Ann. Funct. Anal. 3 (2012), no. 1, 109–120
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

# A CHARACTERISATION OF THE FOURIER TRANSFORM ON THE HEISENBERG GROUP

R. LAKSHMI LAVANYA[1]* AND S. THANGAVELU[2]

Communicated by O. Christensen

ABSTRACT. The aim of this paper is to show that any continuous $*$-homomorphism of $L^1(\mathbb{C}^n)$(with twisted convolution as multiplication) into $\mathcal{B}(L^2(\mathbb{R}^n))$ is essentially a Weyl transform. From this we deduce a similar characterisation for the group Fourier transform on the Heisenberg group, in terms of convolution.

## 1. Introduction and preliminaries

The behaviour of the Fourier transform under translations, dilations, modulations and differentiation is well known. It is an interesting fact that a few of these properties are characteristic of the Fourier transform. Several characterisations of the Fourier transform were done in [3, 4, 8, 9, 10]. A well known property of the Fourier transform is that it takes convolution product into pointwise product. Conversely, is there any relation between the Fourier transform and a map which converts convolution product into pointwise product? Recently, a characterisation for the Fourier transform on $\mathbb{R}^n$ was done in [1, 2] without assuming the map to be linear or continuous. In [7], Jaming proved such characterisations for the groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$([7], Theorem 2.1), $\mathbb{R}^n$ and $\mathbb{T}^n$ ([7], Theorem 3.1). We state below the result of Jaming for the case $\mathbb{R}^n$ and $\mathbb{T}^n$:

**Theorem 1.1.** *Let $n \geq 1$ be an integer and $G = \mathbb{R}^n$ or $G = \mathbb{T}^n$. Let $T$ be a continuous linear operator $L^1(G) \to C(\widehat{G})$(here $\widehat{G}$ denotes the dual group of $G$) such that $T(f * g) = T(f)\ T(g)$. Then there exists a set $E \subset \widehat{G}$ and a function $\varphi : \widehat{G} \to \widehat{G}$ such that $T(f)(\xi) = \chi_E(\xi)\ \widehat{f}(\varphi(\xi))$.*

*Date*: Received: 2 November 2011; Accepted: 6 February 2012.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46K05; Secondary 42A85, 43A32.
*Key words and phrases.* Heisenberg group, Weyl transform, Heisenberg group Fourier transform, Hermite functions.

In the same paper([7]) he posed a question, which leads to that of the characterisation of the Weyl transform in terms of the twisted convolution. Here we attempt to prove such a characterisation and deduce a similar one for the Heisenberg group Fourier transform. An extensive study of Fourier analysis on the Heisenberg group was done in [6]. Before stating our results, we recall a few standard notations and terminology as in [5, 12, 13].

## 2. Notations and preliminaries

The $(2n+1)-$ dimensional Heisenberg group $\mathbb{H}^n$ is the nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. $\mathbb{H}^n$ forms a noncommutative group under the operation

$$(z,t)(w,s) = \left(z+w, t+s+\frac{1}{2}Im(z.\overline{w})\right),\ (z,t),(w,s) \in \mathbb{H}^n.$$

The Haar measure on $\mathbb{H}^n$ is the Lebesgue measure $dz\ dt$ on $\mathbb{C}^n \times \mathbb{R}$. By the Stone-von Neumann theorem, all the infinite-dimensional irreducible unitary representations of $\mathbb{H}^n$, acting on $L^2(\mathbb{R}^n)$, are parametrised by $\lambda \in \mathbb{R}^*$, and are given by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}\ e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\ \varphi(\xi+y),\ \xi \in \mathbb{R}^n,\ \varphi \in L^2(\mathbb{R}^n),$$

and $z = x+iy \in \mathbb{C}^n$. The group Fourier transform of an integrable function $f$ on $\mathbb{H}^n$ is defined as

$$\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\ \pi_\lambda(z,t)\ dz\ dt,\ \lambda \in \mathbb{R}^*.$$

Let $\mathcal{B}(L^2(\mathbb{R}^n))$ be the space of bounded linear operators on $L^2(\mathbb{R}^n)$. Then we have $\widehat{f}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}^n))$, with $\|\widehat{f}(\lambda)\|_{op} \leq \|f\|_1$.

The convolution $f*g$ of functions $f,g$ on $\mathbb{H}^n$ is defined by

$$(f*g)(z,t) = \int_{\mathbb{H}^n} f((z,t)(-w,-s))\ g(w,s)\ dw\ ds,\ (z,t) \in \mathbb{H}^n,$$

whenever the integral exists.

Then the group Fourier transform satisfies

**Property 1.** $(\widehat{f^*})(\lambda) = \widehat{f}(\lambda)^*$ for all $\lambda \in \mathbb{R}^*$, where $f^*(z,t) = \overline{f(-z,-t)}$ and $(\widehat{f}(\lambda))^*$ is the adjoint of the operator in $\mathcal{B}(L^2(\mathbb{R}^n))$.

**Property 2.** $(f*g)\widehat{\ }(\lambda) = \widehat{f}(\lambda)\ \widehat{g}(\lambda),\ \lambda \in \mathbb{R}^*,\ f,g \in L^1(\mathbb{H}^n)$.

**Property 3.** $(R_{(z,t)}f)\widehat{\ }(\lambda) = \widehat{f}(\lambda)\ \pi_\lambda(z,t)^*,\ (z,t) \in \mathbb{H}^n$, where $R_{(z,t)}$ denotes the right translation given by

$$(R_{(z,t)}f)(w,s) = f((w,s)(z,t)),\ (w,s) \in \mathbb{H}^n.$$

We shall prove in Section 3 that the above properties characterise the group Fourier transform on $\mathbb{H}^n$.

