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EFFECTIVE H^∞ INTERPOLATION CONSTRAINED BY WEIGHTED HARDY AND BERGMAN NORMS

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ABSTRACT. Given a finite subset σ of the unit disc $\mathbb{D}$ and a holomorphic function f in $\mathbb{D}$ belonging to a class X, we are looking for a function g in another class Y which satisfies $g_{|\sigma} = f_{|\sigma}$ and is of minimal norm in Y. More precisely, we consider the interpolation constant $c\,(\sigma,\, X,\, Y) = \sup_{f\in X,\, \|f\|_X\leq 1} \inf_{g_{|\sigma}=f_{|\sigma}} \|g\|_Y$. When $Y = H^\infty$, our interpolation problem includes those of Nevanlinna–Pick and Carathéodory–Schur. If X is a Hilbert space belonging to the families of weighted Hardy and Bergman spaces, we obtain a sharp upper bound for the constant $c\,(\sigma,\, X,\, H^\infty)$ in terms of $n = \mathrm{card}\,\sigma$ and $r = \max_{\lambda\in\sigma} |\lambda| < 1$. If X is a general Hardy–Sobolev space or a general weighted Bergman space (not necessarily of Hilbert type), we also establish upper and lower bounds for $c\,(\sigma,\, X,\, H^\infty)$ but with some gaps between these bounds. This problem of constrained interpolation is partially motivated by applications in matrix analysis and in operator theory.

1. INTRODUCTION

1.1. **Statement and historical context of the problem.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and let $\mathrm{Hol}\,(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. We consider here the following problem: given two Banach spaces X and Y of holomorphic functions on the unit disc $\mathbb{D}$, $X,\, Y \subset \mathrm{Hol}\,(\mathbb{D})$, and a finite subset σ of $\mathbb{D}$, what is the best possible interpolation by functions of the space Y for the traces $f_{|\sigma}$ of functions of the space X, in the worst case? The case $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or X and Y are incomparable. Here and later on, H^∞ stands for the

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space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

More precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma,\, X,\, Y) = \sup_{f\in X,\, \|f\|_X\le 1} \inf\left\{\|g\|_Y \,:\, g_{|\sigma} = f_{|\sigma}\right\}.$$

For $r \in [0,\, 1)$ and $n \ge 1$, we also define

$$C_{n,r}(X,Y) = \sup\left\{c(\sigma,\, X,\, Y) \,:\, \operatorname{card}\sigma \le n\,,\, |\lambda| \le r,\ \forall\lambda\in\sigma\right\}.$$

It is explained in [14] why the classical interpolation problems, those of Nevanlinna–Pick and Carathéodory–Schur (see [10, p.231]), on the one hand and Carleson's free interpolation (1958) (see [10, p.158]) on the other hand, are of this nature.

From now on, if $\sigma = \{\lambda_1,\, \cdots,\, \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^{n} b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda - z}{1-\bar{\lambda}z}$, $\lambda\in\mathbb{D}$. With this notation and supposing that X satisfies the division property

$$[f\in X,\, \lambda\in\mathbb{D} \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z-\lambda} \in X\right],$$

we have

$$c(\sigma,\, X,\, Y) = \sup_{\|f\|_X\le 1} \inf\left\{\|g\|_Y \,:\, g\in Y,\, g - f \in B_\sigma X\right\}.$$

1.2. **Motivations in matrix analysis and in operator theory.** A direct relation between the study of the constants $c(\sigma,\, H^\infty,\, W)$ and some numerical analysis problems is mentioned in [14, (b)- p.5]. Here, W is the Wiener algebra of absolutely convergent Fourier series. In the same spirit, for general Banach spaces X containing H^∞, our constants $c(\sigma,\, X,\, H^\infty)$ are directly linked with the well known Von-Neumann's inequality for contractions on Hilbert spaces, which asserts that if A is a contraction on a Hilbert space and $f \in H^\infty$, then the operator $f(A)$ satisfies

$$\|f(A)\| \le \|f\|_\infty.$$

Using this inequality we get the following interpretation of our interpolation constant $c(\sigma,\, X,\, H^\infty)$: it is the best possible constant c such that $\|f(A)\| \le c\|f\|_X$, $\forall f\in X$. That is to say:

$$c(\sigma,\, X,\, H^\infty) = \sup_{\|f\|_X\le 1} \sup\left\{\|f(A)\| \,:\, A : (\mathbb{C}^n,\, |\cdot|_2) \to (\mathbb{C}^n,\, |\cdot|_2)\,,\, \|A\|\le 1,\, \sigma(A)\subset\sigma\right\},$$

where the interior sup is taken over all contractions A on $n-$dimensional Hilbert spaces $(\mathbb{C}^n,\, |.|_2)$, with a given spectrum $\sigma(A) \subset \sigma$.

An interesting case occurs for f such that $f_{|\sigma} = (1/z)_{|\sigma}$ (estimates on condition numbers and the norm of inverses of $n \times n$ matrices) or $f_{|\sigma} = [1/(\lambda - z)]_{|\sigma}$ (estimates on the norm of the resolvent of an $n \times n$ matrix), see for instance [16].

1.3. **Known results.** Let H^p $(1 \le p \le \infty)$ be the standard Hardy spaces and let L^2_a be the Bergman space on $\mathbb{D}$. We obtained in [14] (in which a more general approach to this effective interpolation problem is also given) some estimates on $c\,(\sigma,\, X,\, H^\infty)$ for the cases $X \in \{H^p,\, L^2_a\}$.

