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THE WAVELET CHARACTERIZATION OF HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, using the atomic theory of the Herz-type Hardy spaces with variable exponent, we give their wavelet characterization by means of some discrete tent spaces with variable exponent at the origin.

1. Introduction and preliminaries

The theory of function spaces with variable exponent has developed since the paper [8] of Kováčik and J.Rákosník appeared in 1991. In [3, 10], Hernández, Lu, Weiss and Yang gave the φ-transform and wavelet characterizations of Herz-type spaces. In addition, Kopaliani and Izuki introduced the wavelets inequalities of Lebesgue spaces with variable exponent in [7] and [4], respectively. Recently, the authors [12] defined the Herz-type Hardy spaces with variable exponent and gave their atomic characterizations.

Inspired by the aforementioned references, we give the wavelet characterization of the Herz-type Hardy spaces with variable exponent by using the atomic decomposition theory in Section 3. And for this purpose, firstly in Section 2 we will introduce a kind of discrete tent space with variable exponent.

To be precise, we first briefly recall some standard notations in the remainder of this section. Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

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This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is a constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The variable L^p spaces are a special case of Musielak-Orlicz spaces.

For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \to [1, \infty)$ such that

$$1 < p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} \leq \operatorname{ess\,sup}\{p(x) : x \in \Omega\} = p^+ < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively.

Lemma 1.1. *([8]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. *([5]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_1}$$

and

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Remark 1.3. The conclusions of Lemma 1.2 are true if we replace the balls B by the cubes Q.

Remark 1.4. Throughout this paper δ_2 is the same as in Lemma 1.2.

Lemma 1.5. *([2, 6, 9]) Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all cubes Q in $\mathbb{R}^n$,*

$$\frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Next we give the definition of the Herz spaces with variable exponent. Let $Q_k = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x_i| \leq 2^k\}$ and $A_k = Q_k \setminus Q_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ as the set of positive integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{Q_0}$. Similar to the definition of [5], we have

Definition 1.6. Let $\alpha \in \mathbb{R}, 0 < p < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

In [12], we gave the definitions of Herz-type Hardy spaces with variable exponent and their atomic decomposition characterizations. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N f(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1\}$ and $N > n + 1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.7. ([12]) Let $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n + 1$.

(i)The homogeneous Herz-type Hardy space $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii)The non-homogeneous Herz-type Hardy space $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x. Similar to the results of [12], we have the following definition and lemma.

Definition 1.8. Let $n\delta_2 \le \alpha < \infty, q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and non-negative integer $s \ge [\alpha - n\delta_2]$.

(i) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$, for some $r > 0$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, for any multi-index β with $|\beta| \le s$.

(ii) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and

$(1)'$ $\operatorname{supp} a \subset B(0,r)$, for some $r \ge 1$.

Lemma 1.9. *Let $n\delta_2 \le \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$(or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left(\text{or} \sum_{k=0}^{\infty} \lambda_k a_k \right), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in Q_k and

$\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$*(or* $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$*). Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left(\text{or } \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decompositions of f.

2. The discrete Herz-type tent spaces with variable exponent

In this section, we introduce a kind of discrete tent space with variable exponent to establish the wavelet characterization of Herz-type Hardy spaces with variable exponent. Let $\nu \in \mathbb{Z}$ and $K \in \mathbb{Z}^n$. Define $Q_{\nu,K} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : 2^\nu x - K \in [0,1)^n\}$ and $\mathcal{D} = \{Q_{\nu,K} : \nu \in \mathbb{Z}, K \in \mathbb{Z}^n\}$. Moreover, we set $s(\beta)(x) = \left(\sum_{Q \in \mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2}$ and $\operatorname{supp} \beta = \bigcup_{\{Q \in \mathcal{D} : \beta(Q) \neq 0\}} Q$, where $\beta = \{\beta(Q)\}_{Q \in \mathcal{D}}$ is a complex numerical series.