For $f \in L^1(\mathbb{H}^n)$ we denote by $f^\lambda(z)$, the inverse Fourier transform of $f$ in the $t$-variable, i.e.,

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t)\, e^{i\lambda t}\, dt,\ z \in \mathbb{C}^n.$$

We write $\pi_\lambda(z) = \pi_\lambda(z,0)$ so that $\pi_\lambda(z,t) = e^{i\lambda t}\, \pi_\lambda(z)$ and

$$\widehat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\, \pi_\lambda(z)\, dz.$$

For $\lambda \in \mathbb{R}^*$ and $g \in L^1(\mathbb{C}^n)$, consider the operator

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(z)\, \pi_\lambda(z)\, dz.$$

When $\lambda = 1$, we call this the Weyl transform of $g$. The $\lambda$-twisted convolution of functions $f, g \in L^1(\mathbb{C}^n)$ is defined as

$$(f *_\lambda g)(z) = \int_{\mathbb{C}^n} f(z-w)\, g(w)\, e^{i\frac{\lambda}{2}Im(z.\overline{w})}\, dw,\ z \in \mathbb{C}^n.$$

The convolution of functions on $\mathbb{H}^n$, and the $\lambda$-twisted convolution of functions on $\mathbb{C}^n$, are related as

$$(f * g)^\lambda(z) = (f^\lambda *_\lambda g^\lambda)(z),\ z \in \mathbb{C}^n.$$

The operators $W_\lambda$ are continuous, linear and map $L^1(\mathbb{C}^n)$ into $\mathcal{B}(L^2(\mathbb{R}^n))$. Also, they satisfy the following properties:

**Property A.** $W_\lambda(f^*) = W_\lambda(f)^*,\ f \in L^1(\mathbb{C}^n)$, where $f^*(z) = \overline{f(-z)}$.

**Property B.** $W_\lambda(f *_\lambda g) = W_\lambda(f)\, W_\lambda(g),\ f, g \in L^1(\mathbb{C}^n)$,

i.e., $W_\lambda$ is a continuous $*$-homomorphism from $L^1(\mathbb{C}^n)$ into $\mathcal{B}(L^2(\mathbb{R}^n))$. In Section 3, we shall prove the converse that any continuous $*$-homomorphism from $L^1(\mathbb{C}^n)$ into $\mathcal{B}(L^2(\mathbb{R}^n))$ is essentially a Weyl transform.

We now recall a few properties of the Hermite and special Hermite functions which will be of much use in proving this characterisation.
For $k \in \mathbb{N} = \{0, 1, 2, ...\}$, let

$$h_k(x) = (-1)^k\ (2^k\ k!\ \sqrt{\pi})^{(-1/2)}\ \left(\frac{d^k}{dx^k} e^{-x^2}\right)\ e^{x^2/2},\ x \in \mathbb{R},$$

denote the normalised Hermite functions on $\mathbb{R}$. The multi-dimensional Hermite functions are defined as

$$\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j),\ x = (x_1, ..., x_n) \in \mathbb{R}^n,\ \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n.$$

The collection $\{\Phi_\alpha :\ \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$ and their linear span is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. For $\lambda \in \mathbb{R}^*$, Suppose that

$\Phi^\lambda_\alpha(z) = |\lambda|^{\frac{n}{4}}\ \Phi_\alpha(\sqrt{|\lambda|}x)$. Then the scaled special Hermite functions are defined by

$$\Phi^\lambda_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}}\ |\lambda|^{\frac{n}{2}}\ \left(\pi_\lambda(z)\ \Phi^\lambda_\alpha, \Phi^\lambda_\beta\right),\ z \in \mathbb{C}^n,$$

and they form an orthonormal basis for $L^2(\mathbb{C}^n)$. Further finite linear combinations of special Hermite functions are dense in $L^p(\mathbb{C}^n)$ for $1 \le p < \infty$. Also they satisfy

$$\overline{\Phi}^\lambda_{\alpha\beta} *_\lambda\ \overline{\Phi}^\lambda_{\mu\nu}(z) = (2\pi)^{\frac{n}{2}}\ |\lambda|^{-n}\ \delta_{\alpha\nu}\ \ \overline{\Phi}^\lambda_{\mu\beta}(z),\ \ \alpha, \beta, \mu, \nu \in \mathbb{N}^n. \tag{2.1}$$

We refer to [12, 13] for these properties. We now proceed to prove our main results.

## 3. Characterisation of the Weyl transform

As recalled in Section 2, the Weyl transform is a continuous linear map from $L^1(\mathbb{C}^n)$ into $\mathcal{B}(L^2(\mathbb{R}^n))$ taking twisted convolution into composition of operators. We shall now prove the converse, thus answering a modified version of Jaming's question. We remark that the proof of the following theorem is similar to that of the Stone-von Neumann theorem as in [5]. Indeed, if $\rho_\lambda$ is a primary representation of $\mathbb{H}^n$ with central character $e^{i\lambda t}$, then the operator defined on $L^1(\mathbb{C}^n)$ by

$$T_\lambda(f) = \int_{\mathbb{C}^n} f(z)\ \rho_\lambda(z, 0)\ dz$$

satisfies the hypothesis of the following theorem. By the Stone-von Neumann theorem $\rho_\lambda(z, t)$ is a direct sum of representations each of which is unitarily equivalent to $\pi_\lambda(z, t)$. The proof makes use of the relations

$$T_\lambda f\ \rho_\lambda(z, 0) = T_\lambda(\tau^\lambda_z\ f),\ \ \rho_\lambda(z, 0)\ T_\lambda\ f = T_\lambda(\tau^{-\lambda}_z f)$$

where

$$\tau^\lambda_z\ f(w) = f(w - z)\ e^{-i\frac{\lambda}{2}\Im(w.\bar{z})}$$

is the $\lambda-$twisted translation. The proof given below shows that we really do not need these extra properties in order to prove Stone-von Neumann theorem.