Theorem 1.1. *Let $n \ge 1$, $r \in [0,\,1)$, $p \in [1,\,+\infty]$ and $|\lambda| \le r$. Then*

$$\frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-|\lambda|}\right)^{\frac{1}{p}} \le c\,(\sigma_{n,\lambda},\, H^p, H^\infty) \le C_{n,r}\,(H^p,\, H^\infty) \le A_p\left(\frac{n}{1-r}\right)^{\frac{1}{p}}, \qquad (1)$$

$$\frac{1}{32}\frac{n}{1-|\lambda|} \le c\left(\sigma_{n,\lambda},\, L^2_a,\, H^\infty\right) \le C_{n,r}\left(L^2_a,\, H^\infty\right) \le \sqrt{2}10^{\frac{1}{4}}\frac{n}{1-r}, \qquad (2)$$

where

$$\sigma_{n,\lambda} = \{\lambda,\, \cdots,\, \lambda\},\ (n \text{ times}),$$

is the one-point set of multiplicity n corresponding to λ, A_p is a constant depending only on p and the left-hand side inequality in (1) is valid only for $p \in 2\mathbb{Z}_+$. For $p = 2$, we have $A_2 = \sqrt{2}$.

Note that this theorem was partially motivated by a question posed in an applied situation in [5, 6].

Trying to generalize inequalities (1) and (2) for general Banach spaces X (of analytic functions of moderate growth in $\mathbb{D}$), we formulate the following conjecture: $C_{n,r}\,(X,\, H^\infty) \le a\varphi_X\left(1 - \frac{1-r}{n}\right)$, where a is a constant depending on X only and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X. The aim of this paper is to establish this conjecture for some families of weighted Hardy and Bergman spaces.

2. Main results

Here, we extend Theorem 1.1 to the case where X is a weighted space

$$l^p_a(\alpha) = \left\{ f = \sum_{k\ge 0}\hat{f}(k)z^k :\ \|f\|^p = \sum_{k\ge 0}|\hat{f}(k)|^p(k+1)^{p\alpha} < \infty \right\},\ \alpha \le 0.$$

First, we study the special case $p = 2$, $\alpha \le 0$. Then $l^p_a(\alpha)$ are the spaces of the functions $f = \sum_{k\ge 0}\hat{f}(k)z^k$ satisfying

$$\sum_{k\ge 0}|\hat{f}(k)|^2(k+1)^{2\alpha} < \infty.$$

Notice that $H^2 = l^2_a(1)$. Let $\beta = -2\alpha - 1 > -1$. The scale of weighted Bergman spaces of holomorphic functions

$$X = L^2_a\,(\beta) =$$

$$= L_a^2\left(\left(1-|z|^2\right)^\beta dA\right) = \left\{f \in \mathrm{Hol}(\mathbb{D})\ :\ \int_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^\beta dA < \infty\right\},$$

gives the same spaces, with equivalence of the norms:

$$l_a^2(\alpha) = L_a^2(\beta).$$

In the case $\beta = 0$ we have $L_a^2(0) = L_a^2$.

We start with the following result.

Theorem 2.1. *Let $n \geq 1$, $r \in [0, 1)$, $\alpha \in (-\infty, 0]$ and $|\lambda| \leq r$. Then*

$$B\left(\frac{n}{1-|\lambda|}\right)^{\frac{1-2\alpha}{2}} \leq c\left(\sigma_{n,\lambda},\, l_a^2(\alpha),\, H^\infty\right) \leq C_{n,r}\left(l_a^2(\alpha),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}}.$$

Equivalently, if $\beta \in (-1, +\infty)$ then

$$B'\left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{2}} \leq c\left(\sigma_{n,\lambda},\, L_a^2(\beta),\, H^\infty\right) \leq C_{n,r}\left(L_a^2(\beta),\, H^\infty\right) \leq A'\left(\frac{n}{1-r}\right)^{\frac{\beta+2}{2}},$$

where A and B depend only on α, A' and B' depend only on β, and both of the two left-hand side inequalities are valid only for α and β satisfying $1-2\alpha \in \mathbb{N}$ and $\frac{\beta+1}{2} \in \mathbb{N}$.

The right-hand side inequalities given in Theorem 2.1 are proved in Section 4 whereas the left-hand side ones are proved in Section 5 .

Remark 2.2. If $N = [1-2\alpha]$ is the integer part of $1-2\alpha$, then Theorem 2.1 is valid with B and A such that $B \asymp \frac{1}{2^{3N}(2N)!}$ and $A \asymp N!(4N)^N$. In the same way, if $N' = [2+\beta]$ is the integer part of $2+\beta$, then Theorem 2.1 is valid with B' and A' such that $B' \asymp \frac{1}{2^{3N'}(2N')!}$ and $A' \asymp N'!(4N')^{N'}$. (The notation $x \asymp y$ means that there exist numerical constants $c_1, c_2 > 0$ such that $c_1 y \leq x \leq c_2 y$).

Next, we give an estimate for $C_{n,r}(X, H^\infty)$ in the scale of the spaces $X = l_a^p(\alpha)$, $\alpha \leq 0$, $1 \leq p \leq +\infty$. We start with a result for $1 \leq p \leq 2$.

Theorem 2.3. *Let $r \in [0, 1)$, $n \geq 1$, $p \in [1, 2]$, and let $\alpha \leq 0$. We have*

$$Bn^{1-\alpha-\frac{1}{p}} \leq C_{n,r}\left(l_a^p(\alpha),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}},$$

where $A = A(\alpha, p)$ and $B = B(\alpha, p)$ are constants depending only on α and p.

It is very likely that the bounds stated in Theorem 2.3 are not sharp. The sharp one should be probably $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$. In the same way, for $2 \leq p \leq \infty$, we give the following theorem, in which we feel again that the upper bound $\left(\frac{n}{1-r}\right)^{\frac{3}{2}-\alpha-\frac{2}{p}}$ is not sharp. As before, the sharp one is probably $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$.