Definition 2.1. Let $0 < \alpha < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\beta = \{\beta(Q)\}_{Q \in \mathcal{D}}$. The tent space associated with $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{\beta : s(\beta) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|\beta\|_{T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|s(\beta)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

Similarly, we can define the space $TK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ replacing $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ by $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ in Definition 2.1.

Firstly we establish the central $(\alpha, q(\cdot))$-atom-sequence decomposition characterization of the space $T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Definition 2.2. Let $0 < \alpha < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If there is a cube R with the center at the origin, such that $R \supset \operatorname{supp} \beta$ and

$$\left\| \left(\sum_{Q \in \mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |R|^{-\alpha/n},$$

then $\beta = \{\beta(Q)\}_{Q \in \mathcal{D}}$ is said to be a central $(\alpha, q(\cdot))$-atom-sequence, and the smallest cube R with above property is called the base of β.

Theorem 2.3. *Let $0 < \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The following two statements are equivalent:*

(i) $\beta \in T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(ii) There exist a sequence $\{\beta_j\}_{j=-\infty}^{\infty}$ of central $(\alpha, q(\cdot))$-atom-sequences and a sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ of numbers such that

$$\operatorname{supp} \beta_j \subset \bigcup_{Q \in \mathcal{D}_j} Q, \ \beta = \sum_{j=-\infty}^{\infty} \lambda_j \beta_j \text{ and } \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} < \infty,$$

where $\mathcal{D}_j = \{Q \in \mathcal{D} : Q \subset Q_j \setminus Q_{j-1}\}$. Furthermore, in this case, the following two norms are mutually equivalent:

$$\|\beta\|_{T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \text{ and } \inf \left\{ \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all the central $(\alpha, q(\cdot))$-atom-sequence decompositions of β.

Proof. We prove (i) implies (ii) firstly. Let

$$\lambda_j = |Q_j|^{\alpha/n} \left\| \left(\sum_{Q \in \mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2} \chi_j(x) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Define $\beta_j = \{\beta_j(Q)\}_{Q \in \mathcal{D}}$ by

$$\beta_j(Q) = \begin{cases} \lambda_j^{-1} \beta(Q), & \text{if } Q \in \mathcal{D}_j; \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have $\beta = \sum_{j=-\infty}^{\infty} \lambda_j \beta_j$, $\operatorname{supp} \beta_j \subset \bigcup_{Q \in \mathcal{D}_j} Q$ and

$$\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \le \|s(\beta)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|\beta\|_{T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Moreover, we claim that β_j is a central $(\alpha, q(\cdot))$-atom-sequence, since

$$\begin{aligned}
&\left\| \left(\sum_{Q\in\mathcal{D}_j} |\beta_j(Q)|^2|Q|^{-1}\chi_Q(x) \right)^{1/2} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \left\| \left(\sum_{Q\in\mathcal{D}_j} \lambda_j^{-2}|\beta(Q)|^2|Q|^{-1}\chi_Q(x) \right)^{1/2} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \lambda_j^{-1} \left\| \left(\sum_{Q\in\mathcal{D}_j} |\beta(Q)|^2|Q|^{-1}\chi_Q(x) \right)^{1/2} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \lambda_j^{-1} \left\| \left(\sum_{Q\in\mathcal{D}} |\beta(Q)|^2|Q|^{-1}\chi_Q(x) \right)^{1/2} \chi_j(x) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \lambda_j^{-1}\lambda_j|Q_j|^{-\alpha/n} \le \left| \bigcup_{Q\in\mathcal{D}_j} Q \right|^{-\alpha/n}.
\end{aligned}$$

Next we will prove (ii) implies (i). Suppose β_j is a central $(\alpha, q(\cdot))$-atom-sequence with $\operatorname{supp}\beta_j \subset \bigcup_{Q\in\mathcal{D}_j} Q \subset Q_j$. We consider the two cases $0 < p \le 1$ and $1 < p < \infty$.

