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APPLICATIONS OF FIXED POINT THEOREMS TO THE HYERS–ULAM STABILITY OF FUNCTIONAL EQUATIONS – A SURVEY

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Abstract. The fixed point method, which is the second most popular technique of proving the Hyers–Ulam stability of functional equations, was used for the first time in 1991 by J.A. Baker who applied a variant of Banach's fixed point theorem to obtain the stability of a functional equation in a single variable. However, most authors follow Radu's approach and make use of a theorem of Diaz and Margolis. The main aim of this survey is to present applications of different fixed point theorems to the theory of the Hyers–Ulam stability of functional equations.

1. Introduction

Speaking of the stability of a functional equation we follow the question raised in 1940 by S.M. Ulam: *"when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?".* The first partial answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by D.H. Hyers (see [22]). After his result a great number of papers (see for instance monographs [23, 26, 29], survey articles [1, 18, 24, 39, 40, 43] and the references given there) on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions and to other equations (as the words *"differing slightly"* and *"be close"* may have various meanings, different kinds of stability can be dealt with (see for instance [35])).

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Hyers's method used in [22], which is often called the *direct method*, has been applied for studying the stability of various functional equations (however, as it was shown in [30], this method sometimes does not work). Nevertheless, there are also other approaches proving the Hyers–Ulam stability, for example: the method of *invariant means* (see [42]), the method based on *sandwich theorems* (see [36]), the method using the concept of *shadowing* (see [44]).

It this paper we present the *fixed point method*, which is the second most popular technique of proving the stability of functional equations. Although it was used for the first time by J.A. Baker (see [5]) who applied a variant of Banach's fixed point theorem to obtain the Hyers–Ulam stability of a functional equation in a single variable, most authors follow Radu's approach (see [38]) and make use of a theorem of Diaz and Margolis. Our aim is not to collect the results obtained in this way (some of them can be found for instance in [15, 17, 29, 37]), but to show applications of different fixed point theorems to the theory of the Hyers–Ulam stability.

The article contains both classical and more recent results. In its first part we present applications of theorems (or some their variants) of Banach, Jung, Matkowski, Ćirić, Diaz and Margolis coming from [2, 5, 13, 20, 21, 33, 38]. The second part of the paper follows [4, 8, 9] and shows a somewhat different (but still fixed point) approach, when the results on the stability are simple consequences of the proved (new) fixed point theorems.

In the paper $\mathbb{N}$ denotes the set of positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$.

2. Applications of known fixed point theorems

2.1. **(Some variants of) Banach's theorem.** The fixed point method for studying the Hyers–Ulam stability of functional equations was used for the first time in 1991. Namely, in [5], J.A. Baker proved the following variant of Banach's fixed point theorem.

Theorem 2.1. [5, Theorem 1] *Let (Y, ρ) be a complete metric space and $T : Y \to Y$ be a contraction (that is, there is a $\lambda \in [0, 1)$ such that*

$$\rho(T(x), T(y)) \le \lambda \rho(x, y)$$

for all $x, y \in Y$). If $u \in Y$, $\delta > 0$ and

$$\rho(u, T(u)) \le \delta, \tag{2.1}$$

then T has a unique fixed point $p \in Y$. Moreover,

$$\rho(u, p) \le \frac{\delta}{1 - \lambda}.$$

Next, he applied Theorem 2.1 to obtain the following result concerning the stability of a quite general functional equation in a single variable.

Theorem 2.2. [5, Theorem 2] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $F : S \times X \to X$, $\lambda \in [0, 1)$ and*