The following theorem can also be proved using the Stone-von Neumann theorem and the representation theory of locally compact groups. We attempt to prove it without using these techniques.

**Theorem 3.1.** *Let $T : (L^1(\mathbb{C}^n), *_\lambda) \to \mathcal{B}(L^2(\mathbb{R}^n))$ be a nonzero continuous homomorphism. Then there is a subspace $\mathcal{H}^\lambda$ of $L^2(\mathbb{R}^n)$ and a unitary representation $\rho_\lambda$ of $\mathbb{H}^n$ on $\mathcal{H}^\lambda$ such that*

$$T(f) = \int_{\mathbb{C}^n} f(z)\ \rho_\lambda(z, 0)\ dz,\ \ on\ \mathcal{H}^\lambda,$$

*and there is a decomposition $L^2(\mathbb{R}^n) = \mathcal{H}^\lambda \bigoplus V^\lambda$, where*

$$V^\lambda := \{v \in L^2(\mathbb{R}^n) : (Tf)(v) = 0\ \ for\ \ all\ \ f \in L^1(\mathbb{C}^n)\}.$$

*Proof.* It suffices to prove the result when $\lambda = 1$ as the general case follows similarly. We let $f \times g := f *_\lambda g$ and we will drop all subscripts and superscripts involving $\lambda(= 1)$.

For $\alpha, \beta \in \mathbb{N}^n$, let $Q_{\alpha\beta} = (2\pi)^{-\frac{n}{2}}\, T(\overline{\Phi}_{\alpha\beta})$. Then

$$\begin{aligned} Q_{\alpha\beta}\, Q_{\mu\nu} &= (2\pi)^{-n}\, T(\overline{\Phi}_{\alpha\beta} \times \overline{\Phi}_{\mu\nu}) \quad \text{(by hypothesis)} \\ &= \delta_{\alpha\nu}\, (2\pi)^{-\frac{n}{2}}\, T(\overline{\Phi}_{\mu\beta}) \qquad \text{(by (2.1))} \\ \text{i.e.,} \quad Q_{\alpha\beta}\, Q_{\mu\nu} &= \delta_{\alpha\nu}\, Q_{\mu\beta}. \end{aligned} \tag{3.1}$$

For $\alpha, \beta \in \mathbb{N}^n$ and $v, w \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} (2\pi)^{\frac{n}{2}}\, (Q_{\alpha\beta}\, v, w) &= (v, T(\overline{\Phi}_{\alpha\beta})^*\, w) \\ &= (v, T(\overline{\Phi}_{\beta\alpha})\, w) \\ \text{i.e.,} \qquad {Q_{\alpha\beta}}^* &= Q_{\beta\alpha},\ \alpha,\ \beta \in \mathbb{N}^n. \end{aligned} \tag{3.2}$$

Note that for each $\alpha \in \mathbb{N}^n$, $Q_{\alpha\alpha} \neq 0$. To see this suppose $Q_{\alpha\alpha} = 0$ for some $\alpha \in \mathbb{N}^n$. Then

$$Q_{\beta\alpha}\, u = Q_{\alpha\alpha}\, Q_{\beta\alpha}\, u = 0 \text{ for any } \beta \in \mathbb{N}^n,\ u \in L^2(\mathbb{R}^n).$$

Similarly,

$$Q_{\alpha\gamma}\, u = Q_{\alpha\gamma}\, Q_{\alpha\alpha}\, u = 0 \text{ for any } \gamma \in \mathbb{N}^n,\ u \in L^2(\mathbb{R}^n).$$

For arbitrary $\beta,\ \gamma \in \mathbb{N}^n,\ u \in L^2(\mathbb{R}^n)$,

$$Q_{\beta\gamma}\, u = Q_{\alpha\gamma}\, Q_{\beta\alpha}\, u = 0.$$

This implies $T = 0$, a contradiction. Thus $Q_{\alpha\alpha} \neq 0$ for any $\alpha \in \mathbb{N}^n$.

Let $\alpha \in \mathbb{N}^n$. Then the range $R(Q_{\alpha\alpha})$ of $Q_{\alpha\alpha}$ is non-zero. Let $\{u_\alpha^j\}_{j=1}^\infty$ be an orthonormal basis of $R(Q_{\alpha\alpha})$. For $\beta \in \mathbb{N}^n$, define
$v_{\alpha,\beta}^j = Q_{\alpha\beta}\, u_\alpha^j$. Then

$$\begin{aligned} (v_{\alpha,\beta}^j, v_{\alpha,\gamma}^k) &= (Q_{\gamma\alpha}\, Q_{\alpha\beta}\, u_\alpha^j,\ u_\alpha^k) \qquad \text{(by (3.2))} \\ &= \delta_{\beta\gamma}\ (Q_{\alpha\alpha}\, u_\alpha^j,\ u_\alpha^k) \qquad \text{(by (3.1))} \\ &= \delta_{\beta\gamma}\, \delta_{jk}. \end{aligned} \tag{3.3}$$

In particular, $\{v_{\alpha,\beta}^j\}_{\beta \in \mathbb{N}^n}$ is an orthonormal set.