If $0 < p \le 1$, it suffices to prove that

$$\|s(\beta_j)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C,$$

where C is a positive constant independent of β_j. It is easy to see that $\operatorname{supp} s(\beta_j) \subset Q_j$, and thus we have

$$\begin{aligned}
\|s(\beta_j)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{j} 2^{k\alpha p}\|s(\beta_j)\|^p_{L^{q(\cdot)}(A_k)} \right\}^{1/p} \\
&\le \|s(\beta_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{j} 2^{k\alpha p} \right\}^{1/p} \\
&\le C|Q_j|^{-\alpha/n}2^{j\alpha} \le C.
\end{aligned}$$

If $1 < p < \infty$, by the Minkowski inequality, we have

$$\begin{aligned} s(\beta)(x) &= \left(\sum_{Q\in\mathcal{D}} \left|\sum_{j=-\infty}^{\infty} \lambda_j \beta_j(Q)\right|^2 |Q|^{-1}\chi_Q(x)\right)^{1/2} \\ &\le \sum_{j=-\infty}^{\infty} |\lambda_j| \left(\sum_{Q\in\mathcal{D}} |\beta_j(Q)|^2 |Q|^{-1}\chi_Q(x)\right)^{1/2} \\ &= \sum_{j=-\infty}^{\infty} |\lambda_j| s(\beta_j)(x). \end{aligned}$$

This implies that

$$\begin{aligned} \|s(\beta)\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|s(\beta)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{\infty} |\lambda_j| \|s(\beta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|s(\beta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{k\alpha} |Q_j|^{-\alpha/n}\right)^p \\ &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha}\right)^p \\ &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p 2^{(k-j)p\alpha/2}\right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)p'\alpha/2}\right)^{p/p'} \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+1} 2^{(k-j)p\alpha/2}\right) \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p, \end{aligned}$$

where $1/p + 1/p' = 1$.

Hence, (i) holds, and the proof of Theorem 2.3 is completed. □

Similarly, we also introduce the definition of central $(\alpha, q(\cdot))$-atom-sequence of restricted type.

Definition 2.4. Let $0 < \alpha < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If there is a cube R with the center at the origin and having side length no less than 2, such that $R \supset \operatorname{supp}\beta$ and

$$\left\| \left(\sum_{Q\in\mathcal{D}} |\beta(Q)|^2 |Q|^{-1}\chi_Q(x)\right)^{1/2} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |R|^{-\alpha/n},$$

then $\beta = \{\beta(Q)\}_{Q\in\mathcal{D}}$ is said to be a central $(\alpha, q(\cdot))$-atom-sequence of restricted type, and the smallest cube R with above property is called the base of β.

If the spaces $T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the central $(\alpha, q(\cdot))$-atom-sequence were replaced by the spaces $TK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the central $(\alpha, q(\cdot))$-atom-sequence of restricted type, then there is a similar result. The proof is similar to Theorem 2.3, so we omit it.

Theorem 2.5. *Let $0 < \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The following two statements are equivalent:*

(i) $\beta \in TK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(ii) There are a sequence $\{\beta_j\}_{j=0}^{\infty}$ of central $(\alpha, q(\cdot))$-atom-sequences of restricted type and a sequence $\{\lambda_j\}_{j=0}^{\infty}$ of numbers such that

$$\operatorname{supp}\beta_j \subset \bigcup_{Q\in\mathcal{D}_j} Q,\ \ \beta = \sum_{j=0}^{\infty} \lambda_j\beta_j \text{ and } \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p} < \infty,$$

where $\mathcal{D}_j = \{Q \in \mathcal{D} : Q \subset Q_j \setminus Q_{j-1}\}$. Moreover, in this case, the following norms are mutually equivalent:

$$\|\beta\|_{TK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \text{ and } \inf\left\{\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all the above central $(\alpha, q(\cdot))$-atom-sequence of restricted type decompositions of β.

3. The wavelet characterization of Herz-type Hardy spaces with variable exponent

In this section, we will use the γ-regular compactly support wavelets obtained by Daubechies in [1] (or see [11]) to give the characterization of the elements in Herz-type Hardy spaces with variable exponent.