$$d(F(t, u), F(t, v)) \le \lambda d(u, v), \qquad t \in S,\ u, v \in X.$$

If $g : S \to X$, $\delta > 0$ and

$$d(g(t), F(t, g(\varphi(t)))) \le \delta, \qquad t \in S, \tag{2.2}$$

then there is a unique function $f : S \to X$ such that

$$f(t) = F(t, f(\varphi(t))), \qquad t \in S \tag{2.3}$$

and

$$d(f(t), g(t)) \le \frac{\delta}{1-\lambda}, \qquad t \in S.$$

The proof of Theorem 2.2 proceeds as follows. We put

$$Y := \{a : S \to X : \sup\{d(a(t), g(t)),\ t \in S\} < \infty\} \tag{2.4}$$

and

$$\rho(a, b) := \sup\{d(a(t), b(t)),\ t \in S\}, \qquad a, b \in Y. \tag{2.5}$$

Then $g \in Y$ and (Y, ρ) is a complete metric space. Next, we define

$$T(a)(t) := F(t, a(\varphi(t))), \qquad a \in Y, t \in S \tag{2.6}$$

and show that

$$\rho(T(a), T(b)) \le \lambda \rho(a, b), \qquad a, b \in Y.$$

Since (2.2) yields

$$\rho(g, T(g)) \le \delta, \tag{2.7}$$

Theorem 2.1 finishes the proof.

Theorem 2.2 with

$$F(t, x) := \alpha(t) + \beta(t)x, \qquad t \in S,\ x \in E$$

gives

Corollary 2.3. [5, Theorem 3] *Let S be a nonempty set, E be a real (or complex) Banach space, $\varphi : S \to S$, $\alpha : S \to E$, $\beta : S \to \mathbb{R}$ (or $\mathbb{C}$), $\lambda \in [0, 1)$ and*

$$|\beta(t)| \le \lambda, \qquad t \in S.$$

If $g : S \to E$, $\delta > 0$ and

$$\|g(t) - (\alpha(t) + \beta(t)g(\varphi(t)))\| \le \delta, \qquad t \in S,$$

then there exists a unique function $f : S \to E$ such that

$$f(t) = \alpha(t) + \beta(t)f(\varphi(t)), \qquad t \in S$$

and

$$\|f(t) - g(t)\| \le \frac{\delta}{1-\lambda}, \qquad t \in S. \tag{2.8}$$

Now, following [13], we show how the stability of a functional equation can be deduce from the following variant of Banach's fixed point theorem.

Theorem 2.4. *If (Y, ρ) is a complete metric space and $T : Y \to Y$ is a contraction (with a constant λ), then T has a unique fixed point $p \in Y$. Moreover,*

$$\rho(u, p) \le \frac{\rho(u, T(u))}{1-\lambda}, \qquad u \in Y.$$

Theorem 2.5. [13, Theorem 2.2] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $F : X \times X \to X$, $\lambda, \mu \in \mathbb{R}_+$ and*

$$d(F(s,u), F(t,v)) \le \mu d(s,t) + \lambda d(u,v), \qquad s,t,u,v \in X.$$

Assume also that $g : S \to X$, $\Phi : S \to \mathbb{R}_+$ are such that

$$d(g(t), F(g(t), g(\varphi(t)))) \le \Phi(t), \qquad t \in S$$

and there exists an $L \in [0,1)$ with

$$\lambda\Phi(\varphi(t)) + \mu\Phi(t) \le L\Phi(t), \qquad t \in S.$$

Then there is a unique function $f : S \to X$ such that

$$f(t) = F(f(t), f(\varphi(t))), \qquad t \in S$$

and

$$d(f(t), g(t)) \le \frac{\Phi(t)}{1-L}, \qquad t \in S.$$

To prove the above theorem we put

$$Y := \{a : S \to X : \inf\{k \in [0,\infty] : d(a(t), g(t)) \le k\Phi(t), t \in S\} < \infty\}$$

and

$$\rho(a,b) := \inf\{k \in [0,\infty] : d(a(t), b(t)) \le k\Phi(t), t \in S\}, \qquad a, b \in Y.$$

Then $g \in Y$ and (Y, ρ) is a complete metric space. Moreover, the formula

$$T(a)(t) := F(a(t), a(\varphi(t))), \qquad a \in Y, t \in S$$

defines a mapping $T : Y \to Y$ which is a contraction satisfying $\rho(g, T(g)) \le 1$, and Theorem 2.4 finishes the proof.

Let us mention here that Theorem 2.4 was also applied in [21] to the proof of the following result.

