\} \). \( \square \)

**Theorem 2.15.** Suppose that \( d : \mathcal{A} \to \mathcal{A} \) is a \( \sigma \)-derivation such that \( \sigma \) is an endomorphism and \( \sigma^2 = \sigma \). If \( \sigma d \sigma \) is a continuous mapping and \( \sigma(a)\sigma d \sigma(a) = \sigma d \sigma(a)\sigma(a) \) for all \( a \in \mathcal{A} \) then \( \sigma d \sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq Q(\mathcal{A}) \). In particular if \( d(\mathcal{A}) \subseteq \sigma d \sigma(\mathcal{A}) \) then \( d(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \).

**Proof.** First of all, we define another action on \( \mathcal{A} \) by the following form: \( a \circ b = \sigma(ab) \) for all \( a, b \in \mathcal{A} \). It is clear that \( \mathcal{A} \) is an algebra by this action. We denote this algebra by \( \mathcal{A}_\sigma \). Put \( D = \sigma d \sigma \). It is clear that \( \sigma D = D \sigma = D \) and \( D \) is a
σ-derivation on \( \mathcal{A} \). Moreover, \( D \) is a derivation on \( \tilde{\mathcal{A}}_\sigma \). Because,
\[
D(a \cdot b) = D(\sigma(ab)) = D(\sigma(a)\sigma(b))
\]
\[
= D(\sigma(a))\sigma^2(b) + \sigma^2(a)D(\sigma(b))
\]
\[
= \sigma(D(a))\sigma(b) + \sigma(a)\sigma(D(b))
\]
\[
= D(a) \cdot b + a \cdot D(b)
\]
for all \( a, b \in \tilde{\mathcal{A}}_\sigma \). Suppose that \( a \in \mathcal{A} \) is a non-zero arbitrary element. We define a linear mapping \( d_{La} : B(\tilde{\mathcal{A}}_\sigma) \to B(\tilde{\mathcal{A}}_\sigma) \) by \( d_{La}(T) = TL_a - L_aT \) for all \( T \in B(\tilde{\mathcal{A}}_\sigma) \). We have \( \Delta_{La}(D)(x) = (DL_a - L_aD)(x) = D(a \cdot x) - a \cdot D(x) = L_D(a)(x) \) for all \( x \in \tilde{\mathcal{A}}_\sigma \). Therefore \( \Delta_{La}(D) = \Delta_{La}(L_D(a)) = L_D(a)L_a - L_aL_D(a) = 0 \). Hence \( \Delta_{La}(D) \in Q(B(\tilde{\mathcal{A}}_\sigma)) \). This implies that \( L_D(a) \in Q(L_{\tilde{\mathcal{A}}_\sigma}) \). Since \( D\sigma = \sigma D = D, D(a) \in Q(A) \). It means that \( \sigma d\sigma (A) \subseteq Q(A) \). Since \( D(A) \subseteq Q(A), \sigma D(A) \subseteq \sigma (Q(A)) \). Hence \( \sigma d\sigma (A) \subseteq \sigma (Q(A)) \). Note that \( \sigma(Q(A)) \subseteq Q(A) \).

We know that if \( \sigma : \mathcal{A} \to \mathcal{A} \) is an endomorphism such that \( \sigma^2 = \sigma \) then we can define \( \mathcal{A}_\sigma \) - algebra which introduced in 2.15. We want to define a norm on \( \mathcal{A}_\sigma \) such that it is a Banach algebra. Suppose \( \sigma \) is continuous. Obviously \( \|\sigma\| \geq 1 \). We define \( ||a|| = ||\sigma|| \|a\| \). Clearly \( \mathcal{A}_{\sigma} \) is a Banach algebra with respect to \( ||.|| \).

**Theorem 2.16.** Suppose that \( \mathcal{A} \) is commutative and \( d : \mathcal{A} \to \mathcal{A} \) is a \( \sigma \)-derivation such that \( \sigma \) is a continuous endomorphism and \( \sigma^2 = \sigma \). Then \( \sigma d\sigma (A) \subseteq \sigma (Q(A)) \) \( \subseteq \text{rad}(A) \). In particular if \( d(A) \subseteq \sigma d\sigma (A) \) then \( d(A) \subseteq \sigma (Q(A)) \) \( \subseteq \text{rad}(A) \).

**Proof.** Consider \( \mathcal{A}_\sigma \)-algebra with \( ||.|| \). Clearly it is a commutative Banach algebra. We know that \( D = \sigma d\sigma : \mathcal{A}_\sigma \to \mathcal{A}_\sigma \) is a derivation. By Theorem 4.4 in [9], \( D(\mathcal{A}_\sigma) \subseteq \text{rad}(\mathcal{A}_\sigma) = Q(\mathcal{A}_\sigma) \). Since \( D\sigma = \sigma D = D, D(A) \subseteq Q(A) \). A similar argument to Theorem 2.15 gives the result. \( \square \)

**Definition 2.17.** A Banach algebra \( \mathcal{A} \) has the Cohen’s factorization property if \( \mathcal{A}^2 = \mathcal{A} \), where \( \mathcal{A}^2 = \{bc \mid b, c \in \mathcal{A}\} \).