Let $\mathcal{H}_\alpha^j$ be the Hilbert space with $\{v_{\alpha,\beta}^j\}_{\beta \in \mathbb{N}^n}$ as an orthonormal basis. Define $U_\alpha^j : L^2(\mathbb{R}^n) \to \mathcal{H}_\alpha^j$ by $U_\alpha^j(\Phi_\beta) = v_{\alpha,\beta}^j,\ \beta \in \mathbb{N}^n$. Let

$$S_\alpha^j(f) = U_\alpha^j\, W(f)\, U_\alpha^{j*},\ f \in L^1(\mathbb{C}^n).$$

For $v=\sum_\beta\ c_\beta\ v^j_{\alpha,\beta}\in\mathcal{H}^j_\alpha$, using the relation $W(\overline{\Phi}_{\mu\nu})\ \Phi_\beta=(2\pi)^{\frac{n}{2}}\ \delta_{\beta\mu}\ \Phi_\nu$, we have

$$\begin{aligned} S^j_\alpha(\overline{\Phi}_{\mu\nu})v &= U^j_\alpha\ W(\overline{\Phi}_{\mu\nu})\ \left(\sum_\beta\ c_\beta\ \Phi_\beta\right) \\ &= (2\pi)^{\frac{n}{2}}\ U^j_\alpha\ c_\mu\ \Phi_\nu \\ \text{i.e.,}\ \ S^j_\alpha(\overline{\Phi}_{\mu\nu})v &= (2\pi)^{\frac{n}{2}}\ c_\mu\ v^j_{\alpha,\nu} \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned} T(\overline{\Phi}_{\mu\nu})v &= (2\pi)^{\frac{n}{2}}\ \sum_\beta\ c_\beta\ Q_{\mu\nu}\ Q_{\alpha\beta}\ u^j_\alpha \\ &= (2\pi)^{\frac{n}{2}}\ \ c_\mu\ Q_{\alpha\nu}\ u^j_\alpha \qquad (\text{by}\ \ (3.1)) \\ \text{i.e.,}\ \ T(\overline{\Phi}_{\mu\nu})v &= (2\pi)^{\frac{n}{2}}\ \ c_\mu\ v^j_{\alpha,\nu} \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$T(\overline{\Phi}_{\mu\nu})v \;=\; (U^j_\alpha\ W(\overline{\Phi}_{\mu\nu})\ U^{j*}_\alpha)\ v,\ \text{for all}\ v\in\mathcal{H}^j_\alpha,\ \mu,\nu\in\mathbb{N}^n.$$

This gives

$$T(f)|_{\mathcal{H}^j_\alpha} \;=\; \int_{\mathbb{C}^n} f(z)\ U^j_\alpha\ \pi_1(z)\ U^{j*}_\alpha\ dz,\ \ f\in L^1(\mathbb{C}^n). \tag{3.6}$$

Note that (3.3) implies that the spaces $\mathcal{H}^j_\alpha$ and $\mathcal{H}^k_\alpha$ are orthogonal to each other when $j\neq k$.

Let $\mathcal{H}_\alpha=\bigoplus_{j=1}^\infty\ \mathcal{H}^j_\alpha$ and write $L^2(\mathbb{R}^n)=\mathcal{H}_\alpha\bigoplus V_1$. Equation (3.6) gives a complete description of $T$ on $\mathcal{H}_\alpha$ and our next task is to obtain one for $T|_{\mathcal{H}^\perp_\alpha}$. For this we first show that the range $R(Q_{\alpha\beta})\subseteq\mathcal{H}_\alpha$ for all $\beta\in\mathbb{N}^n$. If $v\in R(Q_{\alpha\beta})$, then using (3.1) we get

$$v \;=\; Q_{\alpha\beta}\ u=Q_{\alpha\beta}\ Q_{\alpha\alpha}\ u\ \ \text{for some}\ \ u\in L^2(\mathbb{R}^n).$$

Since $Q_{\alpha\alpha}\ u\in R(Q_{\alpha\alpha})$, $Q_{\alpha\alpha}\ u=\sum_j\ c_j\ u^j_\alpha$ and so

$$v=Q_{\alpha\beta}\ Q_{\alpha\alpha}\ u=\sum_j\ c_j\ v^j_{\alpha,\beta}\in\mathcal{H}_\alpha.$$

Thus $R(Q_{\alpha\beta})\subseteq\mathcal{H}_\alpha$ for all $\beta\in\mathbb{N}^n$. For $v\in\mathcal{H}^\perp_\alpha$ and $u\in L^2(\mathbb{R}^n)$, this gives $(v,Q_{\alpha\beta}\ u)=0$ for all $\beta\in\mathbb{N}^n$, which implies $Q_{\beta\alpha}\ v=0$ by (3.2). Thus

$$Q_{\beta\alpha}=0\ \ \text{on}\ \ \mathcal{H}^\perp_\alpha\ \ \text{for all}\ \ \beta\in\mathbb{N}^n.$$

By (3.1), for $v\in\mathcal{H}^\perp_\alpha$, $\beta\in\mathbb{N}^n$, $Q_{\beta\beta}\ v=\ Q_{\alpha\beta}\ Q_{\beta\alpha}\ v=0$. Thus

$$Q_{\beta\beta}=0\ \ \text{on}\ \ \mathcal{H}^\perp_\alpha\ \ \text{for all}\ \ \beta\in\mathbb{N}^n.$$

Again for $v\in\mathcal{H}^\perp_\alpha$ and $u\in L^2(\mathbb{R}^n)$,

$$(Q_{\alpha\beta}\ v,u)=(v,Q_{\beta\alpha}\ u)=\ (v,Q_{\alpha\alpha}\ Q_{\beta\alpha}\ u)=0.$$

Thus $Q_{\alpha\beta}=0$ on $\mathcal{H}^\perp_\alpha$ for all $\beta\in\mathbb{N}^n$. Finally, for any $v\in\mathcal{H}^\perp_\alpha$, $\mu,\nu\in\mathbb{N}^n$, $Q_{\mu\nu}\ v=\ Q_{\alpha\nu}\ Q_{\mu\alpha}\ v=0$. This gives $T|_{\mathcal{H}^\perp_\alpha}=0$.