Theorem 2.4. *Let $r \in [0,\,1)$, $n \geq 1$, $p \in [2,\,+\infty]$, and let $\alpha \leq 0$. We have*

$$B^{'} n^{1-\alpha-\frac{1}{p}} \leq C_{n,\,r}\left(l_a^p\left(\alpha\right),\,H^{\infty}\right) \leq A^{'}\left(\frac{n}{1-r}\right)^{\frac{3}{2}-\alpha-\frac{2}{p}},$$

where $A^{'}$ and $B^{'}$ depend only on α and p.

Theorems 2.1, 2.3 and 2.4 were already announced in the note [15]. Let σ be a finite set of $\mathbb{D}$, and let $f \in X$. The technical tools used in the proofs of the upper bounds for the interpolation constants $c\,(\sigma,\,X,\,H^{\infty})$ are: a linear interpolation

$$f \mapsto \sum_{k=1}^{n} \langle f,\,e_k\rangle\,e_k,$$

where $\langle .,. \rangle$ means the Cauchy sesquilinear form $\langle h,\,g\rangle = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)}$, and $(e_k)_{1\leq k\leq n}$ is the explicitly known Malmquist basis (see [11, p. 11]) or Definition 3.1 below) of the space $K_B = H^2 \ominus BH^2$ where $B = B_\sigma$ (Subsection 3.1), a Bernstein-type inequality of Dyakonov (used by induction): $\|f'\|_p \leq c_p\,\|B'\|_{\infty}\,\|f\|_p$, for a (rational) function f in the star-invariant subspace $H^p \cap B\overline{zH^p}$ generated by a (finite) Blaschke product B, (Dyakonov [7, 8]); it is used in order to find an upper bound for $\|\sum_{k=1}^{n}\langle f,\,e_k\rangle\,e_k\|_{\infty}$ (in terms of $\|f\|_X$) (Subsection 3.2), and finally (Subsection 3.3) the complex interpolation between Banach spaces, (see [4] or [12, Theorem 1.9.3-(a)-p.59]).

The lower bound problem (for $C_{n,\,r}\,(X,\,H^{\infty})$) is treated by using the "worst" interpolation $n-$tuple $\sigma = \sigma_{n,\,\lambda} = \{\lambda,\,\cdots,\,\lambda\}$, a one-point set of multiplicity n (the Carathéodory–Schur type interpolation). The "worst" interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ. We note that the spaces $X = l_a^p(\alpha)$ satisfy the condition $X \circ b_\lambda \subset X$ when $p = 2$, whereas this is not the case for $p \neq 2$. That is why our problem of estimating the interpolation constants is more difficult for $p \neq 2$.

The paper is organized as follows. In Section 3, we introduce the three technical tools mentioned above. Section 4 is devoted to the proof of the upper bounds of Theorems 2.1, 2.3 and 2.4. Finally, in Section 5, we prove the lower bounds of these theorems.

3. Preliminaries

In this section, we develop the technical tools mentioned in Section 2, which are used later on to establish an upper bound for $c\,(\sigma,\,X,\,H^{\infty})$.

3.1. **Malmquist basis and orthogonal projection.**

In Definitions 3.1, 3.2, 3.3 and in Remark 3.4 below, $\sigma = \{\lambda_1,\,\cdots,\,\lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and B_σ is the corresponding Blaschke product.

Definition 3.1. *Malmquist family.* For $k \in [1,\,n]$, we set $f_k = \frac{1}{1-\overline{\lambda}_k z}$, and define the family $(e_k)_{1\leq k\leq n}$, (which is known as Malmquist basis, see [11, p.117]), by

$$e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j}\right) \frac{f_k}{\|f_k\|_2}$$

for $k \in [2, n]$; we have $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$.

Definition 3.2. The model space K_{B_σ}. We define K_{B_σ} to be the n-dimensional space:

$$K_{B_\sigma} = \left(B_\sigma H^2\right)^\perp = H^2 \ominus B_\sigma H^2.$$

Definition 3.3. The orthogonal projection P_{B_σ}on

K_{B_σ}. We define P_{B_σ} to be the orthogonal projection of H^2 on its n-dimensional subspace K_{B_σ}.

Remark 3.4. The Malmquist family $(e_k)_{1\leq k\leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ}. In particular,

$$P_{B_\sigma} = \sum_{k=1}^{n} (\cdot,\, e_k)_{H^2}\, e_k,$$

where $(.,\, .)_{H^2}$ means the scalar product on H^2.

We now recall the following lemma already (partially) established in [14, p. 15] which is useful in the proof of the upper bound in Theorem 2.4.

Lemma 3.5. *Let $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$ and let $(e_k)_{1\leq k\leq n}$ be the Malmquist family corresponding to σ. Let also $\langle\cdot,\, \cdot\rangle$ be the Cauchy sesquilinear form $\langle h,\, g\rangle = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)}$, (if $h \in \mathrm{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$, $\hat{h}(k)$ stands for the k^{th} Taylor coefficient of h). The map $\Phi :\, \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ given by*

$$\Phi :\; f \mapsto \sum_{k=1}^{n} \langle f,\, e_k\rangle\, e_k,$$

is well defined and has the following properties:

(a) $\Phi_{|H^2} = P_{B_\sigma}$,

(b) Φ is continuous on $\mathrm{Hol}(\mathbb{D})$ with the topology of the uniform convergence on compact sets of $\mathbb{D}$,

(c) if $X = l^p_a(\alpha)$ with $p \in [1, +\infty]$, $\alpha \in (-\infty, 0]$ and $\Psi = Id_{|X} - \Phi_{|X}$, then $\mathrm{Im}\,(\Psi) \subset B_\sigma X$,

(d) if $f \in Hol(\mathbb{D})$, then

$$|\Phi(f)(\zeta)| = |\langle f,\, P_{B_\sigma} k_\zeta\rangle|\,,$$

for all $\zeta \in \mathbb{D}$, where P_{B_σ} is defined in 3.3 and $k_\zeta = \left(1 - \overline{\zeta} z\right)^{-1}$.