Set $Q_{j,K}$ and $\mathcal{D}$ be as before. Let $E = \{0,1\}^n \setminus \{0, \cdots, 0\}$, φ and ψ be γ-regular compactly supported functions obtained by the multiresolution approximation in [1]. For any $\varepsilon \in E$ and $Q \in \mathcal{D}$, set

$$\psi^{\varepsilon}_Q(x) = 2^{nj/2}\psi^{\varepsilon_1}(2^j x_1 - k_1)\cdots\psi^{\varepsilon_n}(2^j x_n - k_n),$$

where $\psi^0 = \varphi$ and $\psi^1 = \psi$. Let mQ be the cube with the same center as Q and whose sides are m times as long. It is well known that $\{\psi^{\varepsilon}_Q\}_{Q\in\mathcal{D},\varepsilon\in E}$ have the following properties (see [1, 3, 11]):

(A) $\{\psi^{\varepsilon}_Q\}_{Q\in\mathcal{D},\varepsilon\in E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$.

(B) $\operatorname{supp}\psi^{\varepsilon}_Q \subset mQ$ with $m \geq 1$ for all $Q \in \mathcal{D}$.

(C) For any index $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq \gamma$,

$$|\partial^{\alpha}\psi^{\varepsilon}_Q(x)| \leq C2^{nj/2}2^{j|\alpha|}.$$

(D) $\int_{\mathbb{R}^n} x^{\alpha}\psi^{\varepsilon}_Q(x)dx = 0,\ |\alpha| \leq \gamma.$

Let Λ denote the set of indices $\lambda = K2^{-j} + \varepsilon 2^{-j-1}$ corresponding to ψ_Q^ε, where $Q = Q_{j,K}$. For simplicity, we write $\psi_Q^\varepsilon = \psi_\lambda$ with $\lambda \in \Lambda$ and $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$.

Theorem 3.1. *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose $f \in \mathcal{S}(\mathbb{R}^n)$, $\alpha(\lambda) = \langle f, \psi_\lambda\rangle$ and $f(x) = \sum_{\lambda\in\Lambda} \alpha(\lambda)\psi_\lambda(x)$, where ψ is a γ-regular compactly supported wavelet and $\gamma \ge \alpha - n\delta_2 + 1$. Then the following two statements are equivalent:*

(1) $S(f)(x) \equiv \left(\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1}\chi_{Q(\lambda)}(x)\right)^{1/2} \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(2) $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Moreover, the norms

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \text{ and } \|S(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$$

are equivalent.

Remark 3.2. This result is true for the spaces $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

To prove Theorem 3.1, we need the following lemma, which is a kind of wavelet characterization of $L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 3.3. *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C \ge 1$ such that for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \|S(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. The method of proof is similar to [4] or [7], here we omit it. Actually, Lemma 3.3 is one of two cases in [4].

□

Proof of Theorem 3.1 We first show (1) implies (2). It is easy to see that for any given $\varepsilon \in E$, $\{\alpha^\varepsilon(Q)\}_{Q\in\mathcal{D}} \in T\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Therefore, by Theorem 2.3, there are a sequence $\{\alpha_j^\varepsilon\}_{j=-\infty}^\infty$ of central $(\alpha, q(\cdot))$-atom-sequences and a sequence $\{\lambda_j\}$ of numbers such that $\operatorname{supp} \alpha_j^\varepsilon \subset Q_j$, $\alpha^\varepsilon = \sum_{j=-\infty}^\infty \lambda_j^\varepsilon \alpha_j^\varepsilon$ and

$$\left(\sum_{j=-\infty}^\infty |\lambda_j^\varepsilon|^p\right)^{1/p} \le \|S(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