We have thus obtained a collection $\{\mathcal{H}_\alpha^j\}_{j=1,2,\ldots}$ of mutually orthogonal subspaces of $L^2(\mathbb{R}^n)$ and unitary representations $\rho_\alpha^j\ (z,t) = U_\alpha^j\ \pi_1(z,t)\ U_\alpha^{j*}$ of $\mathbb{H}^n$, on $\mathcal{H}_\alpha^j$ such that

$$T(f)|_{\mathcal{H}_\alpha^j} = \int_{\mathbb{C}^n} f(z)\ \rho_\alpha^j(z,0)\ dz,\ f \in L^1(\mathbb{C}^n).$$

Then $\rho_\alpha = \bigoplus_{j=1}^\infty\ \rho_\alpha^j$ is a unitary representation of $\mathbb{H}^n$ on $\mathcal{H}_\alpha$ and

$$T(f)|_{\mathcal{H}_\alpha} = \int_{\mathbb{C}^n} f(z)\ \rho_\alpha(z,0)\ dz,\ f \in L^1(\mathbb{C}^n),$$

which is the required characterisation. □

The following remarks are in order. $(L^p(\mathbb{C}^n), *_\lambda)$ is an algebra as long as $1 \leq p \leq 2$ and for $f \in L^p(\mathbb{C}^n), W_\lambda(f)$ is still a bounded linear operator on $L^2(\mathbb{R}^n)$ and satisfies

$$\|W_\lambda(f)\| \leq C\|f\|_p.$$

This follows from the fact that for $\varphi, \psi \in L^2(\mathbb{R}^n)$ the function $(\pi_\lambda(z,0)\varphi, \psi)$ belongs to $L^{p'}(\mathbb{C}^n)$ whose norm is bounded by $\|\varphi\|_2\|\psi\|_2$. It is therefore natural to ask if an anlaogue of the above theorem is true for $1 < p \leq 2$. A close examination of the proof shows that Theorem 3.1 is true for $(L^p(\mathbb{C}^n), *_\lambda)$ with $1 \leq p \leq 2$.

Let $S_2$ be the algebra of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. In the special case when $T$ maps $L^2(\mathbb{C}^n)$ into $S_2$, the decomposition of $\mathcal{H}^\lambda$, obtained in the above result reduces to a finite sum.

**Corollary 3.2.** *Let $T : (L^2(\mathbb{C}^n), *_\lambda) \to S_2$ be a nonzero continuous homomorphism. Then there is a subspace $\mathcal{H}^\lambda$ of $L^2(\mathbb{R}^n)$ and a unitary representation $\rho_\lambda$ of $\mathbb{H}^n$ on $\mathcal{H}^\lambda$ such that*

$$T(f) = \int_{\mathbb{C}^n} f(z)\ \rho_\lambda(z,0)\ dz,\ \text{on } \mathcal{H}^\lambda,$$

*and there is a decomposition $L^2(\mathbb{R}^n) = \mathcal{H}^\lambda \bigoplus V^\lambda$, where*

$$V^\lambda := \{v \in L^2(\mathbb{R}^n) : (Tf)(v) = 0\ \forall\ f \in L^1(\mathbb{C}^n)\}.$$

*Moreover $\mathcal{H}^\lambda$ is the direct sum of finitely many subspaces of $L^2(\mathbb{R}^n)$.*

*Proof.* Here again we work with $\lambda = 1$ and drop all subscripts and superscripts involving $\lambda$. Proceeding as in the proof of the above theorem we obtain a sequence $\{\mathcal{H}_\alpha^j\}_{j=1,2,\ldots}$ of mutually orthogonal subspaces of $L^2(\mathbb{R}^n)$ and unitary representations $\rho_\alpha^j(z,t) = U_\alpha^j\ \pi_1(z,t)\ U_\alpha^{j*}$, of $\mathbb{H}^n$ on $\mathcal{H}_\alpha^j$ such that

$$T(f)|_{\mathcal{H}_\alpha^j} = \int_{\mathbb{C}^n} f(z)\ \rho_\alpha^j(z,0)\ dz,\ f \in L^2(\mathbb{C}^n),$$

i.e., $T(f) = U_\alpha^j\ W(f)\ U_\alpha^{j*}$ on $\mathcal{H}_\alpha^j$. Then

$$\|T(f)\|_{HS}^2 = \sum_{j=1}^\infty \sum_{\beta\in\mathbb{N}^n} \|T(f)v_{\alpha,\beta}^j\|_2^2.$$

Note that $\sum_{\beta\in\mathbb{N}^n}\ \|T(f)v_{\alpha,\beta}^j\|_2^2 = \|W(f)\|_{HS}^2$ is independent of $j$. Hence the above shows that $\mathcal{H}_\alpha^j \neq \{0\}$ only for finitely many $j$, and the decomposition takes the form $\mathcal{H}_\alpha = \bigoplus_{j=1}^m\ \mathcal{H}_\alpha^j$ for some $m \in \mathbb{N}$. □

## 4. Characterisation of the Fourier transform on $\mathbb{H}^n$

In this section we prove a characterisation of the group Fourier transform using Theorem 3.1 of the previous section.

Let $L^\infty(\mathbb{R}^*, \mathcal{B}(L^2(\mathbb{R}^n)), d\mu)$ denote the space of essentially bounded functions on $\mathbb{R}^*$, taking values in $\mathcal{B}(L^2(\mathbb{R}^n))$, where $\mathbb{R}^*$ is equipped with the measure $d\mu(\lambda) = (2\pi)^{-n-1}\, |\lambda|^n\, d\lambda$.