**Corollary 2.18.** Suppose that \( d : \mathcal{A} \to \mathcal{A} \) is a \( \sigma \)-derivation such that all conditions in Theorem 2.16 are hold and furthermore \( d\sigma = \sigma d = d \). If \( Q(L_A) = \{0\} \) and \( \mathcal{A} \) has the Cohen’s factorization property then \( d \) is identically zero.

**Proof.** By Theorem 2.16, \( d(A) \subseteq Q(A) \). Since \( \mathcal{A} \) is commutative and \( Q(L_A) = \{0\} \), it follows from Theorem 2.14 that \( Q(A) = \text{lan}(A) = \text{ann}(A) \). Suppose that \( a \) is an arbitrary element of \( \mathcal{A} \). Then there exist two elements \( b \) and \( c \) in \( \mathcal{A} \) such that \( a = bc \). We have \( d(a) = d(bc) = d(b)\sigma(c) + \sigma(b)d(c) = 0 \). Since \( a \) was arbitrary, \( d \equiv 0 \).

**Remark 2.19.** Suppose that \( \mathcal{A} \) is commutative and has the Cohen’s factorization property and assume that \( d : \mathcal{A} \to \mathcal{A} \) is a derivation. If \( Q(L_A) = \{0\} \) then by Theorem 4.4 of [9], we have \( d(A) \subseteq Q(A) \). It follows from Theorem 2.14 that \( d \equiv 0 \).
Theorem 2.20. Suppose $\mathcal{B}$ is commutative and $d : \mathcal{A} \to \mathcal{B}$ is a $\sigma$ derivation such that $\sigma$ is an isomorphism. Then $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{B})$.

Proof. We define a map $D : \mathcal{B} \to \mathcal{B}$ by $D(b) = d\sigma^{-1}(b)$ for all $b \in \mathcal{B}$. It is clear that $D$ is a derivation on $\mathcal{B}$. According to Theorem 4.4 of [9], $D(\mathcal{B}) \subseteq \text{rad}(\mathcal{B})$. Hence $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{B})$. □

Proposition 2.21. Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a $\sigma$ derivation such that $\sigma^2 = \sigma$ and $\sigma$ is an endomorphism. If $d\sigma = \sigma d$ then $d^n(\sigma(ab)) = \sum_{k=0}^{n} \binom{n}{k} d^{n-k}\sigma(a) d^k\sigma(b)$ ($n \in \mathbb{N}$ and $a, b \in \mathcal{A}$). With the convention that $d^0 = \text{id}$, the identity operator on $\mathcal{A}$.

Proof. We consider $\tilde{\mathcal{A}}_\sigma$ algebra. Clearly $d : \tilde{\mathcal{A}}_\sigma \to \tilde{\mathcal{A}}_\sigma$ is a derivation. According to part (i) of Proposition 18.4 of [1], we have $d^n(a \cdot b) = \sum_{k=0}^{n} \binom{n}{k} d^{n-k}(a) \cdot d^k(b)$ for all $a, b \in \tilde{\mathcal{A}}_\sigma$. Therefore

$$d^n(\sigma(ab)) = \sum_{k=0}^{n} \binom{n}{k} \sigma(d^{n-k}(a) \cdot d^k(b))$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sigma d^{n-k}(a) \cdot \sigma d^k(b)$$

$$= \sum_{k=0}^{n} \binom{n}{k} d^{n-k}\sigma(a) \cdot d^k\sigma(b).$$

Theorem 2.22. Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a continuous $\sigma$ derivation such that $\sigma$ is an endomorphism and $\sigma^2 = \sigma$. If $d\sigma = \sigma d$ and $d\sigma$ is continuous then $e^{d}\sigma$ is a continuous endomorphism and $e^{d}$ is a continuous bijective mapping on $\mathcal{A}$.

Proof. First, we define a linear mapping $d_1$ by the following form: $d_1^0 = \sigma$ and $d_1^1 = d\sigma$. Clearly $d_1^n = d^n\sigma$ for all non-negative integer $n$. It follows from Proposition 2.21 that $d_1^n(ab) = \sum_{k=0}^{n} \binom{n}{k} d_1^{n-k}(a) d_1^k(b)$ for all $a, b \in \mathcal{A}$. We have

$$e^{d_1} = \sum_{n=0}^{\infty} \frac{d_1^n}{n!} = \sigma + \sum_{n=1}^{\infty} \frac{d^n}{n!}$$

$$= \sigma + \sum_{n=1}^{\infty} \frac{(d\sigma)^n}{n!}$$

$$= \sigma + \sum_{n=1}^{\infty} \frac{d^n\sigma}{n!}$$

$$= (id + \sum_{n=1}^{\infty} \frac{d^n}{n!})\sigma$$

$$= e^{d}\sigma.$$
$e^d = e^d \sigma$ is a continuous endomorphism on $\mathcal{A}$. We know that $d : \tilde{\mathcal{A}}_{\sigma} \to \tilde{\mathcal{A}}_{\sigma}$ is a continuous derivation. By Proposition 18.7 of [1], we obtain $e^d(a \cdot b) = e^d(a) \cdot e^d(b)$, i.e. $e^d : \tilde{\mathcal{A}}_{\sigma} \to \tilde{\mathcal{A}}_{\sigma}$ is a continuous automorphism. Hence $e^d$ is a continuous bijective mapping on $\mathcal{A}$. □

References


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