So we have

$$\begin{aligned} f(x) &= \sum_{\lambda\in\Lambda} \alpha(\lambda)\psi_\lambda(x) \\ &= \sum_{\varepsilon\in E}\sum_{Q\in\mathcal{D}} \alpha^\varepsilon(Q)\psi_Q^\varepsilon(x) \\ &= \sum_{\varepsilon\in E}\sum_{Q\in\mathcal{D}}\sum_{j=-\infty}^{\infty} \lambda_j^\varepsilon \alpha_j^\varepsilon(Q)\psi_Q^\varepsilon(x) \\ &= \sum_{\varepsilon\in E}\sum_{j=-\infty}^{\infty} \lambda_j^\varepsilon \left(\sum_{Q\in\mathcal{D}} \alpha_j^\varepsilon(Q)\psi_Q^\varepsilon(x)\right). \end{aligned}$$

Set $a_j^\varepsilon = \sum_{Q\in\mathcal{D}} \alpha_j^\varepsilon(Q)\psi_Q^\varepsilon(x)$, then $\operatorname{supp} a_j^\varepsilon \subset mQ_j$ and by Lemma 3.3 we have

$$\begin{aligned} \|a_j^\varepsilon\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\left\|\left(\sum_{Q\in\mathcal{D}} |\alpha_j^\varepsilon(Q)|^2|Q|^{-1}\chi_Q\right)^{1/2}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C|Q_j|^{-\alpha/n}. \end{aligned}$$

Since ψ satisfies property (D), it is easy to see that a_j^ε is a central $(\alpha, q(\cdot))$-atom up to an absolute constant with support in mQ_j.

Because we only have a finite number of ε' s, by Lemma 1.9 we know that $f \in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C\|S(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

This finishes the proof of the fact that (1) implies (2).

Next we need to prove that (2) implies (1) to complete the proof of Theorem 3.1. We will consider two cases $0 < p \le 1$ and $1 < p < \infty$.

When $0 < p \le 1$, we only need to prove that $\|S(a)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C$ for each central $(\alpha, q(\cdot))$-atom a with support in Q_k, where C is independent of k. Let m satisfy $2^{m_0} \le m < 2^{m_0+1}$ with $m_0 \in \mathbb{N}$ (see (B) at the beginning of this section). Write

$$\begin{aligned} \|S(a)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}^p &= \sum_{l=-\infty}^{k+2} 2^{l\alpha p}\|S(a)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p + \sum_{l=k+3}^{\infty} 2^{l\alpha p}\|S(a)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \\ &\equiv I_1 + I_2. \end{aligned}$$

For I_1, by Lemma 3.3 we obtain

$$I_1 \le C\sum_{l=-\infty}^{k+2} 2^{l\alpha p}\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \le C\sum_{l=-\infty}^{k+2} 2^{(l-k)\alpha p} \le C < \infty.$$

To deal with I_2, we first estimate $S(a)(x)$ when $x \in A_l$ with $l \ge k+3$. Let $Q(\lambda)$ denote the cube associated with $\lambda \in \Lambda$ and $\Lambda^k = \{\lambda \in \Lambda : mQ(\lambda) \cap Q_k \ne \emptyset \text{ and } \psi_\lambda(x) \ne 0\}$. If $\lambda \in \Lambda^k$, we write the side length $L(Q(\lambda))$ of $Q(\lambda)$ is equal to 2^{-j}, so that $m2^{-j} > 2^{l-1} - 2^{l-2} = 2^{l-2}$, where $l \ge k+3$. So $j \le m_0 + 3 - l$. It is observed that a is a central $(\alpha, q(\cdot))$-atom and ψ is γ-regular with $\gamma \ge \alpha - n\delta_2 + 1$.

Let $\gamma_0 = \alpha - n\delta_2$ and $P_{\gamma_0}(x)$ be the γ_0-order Taylor expansion for $\psi_\lambda(x)$ at 0. Then we have

$$\begin{aligned} |\langle a, \psi_\lambda\rangle| &\leq \int_{\mathbb{R}^n} |a(x)||\psi_\lambda(x) - P_{\gamma_0}(x)|dx \\ &\leq C2^{nj/2}2^{j(\gamma_0+1)} \int_{\mathbb{R}^n} |a(x)||x|^{\gamma_0+1}dx \\ &\leq C2^{nj/2+(j+k)(\gamma_0+1)-k\alpha}\|\chi_{Q_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