**Theorem 4.1.** *Let $T : L^1(\mathbb{H}^n) \to L^\infty(\mathbb{R}^*, S_2, d\mu)$ be a nonzero continuous linear map satisfying*

*(i) $T(f^*)(\lambda) = Tf(\lambda)^*$, for all $\lambda \in \mathbb{R}^*$, $f \in L^1(\mathbb{H}^n)$,*

*(ii) $T(f * g)(\lambda) = (Tf)(\lambda)\ (Tg)(\lambda)$, $\lambda \in \mathbb{R}^*$, $f, g \in L^1(\mathbb{H}^n)$, and*

*(iii) $T(R_{(0,t)}\ f)(\lambda) = (Tf)(\lambda)\ e^{-i\lambda t}$, $\lambda \in \mathbb{R}^*$, $f \in L^1(\mathbb{H}^n)$, $t \in \mathbb{R}$.*

*Then for each $\lambda \in A$, there is a decomposition $L^2(\mathbb{R}^n) = \mathcal{H}^\lambda \bigoplus V^\lambda$, and a unitary representation $\rho_\lambda$ of $\mathbb{H}^n$ on $\mathcal{H}^\lambda$ such that*

$$T(f)(\lambda) = \int_{\mathbb{H}^n} f(z,t)\ \rho_\lambda(z,t)\ dz\ dt, \quad on \ \ \mathcal{H}^\lambda,$$

*where $A := \{\lambda \in \mathbb{R}^* :\ (Tf)(\lambda) \neq 0 \ \ for \ \ some\ f \in L^1(\mathbb{H}^n)\}$.*

*Proof.* Let $T_\lambda(f) = (Tf)(\lambda)$, for $\lambda \in \mathbb{R}^*$, $f \in L^1(\mathbb{H}^n)$. For fixed $\varphi, \psi \in L^2(\mathbb{R}^n)$, the map defined on $L^1(\mathbb{H}^n)$ by $f \mapsto (T_\lambda(f)\ \varphi,\ \psi)$ satisfies

$$\begin{aligned} |\,(T_\lambda(f)\ \varphi,\ \psi)\,| &\leq \|T_\lambda(f)\|\ \|\varphi\|_2\ \|\psi\|_2 \\ &\leq C\ \|f\|_{L^1(\mathbb{H}^n)}\ \|\varphi\|_2\ \|\psi\|_2. \end{aligned}$$

i.e., the above map defines a continuous linear functional on $L^1(\mathbb{H}^n)$, and so there is $F_\lambda \in L^\infty(\mathbb{H}^n)$ such that

$$(T_\lambda(f)\varphi, \psi) = \int_{\mathbb{H}^n} f(z,t)\ F_\lambda((z,t);\varphi,\psi)\ dz\ dt,\ \ f \in L^1(\mathbb{H}^n).$$

Let $f \in L^1(\mathbb{H}^n)$ be of the form $f(z,t) = g(z)\ h(t)$.

Then

$$\begin{aligned} (T_\lambda(f)\varphi, \psi) &= \int_{\mathbb{H}^n} f(z,t)\ F_\lambda((z,t);\varphi,\psi)\ dz\ dt \\ &= \int_{\mathbb{R}} h(t)\ \left(\int_{\mathbb{C}^n} g(z)\ F_\lambda((z,t);\varphi,\psi)\ dz\right)\ dt \\ &= \int_{\mathbb{R}} h(t)\ \Phi_\lambda(t)\ dt. \end{aligned}$$

where $\Phi_\lambda(t) = \int_{\mathbb{C}^n} g(z)\ F_\lambda((z,t);\varphi,\psi)\ dz$. But (iii) gives

$$(T_\lambda(f)\ e^{-i\lambda s}\ \varphi, \psi) = \ (T_\lambda(R_{(0,s)}\ f)\varphi, \psi) = \int_{\mathbb{R}} h(t)\ \Phi_\lambda(t-s)\ dt$$

Thus we get $\Phi_\lambda(t-s) = e^{-i\lambda s}\ \Phi_\lambda(t)$ for all $s \in \mathbb{R}$, *a.e.* $t \in \mathbb{R}$. Let $\Psi$ be a Schwartz class function on $\mathbb{R}$ such that $\widehat{\Psi}(\lambda) \neq 0$. Then

$$\int_{\mathbb{R}} \Phi_\lambda(t-s)\ \Psi(s)\ ds = \int_{\mathbb{R}} e^{-i\lambda s}\ \Phi_\lambda(t)\ \Psi(s)\ ds = \widehat{\Psi}(\lambda)\ \Phi_\lambda(t).$$

As the left hand side is a smooth bounded function of $t$, so is $\Phi_\lambda$. Thus we get that $\Phi_\lambda(t-s) = e^{-i\lambda s}\ \Phi_\lambda(t)$ for all $s, t \in \mathbb{R}$. In particular $\Phi_\lambda(t) = e^{i\lambda t}\ \Phi_\lambda(0)$ for all $t \in \mathbb{R}$. Thus for every $g \in L^1(\mathbb{C}^n)$, the function

$$\int_{\mathbb{C}^n} g(z)\ F_\lambda((z,t);\varphi,\psi)\ dz$$

is continuous and satisfies

$$\int_{\mathbb{C}^n} g(z)\ F_\lambda((z,t);\varphi,\psi)\ dz = e^{i\lambda t} \int_{\mathbb{C}^n} g(z)\ F_\lambda((z,0);\varphi,\psi)\ dz$$

Taking

$$g(z) = |B_r(w)|^{-1}\ \chi_{B_r(w)}(z)$$

where $|B_r(w)|$ is the volume of the ball of radius $r$ centered at $w$ and and letting $r \to 0$, we see that for almost every $w \in \mathbb{C}^n$,

$$F_\lambda((w,t);\varphi,\psi) = e^{i\lambda t}\ F_\lambda((w,0);\varphi,\psi).$$