Theorem 2.6. [21, Theorem 2.1] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $F : S \times X \to X$, $\alpha : S \to (0, \infty)$, $\lambda \in [0, 1)$, and for any $t \in S$, $u, v \in X^S$,*

$$\alpha(\varphi(t))d(F(t, u(\varphi(t))), F(t, v(\varphi(t)))) \le \lambda\alpha(t)d(u(\varphi(t)), v(\varphi(t))).$$

If $g : S \to X$ satisfies the inequality

$$d(g(t), F(t, g(\varphi(t)))) \le \alpha(t), \qquad t \in S,$$

then there is a unique function $f : S \to X$ such that (2.3) holds and

$$d(f(t), g(t)) \le \frac{\alpha(t)}{1-\lambda}, \qquad t \in S.$$

2.2. **Matkowski's theorem and a variant of Ćirić's theorem.** In this section, we present applications of two other fixed point theorems. To formulate the first of them we need two more definitions.

A mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it is nondecreasing and

$$\lim_{n\to\infty} \gamma^n(t) = 0, \qquad t \in (0, \infty)\,.$$

Given such a mapping γ and a metric space (X, d), we say that a function $\psi : X \to X$ is a *Matkowski γ-contraction* if

$$d(\psi(x), \psi(y)) \le \gamma(d(x, y)), \qquad x, y \in X. \tag{2.9}$$

We can now recall Matkowski's fixed point theorem from [32].

Theorem 2.7. *If (X, d) is a complete metric space and $T : X \to X$ is a Matkowski γ-contraction, then T has a unique fixed point $p \in X$ and the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges to p for every $x \in X$.*

In [20], this theorem was applied to the proof of the following generalization of Theorem 2.2.

Theorem 2.8. [20, Theorem 2.2] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $F : S \times X \to X$. Assume also that*

$$d(F(t, u), F(t, v)) \le \gamma(d(u, v)), \qquad t \in S,\ u, v \in X,$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, and let $g : S \to X$, $\delta > 0$ be such that (2.2) holds. Then there is a unique function $f : S \to X$ satisfying equation (2.3) and

$$\rho(f, g) := \sup\{d(f(t), g(t)), t \in S\} < \infty. \tag{2.10}$$

Moreover,

$$\rho(f, g) - \gamma(\rho(f, g)) \le \delta. \tag{2.11}$$

To prove Theorem 2.8, we define Y, ρ, T by (2.4), (2.5), (2.6), respectively, and we note that (Y, ρ) is a complete metric space and $T : Y \to Y$ is a Matkowski γ-contraction. Thus Theorem 2.7 implies the existence of a unique $f : S \to X$ satisfying (2.3) and (2.10). Moreover, since for every $t \in S$ we have

$$d(g(t), f(t)) \le d(g(t), (Tg)(t)) + d((Tg)(t), (Tf)(t)) \le \delta + \gamma(\rho(f, g)),$$

(2.11) follows.

Theorem 2.8 with

$$F(t, x) := \psi(x), \qquad t \in S,\ x \in X$$

gives

Corollary 2.9. [20, Corollary 2.3] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$. Assume also that $\psi : X \to X$ is a Matkowski γ-contraction and let $g : S \to X$, $\delta > 0$ be such that*

$$d((\psi \circ g \circ \varphi)(t), g(t)) \le \delta, \qquad t \in S. \tag{2.12}$$

Then there is a unique function $f : S \to X$ satisfying the equation

$$\psi \circ f \circ \varphi = f \tag{2.13}$$

and condition (2.10)*. The function* f *is given by*

$$f(t) = \lim_{n\to\infty} \psi^n(g(\varphi^n(t))), \qquad t \in S. \tag{2.14}$$

On the other hand, in [2], the following variant of Ćirić's fixed point theorem was proved.

Theorem 2.10. [2, Theorem 2.1] *Let* (Y, ρ) *be a complete metric space and* $T : Y \to Y$ *be a mapping such that*

$$\begin{aligned}\rho(T(x), T(y)) &\le \alpha_1(x,y)\rho(x,y) + \alpha_2(x,y)\rho(x,T(x))\\ &+ \alpha_3(x,y)\rho(y,T(y)) + \alpha_4(x,y)\rho(x,T(y))\\ &+ \alpha_5(x,y)\rho(y,T(x)), \qquad x,y \in Y,\end{aligned}$$

where $\alpha_1, \ldots, \alpha_5 : Y \times Y \to \mathbb{R}_+$ *satisfy*

$$\sum_{i=1}^{5} \alpha_i(x,y) \le \lambda \tag{2.15}$$

for all $x, y \in Y$ *and a* $\lambda \in [0,1)$*. If* $u \in Y$*,* $\delta > 0$ *and* (2.1) *holds, then* T *has a unique fixed point* $p \in Y$*. Moreover,*

$$\rho(u,p) \le \frac{(2+\lambda)\delta}{2(1-\lambda)}.$$

Next, Baker's idea and Theorem 2.10 were used to obtain the following result concerning the stability of equation (2.3).