On the other hand, it is easy to show that for any given j, the number of λ in Λ^k is less than an absolute constant C_0. Therefore, if $x \in A_l$ with $l \geq k+3$, then by the generalized Hölder inequality we have

$$\begin{aligned} [S(a)(x)]^2 &= \sum_{\lambda\in\Lambda} |\langle a, \psi_\lambda\rangle|^2|Q(\lambda)|^{-1}\chi_{Q(\lambda)}(x) \\ &= \sum_{\lambda\in\Lambda^k} |\langle a, \psi_\lambda\rangle|^2|Q(\lambda)|^{-1}\chi_{Q(\lambda)}(x) \\ &\leq C_0 \sum_{j=-\infty}^{m_0+3-l} 2^{2j(n/2+\gamma_0+1)+2k(\gamma_0+1-\alpha)+nj}\|\chi_{Q_k}\|^2_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\leq C_0 2^{2k(\gamma_0+1-\alpha)-2l(n+\gamma_0+1)}\|\chi_{Q_k}\|^2_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

That is,

$$S(a)(x) \leq C_0 2^{k(\gamma_0+1-\alpha)-l(n+\gamma_0+1)}\|\chi_{Q_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So by Lemma 1.2 and Lemma 1.5 we have

$$\begin{aligned} I_2 &\leq C \sum_{l=k+3}^{\infty} 2^{(l-k)(\alpha-\gamma_0-1)p}2^{-lnp}\|\chi_{Q_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{Q_l}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{l=k+3}^{\infty} 2^{(l-k)(\alpha-\gamma_0-1)p}2^{-lnp}\|\chi_{Q_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}(|Q_l|\|\chi_{Q_l}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)})^p \\ &\leq C \sum_{l=k+3}^{\infty} 2^{(l-k)(\alpha-\gamma_0-1)p} \left(\frac{\|\chi_{Q_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\ &\leq C \sum_{l=k+3}^{\infty} 2^{(l-k)(\alpha-n\delta_2-\gamma_0-1)p} = C < \infty. \end{aligned}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Let $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Then $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ and

$$\left\{\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right\}^{1/p} \leq C\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},$$

where a_k is a central $(\alpha, q(\cdot))$-atom with support in Q_k. It follows from the Minkowski inequality that

$$S(f)(x) \leq \sum_{k=-\infty}^{\infty} |\lambda_k| S(a_k)(x).$$

Write

$$\begin{aligned} \|S(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\leq \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|S(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|S(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\equiv II_1 + II_2. \end{aligned}$$

Similarly to the estimate for I_1, we obtain

$$\begin{aligned} II_2^p &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha} \right)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{(k-j)\alpha p/2} \right) \left(\sum_{j=k-2}^{\infty} 2^{(k-j)\alpha p'/2} \right)^{p/p'} \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

That is,

$$II_2 \leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Similar to the estimate for I_2, we obtain that if $x \in A_k$ with $k \geq j+3$, then

$$S(a_j)(x) \leq C 2^{j(\gamma_0+1-\alpha)-k(n+\gamma_0+1)} \|\chi_{Q_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So by Lemma 1.2 and Lemma 1.5 we have

$$
\begin{aligned}
II_1^p &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{j(\gamma_0+1-\alpha)-k(n+\gamma_0+1)} \|\chi_{Q_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{Q_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \\
&\quad \times \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{j(\gamma_0+1-\alpha)-k(n+\gamma_0+1)} \|\chi_{Q_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \left(|Q_k| \|\chi_{Q_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)} \right) \right)^p \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{j(\gamma_0+1-\alpha)-k(\gamma_0+1)} \left(\frac{\|\chi_{Q_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \right) \right)^p \\
&\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{(j-k)(\gamma_0+1+n\delta_2-\alpha)} \right)^p \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}
$$

That is,

$$
II_1 \le C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \le C \|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
$$

This finishes the proof of the fact that (2) implies (1).

Thus we finish the proof of Theorem 3.1.

□

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