Proof. Points (a), (b) and (c) were already proved in [14]. In order to prove (d), we simply need to write that

$$\Phi(f)(\zeta) = \sum_{k=1}^{n} \langle f,\, e_k\rangle\, e_k(\zeta) = \left\langle f,\, \sum_{k=1}^{n} \overline{e_k(\zeta)} e_k \right\rangle,$$

$\forall f \in \mathrm{Hol}(\mathbb{D})$, $\forall \zeta \in \mathbb{D}$ and to notice that $\sum_{k=1}^{n} \overline{e_k(\zeta)} e_k = \sum_{k=1}^{n} (k_\zeta, e_k)_{H^2} e_k = P_{B_\sigma} k_\zeta$. □

3.2. **Bernstein-type inequalities for rational functions.** Bernstein-type inequalities for rational functions are the subject of a number of papers and monographs (see, for instance, [2, 3, 7, 8, 9]). We use here a result going back to Dyakonov [7, 8].

Lemma 3.6. *Let* $B = \prod_{j=1}^{n} b_{\lambda_j}$, *be a finite Blaschke product (of order* n*),* $r = \max_j |\lambda_j|$, *and let* $f \in K_B$. *Then*

$$\|f'\|_{H^2} \le 3\frac{n}{1-r} \|f\|_{H^2}.$$

Lemma 3.6 is a partial case ($p = 2$) of the following K. Dyakonov's result [8] (which is, in turn, a generalization of Levin's inequality [9] corresponding to the case $p = \infty$): the norm $\|D\|_{K_B^p \to H^p}$ of the differentiation operator $Df = f'$ on the star-invariant subspace of the Hardy space H^p, $K_B^p := H^p \cap B\overline{zH^p}$, (where the bar denotes complex conjugation) satisfies the following estimate:

$$\|D\|_{K_B^p \to H^p} \le c_p \|B'\|_\infty,$$

for every p, $1 \le p \le \infty$, where c_p is a positive constant depending only on p, B is a finite Blaschke product and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. In the case $p = 2$, Dyakonov's result gives $c_p = \frac{36+2\sqrt{3\pi}}{2\pi}$, which entails an estimate similar to that of Lemma 3.6, but with a larger constant $\left(\frac{13}{2}\text{ instead of } 3\right)$. Our lemma is proved in [14, Proposition 6.1.1].

The sharpness of the inequality stated in Lemma 3.6 is discussed in [13]. Here we use it by induction in order to get the following corollary.

Corollary 3.7. *Let* $B = \prod_{j=1}^{n} b_{\lambda_j}$, *be a finite Blaschke product (of order* n*),* $r = \max_j |\lambda_j|$, *and* $f \in K_B$. *Then,*

$$\left\|f^{(k)}\right\|_{H^2} \le k! 4^k \left(\frac{n}{1-r}\right)^k \|f\|_{H^2},$$

for every $k = 0, 1, \cdots$

Proof. Indeed, since $z^{k-1} f^{(k-1)} \in K_{B^k}$, we obtain applying Lemma 3.6 with B^k instead of B,

$$\left\|z^{k-1} f^{(k)} + (k-1) z^{k-2} f^{(k-1)}\right\|_{H^2} \le 3\frac{kn}{1-r} \left\|z^{k-1} f^{(k-1)}\right\|_{H^2} = 3\frac{kn}{1-r} \left\|f^{(k-1)}\right\|_{H^2}.$$

In particular,

$$\left|\left\|z^{k-1} f^{(k)}\right\|_{H^2} - \left\|(k-1) z^{k-2} f^{(k-1)}\right\|_{H^2}\right| \le 3\frac{kn}{1-r} \left\|f^{(k-1)}\right\|_{H^2},$$

which gives

$$\left\|f^{(k)}\right\|_{H^2} \le 3\frac{kn}{1-r} \left\|f^{(k-1)}\right\|_{H^2} + (k-1) \left\|f^{(k-1)}\right\|_{H^2} \le 4\frac{kn}{1-r} \left\|f^{(k-1)}\right\|_{H^2}.$$

By induction,

$$\left\|f^{(k)}\right\|_{H^2} \leq k!\left(\frac{4n}{1-r}\right)^k \|f\|_{H^2}.$$

□

3.3. **Interpolation between Banach spaces (the complex method).** In Section 4 we use the following lemma which was already proved (and used) in [14].

Lemma 3.8. *Let X_1 and X_2 be two Banach spaces of holomorphic functions in the unit disc $\mathbb{D}$. Let also $\theta \in [0,\, 1]$ and $(X_1,\, X_2)_{[\theta]}$ be the corresponding intermediate Banach space resulting from the classical complex interpolation method applied between X_1 and X_2, (we use the notation of* [4, Chapter 4]*). Then,*

$$C_{n,r}\left((X_1,\, X_2)_{[\theta]}\,,\, H^\infty\right) \leq C_{n,r}\left(X_1,\, H^\infty\right)^{1-\theta} C_{n,r}\left(X_2,\, H^\infty\right)^{\theta},$$

for all $n \geq 1$, $r \in [0,\, 1)$.