Theorem 2.11. [2, Theorem 2.2] *Let* S *be a nonempty set,* (X, d) *be a complete metric space,* $\varphi : S \to S$*,* $F : S \times X \to X$ *and*

$$\begin{aligned}d(F(t,x), F(t,y)) &\le \alpha_1(x,y)d(x,y) + \alpha_2(x,y)d(x,F(t,x))\\ &+ \alpha_3(x,y)d(y,F(t,y)) + \alpha_4(x,y)d(x,F(t,y))\\ &+ \alpha_5(x,y)d(y,F(t,x)), \qquad t \in S,\ x,y \in X,\end{aligned}$$

where $\alpha_1, \ldots, \alpha_5 : X \times X \to \mathbb{R}_+$ *satisfy* (2.15) *for all* $x, y \in X$ *and a* $\lambda \in [0,1)$*. If* $g : S \to X$*,* $\delta > 0$ *and* (2.2) *holds, then there is a unique function* $f : S \to X$ *satisfying equation* (2.3) *and*

$$d(f(t), g(t)) \le \frac{(2+\lambda)\delta}{2(1-\lambda)}, \qquad t \in S.$$

A consequence of Theorem 2.11 is the following

Corollary 2.12. [2, Theorem 2.3] *Let* S *be a nonempty set,* E *be a real or complex Banach space,* $\varphi : S \to S$*,* $\alpha : S \to E$*,* $B : S \to \mathfrak{L}(E)$ *(here* $\mathfrak{L}(E)$ *denotes the Banach algebra of all bounded linear operators on* E*),* $\lambda \in [0,1)$ *and*

$$\|B(t)\| \le \lambda, \qquad t \in S.$$

If $g : S \to E$*,* $\delta > 0$ *and*

$$\|g(t) - (\alpha(t) + B(t)(g(\varphi(t))))\| \le \delta, \qquad t \in S,$$

then there exists a unique function $f : S \to E$ *satisfying equation*

$$f(t) = \alpha(t) + B(t)(f(\varphi(t))), \qquad t \in S$$

and condition (2.8).

2.3. **Fixed point theorems of the alternative on generalized metric space.** In this part of the paper, we show how two fixed points alternatives can be used to get some Hyers–Ulam stability results.

In order to do this let us first recall (see [31, 33]) that a mapping $d : X^2 \to [0, \infty]$ is said to be a *generalized metric* on a nonempty set X if and only if for any $x, y, z \in X$ we have:

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

$$d(x, y) = d(y, x),$$

$$d(x, z) \leq d(x, y) + d(y, z).$$

Now, following [38], we show how the stability of Cauchy's functional equation can be deduce from the following theorem of Diaz and Margolis from [16].

Theorem 2.13. *Let* (X, d) *be a complete generalized metric space. Assume that* $T : X \to X$ *is a strictly contractive operator with the Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* n_0 *such that* $d(T^{n_0+1}(x), T^{n_0}(x)) < \infty$ *for an* $x \in X$, *then the following three statements are true.*

(*i*) *The sequence* $(T^n(x))_{n\in\mathbb{N}}$ *converges to a fixed point* p *of* T.

(*ii*) p *is the unique fixed point of* T *in*

$$Z := \{y \in X : d(T^{n_0}(x), y) < \infty\}.$$

(*iii*) *If* $y \in Z$, *then*

$$d(y, p) \leq \frac{1}{1 - L} d(T(y), y).$$

Let us mention here that the below theorem was first proved by the direct method: for $p \in [0, 1)$ in [3], and for $p \in (1, \infty)$ in [19] (see also for instance [29]).