This leads to the equation

$$\begin{aligned}(T_\lambda(f)\varphi,\psi) &= \int_{\mathbb{H}^n} f(z,t)\ e^{i\lambda t}\ F_\lambda((z,0);\varphi,\psi)\ dz\ dt.\\ &= \int_{\mathbb{C}^n} f^\lambda(z)\ F_\lambda((z,0);\varphi,\psi)\ dz.\end{aligned}$$

Hence $T_\lambda(f)$ depends only on $f^\lambda$ and satisfies

$$\|T_\lambda(f)\| \leq C\ \|f^\lambda\|_{L^1(\mathbb{C}^n)}.$$

For a given $\lambda$, fix $\psi \in L^1(\mathbb{R})$ such that $\widehat{\psi}(-\lambda) = 1$ and define

$$S_\lambda(g) = T_\lambda(g(z)\ \psi(t)) = T_\lambda(f),\ f(z,t) = g(z)\ \psi(t).$$

Then it is clear that $\|S_\lambda(g)\| \leq C\ \|g\|_{L^1(\mathbb{C}^n)}$. Moreover, for $g_1, g_2 \in L^1(\mathbb{C}^n)$, with $f_j(z,t) = g_j(z)\ \psi(t),\ j = 1,2$, we have

$$(f_1 * f_2)^\lambda(z) = g_1 *_\lambda\ g_2(z) = g_1 *_\lambda g_2(z)\ \widehat{\psi}(-\lambda)$$

and hence $(f_1 * f_2)^\lambda = ((g_1 *_\lambda g_2)\ \psi)^\lambda$. Since $T_\lambda(f)$ depends only on $f^\lambda$, this gives

$$S_\lambda(g_1 *_\lambda g_2) = T_\lambda((g_1 *_\lambda g_2)(z)\ \psi(t)) = T_\lambda(f_1 * f_2),$$

and as $T_\lambda(f_1 * f_2) = T_\lambda(f_1)\ T_\lambda(f_2) = S_\lambda(g_1)\ S_\lambda(g_2)$, we get $S_\lambda(g_1 *_\lambda g_2) = S_\lambda(g_1)\ S_\lambda(g_2)$.

As the operator $S_\lambda$ satisfies the hypotheses of Theorem 3.1, for each $\lambda \in A$, there is a decomposition $L^2(\mathbb{R}^n) = \mathcal{H}^\lambda \bigoplus V^\lambda$ and a unitary representation $\rho_\lambda$ of $\mathbb{H}^n$ on $\mathcal{H}^\lambda$ such that

$$S_\lambda(f)|_{\mathcal{H}^\lambda} = \int_{\mathbb{C}^n} f(z)\ \rho_\lambda(z,0)\ dz,\ f \in L^1(\mathbb{C}^n).$$

In particular, for $f \in L^1(\mathbb{H}^n)$

$$\begin{aligned} S_\lambda(f^\lambda)|_{\mathcal{H}^\lambda} &= \int_{\mathbb{C}^n} f^\lambda(z)\ \rho_\lambda(z,0)\ dz, \\ &= \int_{\mathbb{H}^n} f(z,t)\ \rho_\lambda(z,t)\ dz\ dt. \end{aligned}$$

This gives for all $f \in L^1(\mathbb{H}^n)$ and $\lambda \in A$,

$$(Tf)(\lambda)|_{\mathcal{H}^\lambda} = \int_{\mathbb{H}^n} f(z,t)\ \rho_\lambda(z,t)\ dz\ dt.$$

□

In the above theorem, replacing hypothesis (iii) with a stronger assumption, we obtain the following

**Theorem 4.2.** *Let $T : L^1(\mathbb{H}^n) \to L^\infty(\mathbb{R}^*, \mathcal{B}(L^2(\mathbb{R}^n)), d\mu)$ be a nonzero continuous linear map satisfying*

*(i)* $T(f^*)(\lambda) = (Tf)(\lambda)^*,\ \lambda\ \in \mathbb{R}^*,\ f \in L^1(\mathbb{H}^n),$

*(ii)* $T(f * g)(\lambda) = (Tf)(\lambda)\ (Tg)(\lambda),\ \lambda \in \mathbb{R}^*,\ f, g \in L^1(\mathbb{H}^n),$ *and*

*(iii)* $T(R_{(z,t)}\ f)(\lambda) = (Tf)(\lambda)\ \pi_\lambda(z,t)^*,\ \lambda \in \mathbb{R}^*,\ f \in L^1(\mathbb{H}^n), (z,t) \in\ \mathbb{H}^n.$

*Then* $\quad (Tf)(\lambda) = \widehat{f}(\lambda),\ \lambda \in A,\ f \in L^1(\mathbb{H}^n),$
*where* $A := \{\lambda \in \mathbb{R}^* :\ (Tf)(\lambda) \neq 0 \quad \text{for} \quad \text{some} \quad f \in L^1(\mathbb{H}^n)\}.$

*Proof.* By the previous theorem, for each $\lambda \in A$, there is a decomposition $L^2(\mathbb{R}^n) = \mathcal{H}^\lambda \bigoplus V^\lambda$, and a unitary representation $\rho_\lambda$ of $\mathbb{H}^n$ on $\mathcal{H}^\lambda$ such that

$$T(f)(\lambda) = \int_{\mathbb{H}^n} f(z,t)\ \rho_\lambda(z,t)\ dz\ dt, \quad \text{on} \quad \mathcal{H}^\lambda. \tag{4.2}$$

Let $V^\lambda = \{v \in L^2(\mathbb{R}^n) :\ T_\lambda(f)(v) = 0\ \forall\ f \in L^1(\mathbb{H}^n)\}$. Let $v \in V^\lambda$. Then

$$\begin{aligned} T_\lambda(f)\ v &= 0 \quad \text{for} \quad \text{all} \quad f \in L^1(\mathbb{H}^n) \\ \text{gives} \qquad T_\lambda(f)\ \pi_\lambda(z,t)^* v &= 0 \text{ for } \ \text{all}\ f \in L^1(\mathbb{H}^n),\ \text{for}\ \ \text{all}\ (z,t) \in \mathbb{H}^n. \end{aligned}$$