Proof. For the proof of this lemma, we refer to [14, p.19]. □

4. UPPER BOUNDS FOR $C_{n,r}\left(X,\, H^\infty\right)$

The aim of this section is to prove the upper bounds stated in Theorems 2.1, 2.3 and 2.4.

4.1. **The case $X = l_a^2\left(\alpha\right),\, \alpha \leq 0$.** We start with the following result.

Corollary 4.1. *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r}\left(l_a^2\left(-N\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2N+1}{2}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where A depends only on N (of order $N!(4N)^N$, see the proof below).

Proof. Indeed, let $X = l_a^2\left(-N\right)$, σ a finite subset of $\mathbb{D}$ and $B = B_\sigma$. If $f \in X$, then using part (c) of Lemma 3.5, we get that $\Phi(f)_{|\sigma} = f_{|\sigma}$. Now, denoting $X^\star$ the dual of X with respect to the Cauchy pairing $\langle \cdot,\, \cdot \rangle$ (defined in Lemma 3.5). Applying point (d) of the same lemma, we obtain $X^\star = l_a^2\left(N\right)$ and

$$\left|\Phi(f)(\zeta)\right| \leq \|f\|_X \left\|P_B k_\zeta\right\|_{X^\star} \leq \|f\|_X\, K_N\left(\left\|P_B k_\zeta\right\|_{H^2}^2 + \left\|\left(P_B k_\zeta\right)^{(N)}\right\|_{H^2}^2\right)^{\frac{1}{2}},$$

for all $\zeta \in \mathbb{D}$, where

$$K_N = \max\left\{N^N,\, \sup_{k \geq N} \frac{(k+1)^N}{k(k-1)\cdots(k-N+1)}\right\}$$
$$= \max\left\{N^N,\, \frac{(N+1)^N}{N!}\right\} = \left\{\begin{array}{c} N^N,\ if\ N \geq 3 \\ \frac{(N+1)^N}{N!},\ if\ N = 1,\, 2 \end{array}\right. .$$

(Indeed, the sequence $\left(\frac{(k+1)^N}{k(k-1)\cdots(k-N+1)}\right)_{k\geq N}$ is decreasing and $\left[N^N > \frac{(N+1)^N}{N!}\right] \Longleftrightarrow N \geq 3$). Since $P_B k_\zeta \in K_B$, Corollary 3.7 implies

$$|\Phi(f)(\zeta)| \leq \|f\|_X K_N \|P_B k_\zeta\|_{H^2} \left(1 + (N!)^2 \left(4\frac{n}{1-r}\right)^{2N}\right)^{\frac{1}{2}}$$

$$\leq A(N) \left(\frac{n}{1-r}\right)^{N+\frac{1}{2}} \|f\|_X ,$$

where $A(N) = \sqrt{2} K_N \left(1 + (N!)^2 4^{2N}\right)^{\frac{1}{2}}$, since

$$\|P_B k_\zeta\|_{H^2} = \left\|\sum_{k=1}^{n} (k_\zeta,\, e_k)_{H^2}\, e_k\right\|_{H^2} = \sqrt{\sum_{k=1}^{n} |e_k(\zeta)|^2} \leq \sqrt{\frac{2n}{1-r}}. \qquad (4.1.2)$$

□

Proof of Theorem 2.1 (the right-hand side inequality). There exists an integer N such that $N-1 \leq -\alpha \leq N$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha = (1-\theta)(1-N) + \theta.(-N)$. Since

$$\left(l^2_a(1-N)\,,\, l^2_a(-N)\right)_{[\theta]} = l^2_a(\alpha)\,,$$

(see [4, 12]), this gives, using Lemma 3.8 with $X_1 = l^2_a(1-N)$ and $X_2 = l^2_a(-N)$, and Corollary 4.1, that

$$C_{n,r}\left(l^2_a(\alpha)\,,\, H^\infty\right) \leq A(N-1)^{1-\theta} A(N)^\theta \left(\frac{n}{1-r}\right)^{\frac{(2N-1)(1-\theta)}{2} + \frac{(2N+1)\theta}{2}}.$$

It remains to use that $\theta = 1 - \alpha - N$ and set $A(\alpha) = A(N-1)^{1-\theta} A(N)^\theta$. □

4.2. **An upper bound for** $c\left(\sigma,\, l^p_a(\alpha)\,,\, H^\infty\right),\ 1 \leq p \leq 2$**.** The purpose of this subsection is to prove the right-hand side inequality of Theorem 2.3 We start with a partial case.

Lemma 4.2. *Let $N \geq 0$ be an integer. Then*

$$C_{n,r}\left(l^1_a((-N)\,,\, H^\infty\right) \leq A_1 \left(\frac{n}{1-r}\right)^{N+\frac{1}{2}},$$

for all $r \in [0,\, 1)$, $n \geq 1$, where A_1 depends only on N (it is of order $N!(4N)^N$, see the proof below).

Proof. In fact, the proof is exactly the same as in Corollary 4.1: if σ is a sequence in $\mathbb{D}$ with card $\sigma \leq n$, and $f \in l^1_a(-N) = X$, then $X^\star = l^\infty_a(N)$ (the dual of X

with respect to the Cauchy pairing). Using Lemma 3.5 we still have $\Phi(f)_{|\sigma} = f_{|\sigma}$, and for every $\zeta \in \mathbb{D}$,

$$\begin{aligned}|\Phi(f)(\zeta)| &\leq \|f\|_X \|P_B k_\zeta\|_{X^\star} \\ &\leq \|f\|_X K_N \max\left\{\sup_{0\leq k\leq N-1}\left|\widehat{P_B k_\zeta}(k)\right|, \sup_{k\geq N}\left|\widehat{(P_B k_\zeta)^{(N)}}(k-N)\right|\right\} \\ &\leq \|f\|_X K_N \max\left\{\|P_B k_\zeta\|_{H^2}, \left\|(P_B k_\zeta)^{(N)}\right\|_{H^2}\right\},\end{aligned}$$

where K_N is defined in the the proof of Corollary 4.1. Since $P_B k_\zeta \in K_B$, Corollary 3.7 implies that

$$|\Phi(f)(\zeta)| \leq \|f\|_X K_N \|P_B k_\zeta\|_{H^2}\left(1 + N!4^N\left(\frac{n}{1-r}\right)^N\right),$$

for all $\zeta \in \mathbb{D}$, which completes the proof using (4.1.2) and setting $A_1(N) = 2\sqrt{2}N!4^N K_N$. □