Theorem 2.14. [38] *Let* E *be a real normed space,* F *be a real Banach space,* $\theta \in [0, \infty)$, $p \in [0, \infty) \setminus \{1\}$ *and* $f : E \to F$ *be such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \qquad x, y \in E. \tag{2.16}$$

Then there exists a unique additive mapping $a : E \to F$ *such that*

$$\|f(x) - a(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p, \qquad x \in E. \tag{2.17}$$

Radu's proof of Theorem 2.14 (see [38] and also [29]) proceeds as follows. We put

$$X := \{g : E \to F : pg(0) = 0\}$$

and

$$d_p(g, h) := \sup\{\frac{\|g(x) - h(x)\|}{\|x\|^p}, x \in E \setminus \{0\}\}, \qquad g, h \in X.$$

Thus, we obtain a complete generalized metric space (X, d_p). Next, we define

$$T(g)(x) := \frac{1}{q} g(qx), \qquad g \in X, x \in E,$$

where $q = 2$ if $p < 1$, while $q = \frac{1}{2}$ if $p > 1$, and prove that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L = q^{p-1} < 1$ and $d_p(f, T(f)) < \infty$. Using Theorem 2.13 one can now show that there exists a unique mapping $a : E \to F$ such that

$$a(2x) = 2a(x), \qquad x \in E$$

and

$$\|a(x) - f(x)\| \leq c\|x\|^p, \qquad x \in E$$

for a $c \in (0, \infty)$. Moreover,

$$a(x) = \lim_{n\to\infty} q^{-n} f(q^n x), \qquad x \in E$$

and (2.17) holds. Finally, the proof of the additivity of a is standard (we replace x and y in (2.16) by $q^n x$ and $q^n y$, respectively, divide by q^n the obtained inequality, and let $n \to \infty$).

In [33], D. Miheţ gave one more generalization of Theorem 2.2. To do this he proved another fixed point alternative. Now, we present these results.

Recall that a mapping $\gamma : [0, \infty] \to [0, \infty]$ is called a *generalized strict comparison function* if it is nondecreasing, $\gamma(\infty) = \infty$,

$$\lim_{n\to\infty} \gamma^n(t) = 0, \qquad t \in (0, \infty)$$

and $\lim_{t\to\infty}(t - \gamma(t)) = \infty$. Given such a mapping γ and a generalized metric space (X, d), we say that a function $\psi : X \to X$ is a *strict γ-contraction* if it satisfies inequality (2.9).

Now, we can formulate the following fixed point result.

Theorem 2.15. [33, Theorem 2.2] *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a strict γ-contraction such that $d(x, T(x)) < \infty$ for an $x \in X$. Then T has a unique fixed point p in the set*

$$Z := \{y \in X : d(x, y) < \infty\}$$

and the sequence $(T^n(y))_{n\in\mathbb{N}}$ converges to p for every $y \in Z$. Moreover, $d(x, T(x)) \leq \delta$ implies

$$d(p, x) \leq \sup\{s > 0 : s - \gamma(s) \leq \delta\}.$$

Using this theorem we can get the following generalization of Theorem 2.2.

Theorem 2.16. [33, Theorem 3.1] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $F : S \times X \to X$. Assume also that*

$$d(F(t, u), F(t, v)) \leq \gamma(d(u, v)), \qquad t \in S,\ u, v \in X,$$

where $\gamma : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function, and let $g : S \to X$, $\delta > 0$ be such that (2.2) holds. Then there is a unique function $f : S \to X$ satisfying (2.3) and

$$d(f(t), g(t)) \leq \sup\{s > 0 : s - \gamma(s) \leq \delta\}, \qquad t \in S. \tag{2.18}$$

The proof of Theorem 2.16 is similar to that of Theorem 2.2. We put $Y := \{a : S \to X\}$ and define ρ by (2.5). Then one can check that (Y, ρ) is a complete generalized metric space and $T : Y \to Y$, given by (2.6), is a strict γ-contraction satisfying (2.7), and thus our assertion follows from Theorem 2.15.