Proof of Theorem 2.3 (the right-hand inequality) . **Step 1.** We start by proving the result for $p = 1$ and for all $\alpha \leq 0$. We use the same reasoning as in Theorem 2.1 except that we replace $l_a^2(\alpha)$ by $l_a^1(\alpha)$.

Step 2. We now prove the result for $p \in [1, 2]$ and for all $\alpha \leq 0$: the scheme of this step is completely the same as in Step 1, but we use this time the complex interpolation between $l_a^1(\alpha)$ and $l_a^2(\alpha)$ (the classical Riesz-Thorin Theorem [4, 12]). Applying Lemma 3.8 with $X_1 = l_a^1(\alpha)$ and $X_2 = l_a^2(\alpha)$, it suffices to use Theorem 2.1 and Theorem 2.3 for the special case $p = 1$ (already proved in Step 1), to complete the proof of the right-hand side inequality. □

4.3. **An upper bound for $c(\sigma, l_a^p(\alpha), H^\infty)$, $2 \leq p \leq +\infty$.** Here, we prove the upper bound stated in Theorem 2.4. As before, the upper bound $\left(\frac{n}{1-r}\right)^{\frac{3}{2}-\alpha-\frac{2}{p}}$ is not as sharp as in Subsection 4.1. As in Subsection 4.2, we can suppose the constant $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$should be again a sharp upper (and lower) bound for the quantity $C_{n,r}(l_a^p(\alpha), H^\infty)$, $2 \leq p \leq +\infty$.

First we prove the following partial case of Theorem 2.4.

Corollary 4.3. *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r}(l_a^\infty(-N), H^\infty) \leq A_\infty\left(\frac{n}{1-r}\right)^{N+\frac{3}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where A_∞ depends only on N (it is of order $N!(4N)^N$, see the proof below).

Proof. We use literally the same method as in Corollary 4.1 and Lemma 4.2. Indeed, if $\sigma = \{\lambda_1, \cdots, \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and $f \in l_a^\infty(-N) = X$, then $X^\star = l_a^1(N)$ and applying again Lemma 3.5 we get $\Phi(f)_{|\sigma} = f_{|\sigma}$. For every $\zeta \in \mathbb{D}$, we have

$$|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k_\zeta\|_{X^\star} \leq \|f\|_X K_N\left(\|P_B k_\zeta\|_W + \left\|(P_B k_\zeta)^{(N)}\right\|_W\right),$$

where

$$W=\left\{f=\sum_{k\geq 0}\hat{f}(k)z^k:\ \|f\|_W:=\sum_{k\geq 0}\left|\hat{f}(k)\right|<\infty\right\},$$

stands for the Wiener algebra, and K_N is defined in the proof of Corollary 4.1. Now, applying Hardy's inequality (see [11, p.370],), we obtain

$$\begin{aligned}&|\Phi(f)(\zeta)|\\ &\leq \|f\|_X K_N\left(\pi\left\|(P_Bk_\zeta)'\right\|_{H^1}+|(P_Bk_\zeta)(0)|+\pi\left\|(P_Bk_\zeta)^{(N+1)}\right\|_{H^1}+\left|(P_Bk_\zeta)^{(N)}(0)\right|\right)\\ &\leq \|f\|_X K_N\pi\left(\left\|(P_Bk_\zeta)'\right\|_{H^2}+\|(P_Bk_\zeta)\|_{H^2}+\left\|(P_Bk_\zeta)^{(N+1)}\right\|_{H^2}+\left\|(P_Bk_\zeta)^{(N)}\right\|_{H^2}\right),\end{aligned}$$

for all $\zeta\in\mathbb{D}$. Using Lemma 3.6 and Corollary 3.7, we get

$$|\Phi(f)(\zeta)|\leq \|f\|_X K_N\pi\,\|P_Bk_\zeta\|_{H^2}\left(\frac{3n}{1-r}+1+(N+1)!\left(\frac{4n}{1-r}\right)^{N+1}+N!\right),$$

for all $\zeta\in\mathbb{D}$, which completes the proof using (4.1.2). □

We now prove the right-hand side inequality of our Theorem 2.4

Proof. The proof repeates the scheme from Theorem 2.3 (the two steps) excepted that this time, we replace (in both steps) the space $X=l^1_a(\alpha)$ by $X=l^\infty_a(\alpha)$. □

5. Lower bounds for $C_{n,r}(X,\,H^\infty)$

Here we prove the left-hand side inequalities stated in Theorems 2.1, 2.3 and 2.4.