The same proof, with T given by

$$T(a)(t) := \psi(a(\varphi(t))),$$

also gives

Theorem 2.17. [33, Theorem 4.1] *Let S be a nonempty set, (X, d) be a complete metric space, $\varphi : S \to S$, $\psi : X \to X$. Assume also that $g : S \to X$ and $\delta > 0$ are such that (2.12) holds. If $\gamma : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function satisfying inequality (2.9), then there is a unique mapping $f : S \to X$ such that (2.13) and (2.18) hold. The function f is given by formula (2.14).*

3. New fixed point theorems and their applications

In this section, we present a somewhat different fixed point approach to the Hyers–Ulam stability of functional equations, in which the stability results are simple consequences of some new fixed point theorems.

Given a nonempty set S and a metric space (X, d), we define $\Delta : \left(X^S\right)^2 \to {\mathbb{R}_+}^S$ (A^B denotes the family of all functions mapping a set B into a set A) by

$$\Delta(\xi, \mu)(t) := d(\xi(t), \mu(t)), \qquad \xi, \mu \in X^S,\ t \in S.$$

With this notation, we have

Theorem 3.1. [8, Theorem 1] *Let S be a nonempty set, (X, d) be a complete metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$ and $\Lambda : {\mathbb{R}_+}^S \to {\mathbb{R}_+}^S$ be given by*

$$(\Lambda\delta)(t) := \sum_{i=1}^{k} L_i(t)\delta(f_i(t)), \qquad \delta \in {\mathbb{R}_+}^S,\ t \in S. \tag{3.1}$$

If $\mathcal{T} : X^S \to X^S$ is an operator satisfying the inequality

$$\Delta\big(\mathcal{T}\xi, \mathcal{T}\mu\big)(t) \le \Lambda\big(\Delta(\xi, \mu)\big)(t), \qquad \xi, \mu \in X^S,\ t \in S \tag{3.2}$$

and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that

$$\Delta\big(\mathcal{T}g, g\big)(t) \le \varepsilon(t), \qquad t \in S \tag{3.3}$$

and

$$\sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(t) =: \sigma(t) < \infty, \qquad t \in S, \tag{3.4}$$

then for every $t \in S$ the limit

$$\lim_{n\to\infty}(\mathcal{T}^n g)(t) =: f(t) \tag{3.5}$$

exists and the function $f : S \to X$, defined in this way, is a unique fixed point of $\mathcal{T}$ with

$$\Delta\big(g, f\big)(t) \le \sigma(t), \qquad t \in S. \tag{3.6}$$

A consequence of Theorem 3.1 is the following result on the stability of a quite wide class of functional equations in a single variable.

Corollary 3.2. [8, Corollary 3] *Let S be a nonempty set, (X, d) be a complete metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$, a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d\big(\Phi(t, y_1, \cdots, y_k), \Phi(t, z_1, \cdots, z_k)\big) \leq \sum_{i=1}^{k} L_i(t) d(y_i, z_i)$$

for any $(y_1, \cdots, y_k), (z_1, \cdots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by

$$(\mathcal{T}\varphi)(t) := \Phi(t, \varphi(f_1(t)), \cdots, \varphi(f_k(t))), \qquad \varphi \in X^S, \, t \in S. \tag{3.7}$$

Assume also that Λ is given by (3.1) and functions $g : S \to X$ and $\varepsilon : S \to \mathbb{R}_+$ are such that

$$d\big(g(t), \Phi(t, g(f_1(t)), \cdots, g(f_k(t)))\big) \leq \varepsilon(t), \qquad t \in S \tag{3.8}$$

and (3.4) holds. Then for every $t \in S$ limit (3.5) exists and the function $f : S \to X$ is a unique solution of the functional equation

$$\Phi(t, f(f_1(t)), \cdots, f(f_k(t))) = f(t), \qquad t \in S \tag{3.9}$$

satisfying inequality (3.6).

Next, following [9], we consider the case of non-Archimedean metric spaces (let us mention here that the first paper dealing with the Hyers–Ulam stability of functional equations in non-Archimedean normed spaces was [34], whereas [41] seems to be the first one in which the stability problem in a particular type of these spaces was considered). In order to do this, we introduce some notations and definitions.