5.1. **The case** $X=l^2_a(\alpha)$, $\alpha\leq 0$**.** We start with verifying the sharpness of the upper estimate for the quantity

$$C_{n,r}\left(l^2_a\left(\frac{1-N}{2}\right),\,H^\infty\right),$$

(where $N\geq 1$ is an integer), in Theorem 2.1. This lower bound problem is treated by estimating our interpolation constant $c(\sigma,\ X,\ H^\infty)$ for the one-point interpolation set $\sigma_{n,\lambda}=\underbrace{\{\lambda,\lambda,\cdots,\lambda\}}_{n}$, $\lambda\in\mathbb{D}$:

$$c(\sigma_{n,\lambda},\ X,\ H^\infty)=\sup\left\{\|f\|_{H^\infty/b^n_\lambda H^\infty}:\ f\in X,\ \|f\|_X\leq 1\right\},$$

where $\|f\|_{H^\infty/b^n_\lambda H^\infty}=\inf\{\|f+b^n_\lambda g\|_\infty\,:\,g\in X\}$. In the proof, we notice that $l^2_a(\alpha)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$ (RKHS) and we use the fact that this space has some special properties for particular values of α $\left(\alpha=\frac{1-N}{2},\ N=1,\,2,\,\cdots\right)$. Before giving this proof (see Paragraph 5.1.2 below), we show in Subsection 5.1.1 that $l^2_a(\alpha)$ is a RKHS and we focus on the special case $\alpha=\frac{1-N}{2},\ N=1,\,2,\,\cdots$.

5.1.1. *The spaces $l_a^2(\alpha)$ are RKHS.* The reproducing kernel of $l_a^2(\alpha)$, by definition, is a $l_a^2(\alpha)$-valued function $\lambda \longmapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f,\, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2(\alpha)$, where $(.,.)$ means the scalar product $(f,\, g) = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)}(k+1)^{2\alpha}$. Since one has $f(\lambda) = \sum_{k\geq 0} \hat{f}(k)\lambda^k \frac{1}{(k+1)^{2\alpha}}(k+1)^{2\alpha}$ $(\lambda \in \mathbb{D})$, it follows that

$$k_\lambda^\alpha(z) = \sum_{k\geq 0} \frac{\overline{\lambda}^k z^k}{(k+1)^{2\alpha}},\ z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$, we get the Szegö kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

and for the Bergman space $L_a^2 = l_a^2\left(-\frac{1}{2}\right)$, the Bergman kernel $k_\lambda^{-1/2}(z) = (1 - \overline{\lambda}z)^{-2}$.

Now let us explain that more generally if $\alpha = \frac{1-N}{2}$, $N \in \mathbb{N}\backslash\{0\}$, the space $l_a^2(\alpha)$ coincides (topologically) with the RKHS whose reproducing kernel is $(k_\lambda(z))^N = (1 - \overline{\lambda}z)^{-N}$. Following the Aronszajn theory of RKHS (see, for example [1, 10]), given a positive definite function $(\lambda, z) \longmapsto k(\lambda,\, z)$ on $\mathbb{D} \times \mathbb{D}$ (i.e. such that $\sum_{i,j} \overline{a}_i a_j k(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers (a_i)) one can define the corresponding Hilbert spaces $H(k)$ as the completion of finite linear combinations $\sum_i \overline{a}_i k(\lambda_i, \cdot)$ endowed with the norm

$$\left\| \sum_i \overline{a}_i k(\lambda_i, \cdot) \right\|^2 = \sum_{i,j} \overline{a}_i a_j k(\lambda_i, \lambda_j).$$

When k is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions $H(k)$ embedded into $\mathrm{Hol}(\mathbb{D})$. Now, choosing for k the reproducing kernel of H^2, $k : (\lambda,\, z) \mapsto k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$, and $\varphi = z^N$, $N = 1, 2, \cdots$, the function $\varphi \circ k$ is also positive definite and the corresponding Hilbert space is

$$H(\varphi \circ k) = l_a^2\left(\frac{1-N}{2}\right). \tag{5.1}$$

(Another notation for the space $H(\varphi \circ k)$ is $\varphi(H^2)$ since k is the reproducing kernel of H^2). The equality (5.1) is a topological identity: the spaces coincide as sets of functions, and the norms are equivalent. Moreover, the space $H(\varphi \circ k)$ satisfies the following property: for every $f \in H^2$, $\varphi \circ f \in \varphi(H^2)$, and

$$\|\varphi \circ f\|_{H(\varphi\circ k)}^2 \leq \varphi(\|f\|_{H^2}^2), \tag{5.2}$$

(the Aronszajn-deBranges inequality, see [10, p.320]). The link between spaces of type $l_a^2\left(\frac{1-N}{2}\right)$ and of type $H(z^N \circ k)$ being established, we give the proof of the left-hand side inequality in Theorem 2.1.

5.1.2. *The proof of Theorem 2.1 (the lower bound).*

Proof. 0) We set $N = 1 - 2\alpha,\ N = 1,\, 2,\, \cdots$ and $\varphi(z) = z^N$.

1) Let $b > 0$, $b^2 n^N = 1$. We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} z},\ H_n = \varphi \circ Q_n,\ \Psi = bH_n.$$

Then $\|Q_n\|_2^2 = n$, and hence by (5.2),

$$\|\Psi\|_{H_\varphi}^2 \le b^2 \varphi\left(\|Q_n\|_2^2\right) = b^2\varphi(n) = 1.$$

Let $b > 0$ such that $b^2\varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c\,(\sigma_{n,\,\lambda},\ H_\varphi,\ H^\infty)$ $= c\,(\sigma_{n,\,\mu},\ H_\varphi,\ H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi / b_\lambda^n H_\varphi}$ consider $G \in H^\infty$ such that $\Psi - G \in b_\lambda^n \mathrm{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \mathrm{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =:\ \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree nN) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} b_\lambda(z)} \\ &= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \left(1 + (1-\overline{\lambda}) \sum_{k=1}^{n-1} z^k - \overline{\lambda} z^n\right) \\ &= (1-r^2)^{-1/2} \left(1 + (1+r) \sum_{k=1}^{n-1} z^k + r z^n\right) =: (1-r^2)^{-1/2} \psi_1. \end{aligned}$$