Let S be a nonempty set. For any $\delta_1, \delta_2 \in {\mathbb{R}_+}^S$ we write $\delta_1 \leq \delta_2$ provided

$$\delta_1(t) \leq \delta_2(t), \qquad t \in S,$$

and we say that an operator $\Lambda : {\mathbb{R}_+}^S \to {\mathbb{R}_+}^S$ is *non-decreasing* if it satisfies the condition

$$\Lambda\delta_1 \leq \Lambda\delta_2, \qquad \delta_1, \delta_2 \in {\mathbb{R}_+}^S, \delta_1 \leq \delta_2.$$

Moreover, given a sequence $(g_n)_{n\in\mathbb{N}}$ in ${\mathbb{R}_+}^S$, we write $\lim_{n\to\infty} g_n = 0$ provided

$$\lim_{n\to\infty} g_n(t) = 0, \qquad t \in S.$$

We will also use the following hypothesis concerning operators $\Lambda : {\mathbb{R}_+}^S \to {\mathbb{R}_+}^S$:

($\mathcal{C}$) $\lim_{n\to\infty} \Lambda\delta_n = 0$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in ${\mathbb{R}_+}^S$ with $\lim_{n\to\infty} \delta_n = 0$.

Finally, let us recall that a metric d on a nonempty set X is called *non-Archimedean* (or an *ultrametric*) provided

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \qquad x, y, z \in X.$$

We can now formulate the following fixed point theorem.

Theorem 3.3. [9, Theorem 1] *Let S be a nonempty set, (X,d) be a complete non-Archimedean metric space and $\Lambda : {\mathbb{R}_+}^S \to {\mathbb{R}_+}^S$ be a non-decreasing operator satisfying hypothesis $(\mathcal{C})$. If $\mathcal{T} : X^S \to X^S$ is an operator satisfying inequality (3.2) and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that*

$$\lim_{n\to\infty} \Lambda^n \varepsilon = 0 \tag{3.10}$$

and (3.3) holds, then for every $t \in S$ limit (3.5) exists and the function $f : S \to X$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$\Delta\big(g, f\big)(t) \le \sup_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(t) =: \sigma(t), \qquad t \in S. \tag{3.11}$$

If, moreover,

$$(\Lambda\sigma)(t) \le \sup_{n\in\mathbb{N}_0} (\Lambda^{n+1}\varepsilon)(t), \qquad t \in S,$$

then f is the unique fixed point of $\mathcal{T}$ satisfying (3.11).

An immediate consequence of Theorem 3.3 is the following result on the stability of equation (3.9) in complete non-Archimedean metric spaces.

Corollary 3.4. [9, Corollary 3] *Let S be a nonempty set, (X,d) be a complete non-Archimedean metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$, a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d\big(\Phi(t, y_1, \cdots, y_k), \Phi(t, z_1, \cdots, z_k)\big) \le \max_{i\in\{1,\ldots,k\}} L_i(t) d(y_i, z_i)$$

for any $(y_1, \cdots, y_k), (z_1, \cdots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by (3.7). Assume also that Λ is given by

$$(\Lambda\delta)(t) := \max_{i\in\{1,\ldots,k\}} L_i(t)\delta(f_i(t)), \qquad \delta \in {\mathbb{R}_+}^S, t \in S$$

and functions $g : S \to X$ and $\varepsilon : S \to \mathbb{R}_+$ are such that (3.8) and (3.10) hold. Then for every $t \in S$ limit (3.5) exists and the function $f : S \to X$ is a solution of functional equation (3.9) satisfying inequality (3.11).

Given nonempty sets S, Z and functions $\varphi : S \to S$, $F : S \times Z \to Z$, we define an operator $\mathcal{L}^F_\varphi : Z^S \to Z^S$ by

$$\mathcal{L}^F_\varphi(g)(t) := F(t, g(\varphi(t))), \qquad g \in Z^S, t \in S,$$

and we say that $\mathcal{U} : Z^S \to Z^S$ is an *operator of substitution* provided $\mathcal{U} = \mathcal{L}^G_\psi$ with some $\psi : S \to S$ and $G : S \times Z \to Z$. Moreover, if $G(t, \cdot)$ is continuous for each $t \in S$ (with respect to a topology in Z), then we say that $\mathcal{U}$ is *continuous*.