Then, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1-r^2)^{-\frac{1}{2}}\, \psi_1\right)$. Furthermore,

$$\varphi \circ \psi_1 = \psi_1^N(z).$$

Now, it is clear that ψ is a polynomial of degree Nn such that

$$\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b\varphi\left((1-r^2)^{-1/2}(1+r)n\right) = b\left(\sqrt{\frac{1+r}{1-r}}\, n\right)^N > 0.$$

4) Next, we show that there exists $c = c(N) > 0$ (for example, $c = K/\left[2^{2N}(N-1)!\right]$, K being a numerical constant) such that

$$\sum^{m}(\psi) := \sum_{j=0}^{m}\hat{\psi}(j) \geq c\sum_{j=0}^{Nn}\hat{\psi}(j) = c\psi(1),$$

where $m \geq 1$ is such that $2m = n$ if n is even and $2m-1 = n$ if n is odd.

Indeed, setting

$$S_n = \sum_{j=0}^{n} z^j,$$

we have

$$\sum^{m}\left(\psi_1^N\right) = \sum^{m}\left(\left(1+(1+r)\sum_{k=1}^{n-1}z^k + rz^n\right)^N\right) \geq \sum^{m}\left(S_{n-1}^N\right).$$

Next, we obtain

$$\begin{aligned}\sum^{m}\left(S_{n-1}^N\right) &= \sum^{m}\left(\left(\frac{1-z^n}{1-z}\right)^N\right)\\ &= \sum^{m}\left(\frac{1}{(1-z)^N}\right) = \frac{1}{(N-1)!}\sum^{m}\left(\frac{d^{N-1}}{dz^{N-1}}\frac{1}{1-z}\right)\\ &= \sum_{j=0}^{m} C_{N+j-1}^{j} \geq \sum_{j=0}^{m}\frac{(j+1)^{N-1}}{(N-1)!} \geq K\frac{m^N}{(N-1)!},\end{aligned}$$

where $K > 0$ is a numerical constant. Finally,

$$\begin{aligned}\sum^{m}\left(\psi_1^N\right) &\geq K\frac{m^N}{(N-1)!} \geq K\frac{(n/2)^N}{(N-1)!}\\ &= \frac{K}{2^N(N-1)!}\cdot\frac{((1+r)n)^N}{(1+r)^N}\\ &= \frac{K}{2^N(1+r)^N(N-1)!}\cdot(\psi_1(1))^N,\end{aligned}$$

which gives our estimate.

5) Let $F_n = \Phi_m + z^m\Phi_m$, where Φ_k stands for the k-th Fejer kernel. We have $\|g\|_\infty\|F_n\|_{L^1} \geq \|g \star F_n\|_\infty$ for every $g \in L^\infty(\mathbb{T})$, and taking the infimum over all $g \in H^\infty$ satisfying $\hat{g}(k) = \hat{\psi}(k)$, $\forall k \in [0,\, n-1]$, we obtain

$$\|\psi\|_{H^\infty/z^nH^\infty} \geq \frac{1}{2}\|\psi \star F_n\|_\infty,$$

where $\star$ stands for the usual convolution product. Now using part 4),

$$\begin{aligned}\|\Psi\|_{H^\infty/b_\lambda^n H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2}\|\psi\star F_n\|_\infty \\ &\geq \frac{1}{2}\left|(\psi\star F_n)(1)\right| \geq \frac{1}{2}\sum_{j=0}^{m}\hat{\psi}(j) \geq \frac{c}{2}\psi(1) = \frac{c}{2}b\left(\sqrt{\frac{1+r}{1-r}}n\right)^N \\ &\geq B\left(\frac{n}{1-r}\right)^{\frac{N}{2}}.\end{aligned}$$

6) In order to conclude, it remains to use (5.1). □

5.2. **The case $X = l_a^p(\alpha)$, $1 \leq p \leq \infty$.**

Proof of Theorems 2.3 and 2.4 (the lower bound). We first notice that

$$r \mapsto C_{n,r}(X,\, H^\infty)$$

increases. As a consequence, if $X = l_a^p(\alpha)$, $1 \leq p \leq \infty$, then

$$C_{n,r}\left(l_a^p(\alpha),\, H^\infty\right) \geq C_{n,0}\left(l_a^p(\alpha),\, H^\infty\right) = c\left(\sigma_{n,0},\, l_a^p(\alpha),\, H^\infty\right),$$

where $\sigma_{n,0} = \underbrace{\{0,\, 0, \cdots,\, 0\}}_{n}$. Now let $f = \frac{1}{n^{1/p}}\sum_{k=0}^{n-1}(k+1)^{-\alpha}z^k$. Then $\|f\|_X = 1$, and

$$\begin{aligned}c\left(\sigma_{n,0},\, l_a^p(\alpha),\, H^\infty\right) &\geq \|f\|_{H^\infty/z^n H^\infty} \\ &\geq \frac{1}{2}\|f\star F_n\|_\infty \\ &\geq \frac{1}{2}\left|(f\star F_n)(1)\right| \\ &\geq \frac{1}{2}\sum_{j=0}^{m}\hat{f}(j),\end{aligned}$$

where $\star$ and F_n are defined in part 5) of the proof of Theorem 2.1 (lower bound) in Subsection 5.1 and where $m \geq 1$ is such that $2m = n$ if n is even and $2m-1 = n$ if n is odd as in part 4) of the proof of the same Theorem. Now, since

$$\sum_{j=0}^{m}\hat{f}(j) = \frac{1}{n^{1/p}}\sum_{k=0}^{m}(k+1)^{-\alpha},$$

we get the result. □

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