Theorem 3.5. [4, Theorem 2.1] *Let S be a nonempty set, (X,d) be a complete metric space, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{T} : X^S \to X^S$, $\varphi : S \to S$ and*

$$\Delta(\mathcal{T}\alpha, \mathcal{T}\beta)(t) \le \Lambda(t, \Delta(\alpha\circ\varphi, \beta\circ\varphi)(t)), \qquad \alpha, \beta \in X^S,\ t \in S.$$

Assume also that for every $t \in S$, $\Lambda_t := \Lambda(t, \cdot)$ is nondecreasing, $\varepsilon : S \to \mathbb{R}_+$, $g : S \to X$,

$$\sum_{n=0}^{\infty} \big((\mathcal{L}^\Lambda_\varphi)^n \varepsilon\big)(t) =: \sigma(t) < \infty, \qquad t \in S \tag{3.12}$$

and (3.3) *holds. Then for every* $t \in S$ *limit* (3.5) *exists and inequality* (3.6) *is satisfied. Moreover, the following two statements are true.*

(*i*) *If* $\mathcal{T}$ *is a continuous operator of substitution or* Λ_t *is continuous at* 0 *for each* $t \in S$, *then* f *is a fixed point of* $\mathcal{T}$.

(*ii*) *If* Λ_t *is subadditive (that is,*

$$\Lambda_t(a+b) \le \Lambda_t(a) + \Lambda_t(b)$$

for all $a, b \in \mathbb{R}_+$*) for each* $t \in S$, *then* $\mathcal{T}$ *has at most one fixed point* $f \in X^S$ *such that*

$$\Delta(g,f)(t) \le M\sigma(t), \qquad t \in S \tag{3.13}$$

for a positive integer M.

Theorem 3.5 with $\mathcal{T} = \mathcal{L}_\varphi^F$ immediately gives the following generalization of Theorem 2.2.

Corollary 3.6. [4, Corollary 2.1] *Let* S *be a nonempty set,* (X, d) *be a complete metric space,* $F : S \times X \to X$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$ *and*

$$d(F(t,x), F(t,y)) \le \Lambda(t, d(x,y)), \qquad t \in S,\ x, y \in X.$$

Assume also that $\varphi : S \to S$, $\varepsilon : S \to \mathbb{R}_+$, (3.12) *holds,* $g : S \to X$, *for every* $t \in S$, $\Lambda_t := \Lambda(t, \cdot)$ *is nondecreasing,* $F(t, \cdot)$ *is continuous and*

$$d(g(t), F(t, g(\varphi(t)))) \le \varepsilon(t), \qquad t \in S.$$

Then for every $t \in S$ *the limit*

$$f(t) := \lim_{n\to\infty} \left(\mathcal{L}_\varphi^F\right)^n (g)(t)$$

exists, (3.6) *holds and* f *is a solution of equation* (2.3). *Moreover, if for every* $t \in S$, Λ_t *is subadditive and* $M \in \mathbb{N}$, *then* $f : S \to X$ *is the unique solution of* (2.3) *fulfilling* (3.13).

4. Final remarks

We end the paper with a few general remarks giving some further information on the connections between the fixed point theory and the Hyers–Ulam stability.

Remark 4.1. In this survey applications of different fixed point theorems to the Hyers–Ulam stability of functional equations have been presented. On the other hand, fixed point theorems can be obtained from some stability results. We refer to [10, 11, 25] for such an approach.

Remark 4.2. The fixed point method is also a useful tool for proving the Hyers–Ulam stability of differential (see [7, 28]) and integral equations (see for instance [12, 21, 27]).

Remark 4.3. Recently, during the 49th International Symposium on Functional Equations (Graz, June 19-26, 2011), B. Przebieracz presented an application of a Markov–Kakutani fixed point theorem to the proof of the stability of Cauchy's functional equation. She also mentioned that, according to R. Badora's communication, using in her proof a Day's fixed point theorem instead of the Markov–Kakutani one we obtain a similar result.

Remark 4.4. In [14], the fixed point alternatives of theorems of Bianchini–Grandolfi (see [6]) and Matkowski (see [32]) were used to get the generalized stability of Cauchy's functional equation in complete β-normed spaces.

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