This implies that $V^\lambda$ is invariant under $\pi_\lambda$. Now the irreducibility of $\pi_\lambda$ forces $V^\lambda = L^2(\mathbb{R}^n)$ or $V^\lambda = (0)$. If $\lambda \in A$, then $V^\lambda \neq L^2(\mathbb{R}^n)$ and so $V^\lambda = (0)$. But equation (4.2) gives $T(R_{(z,t)}\ f)(\lambda) = (Tf)(\lambda)\ \rho_\lambda(z,t)^*$. This, combined with (iii) of the hypothesis implies for each $f \in L^1(\mathbb{H}^n), \lambda \in A$ and $\varphi \in L^2(\mathbb{R}^n)$,

$$(Tf)(\lambda)\ \pi_\lambda(z,t)^*\ \varphi = (Tf)(\lambda)\ \rho_\lambda(z,t)^*\ \varphi,$$

which gives

$$(Tf)(\lambda)\ [(\pi_\lambda(z,t)^* - \rho_\lambda(z,t)^*)\ \ \varphi] = 0.$$

That is, the term in the square bracket is in $V^\lambda$ and so it is 0. Thus for all $\lambda \in A$ and $(z,t) \in \mathbb{H}^n,\ \rho_\lambda(z,t) = \pi_\lambda(z,t)$. This gives

$$(Tf)(\lambda) = \widehat{f}(\lambda),\ \lambda \in A,\ f \in L^1(\mathbb{H}^n).$$

□

When $T$ is an operator from $L^2(\mathbb{H}^n)$ onto $L^2(\mathbb{R}^*, S_2, d\mu)$, we obtain the following characterization.

**Theorem 4.3.** *Let $T : L^2(\mathbb{H}^n) \to L^2(\mathbb{R}^*, S_2, d\mu)$ be a nonzero surjective continuous linear map satisfying*

*(i) $T(f * g)(\lambda) = (Tf)(\lambda)\ (Tg)(\lambda), \lambda \in \mathbb{R}^*,\ f, g, f * g \in L^2(\mathbb{H}^n)$, and*

*(ii) $T(R_{(z,t)}\ f)(\lambda) = (Tf)(\lambda)\ \pi_\lambda(z,t)^*, \lambda \in \mathbb{R}^*, f \in L^2(\mathbb{H}^n), (z,t) \in \mathbb{H}^n$.*

*Then $T(f)(\lambda) = \widehat{f}(\lambda)$ for all $\lambda \in \mathbb{R}^*,\ f \in L^2(\mathbb{H}^n)$.*

*Proof.* Define $U : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ as $Uf = g$ if $Tf = \widehat{g}$, i.e., $U$ is such that $(Tf)(\lambda) = \widehat{(Uf)}(\lambda)$. Then $U$ is surjective, linear and continuous.

If $f_1, f_2, f_1 * f_2 \in L^2(\mathbb{H}^n)$ are such that $Uf_1 = g_1,\ Uf_2 = g_2$, and $U(f_1 * f_2) = g$, then $\widehat{g} = T(f_1 * f_2) = \widehat{g_1}\ \widehat{g_2} = (g_1 * g_2)\widehat{\ }$. This gives

$$U(f_1 * f_2) = U(f_1) * U(f_2) \quad \text{for all} \quad f_1, f_2, f_1 * f_2 \in L^2(\mathbb{H}^n). \tag{4.3}$$

Now, (ii) of the hypothesis and the similar property of the Fourier transform give

$$(U\ R_{(z,t)}\ f)\widehat{\ }(\lambda) = (Uf)\widehat{\ }(\lambda)\ \pi_\lambda(z,t)^* = (R_{(z,t)}\ Uf)\widehat{\ }(\lambda)$$

This gives $U\ R_{(z,t)}\ f = R_{(z,t)}\ Uf$ for all $f \in L^2(\mathbb{H}^n)$, i.e., $U$ is right-translation invariant. This implies from [11] that

$$\widehat{(Uf)}(\lambda) = m(\lambda)\ \widehat{f}(\lambda), \quad \text{for some} \quad m \in L^\infty(\mathbb{R}^*, \mathcal{B}(L^2(\mathbb{R}^n)), d\mu).$$

This gives

$$(U(f * g))\widehat{\ }(\lambda) = m(\lambda)\ \widehat{f}(\lambda)\ \widehat{g}(\lambda) = (Uf)\widehat{\ }(\lambda)\ \widehat{g}(\lambda). \tag{4.4}$$

But by (4.3),

$$(U(f * g))\widehat{\ }(\lambda) = (Uf * Ug)\widehat{\ }(\lambda) = \widehat{(Uf)}(\lambda)\ \widehat{(Ug)}(\lambda). \tag{4.5}$$

From (4.4), (4.5) and the surjectivity of $U$, we get

$$\widehat{h}(\lambda)\ ((\widehat{g}(\lambda) - \widehat{(Ug)}(\lambda)) = 0, \text{ for all } g, h \in L^2(\mathbb{H}^n).$$

This implies that the range $R((g - Ug)\widehat{\ }(\lambda))$ is contained in the kernel of $\widehat{h}(\lambda)$ for all $h \in L^2(\mathbb{H}^n)$, which forces $(g - Ug)\widehat{\ }(\lambda) = 0$ for every $\lambda \in L^2(\mathbb{H}^n)$. Hence $Ug = g$ for all $g \in L^2(\mathbb{H}^n)$, and thus

$$(Tf)(\lambda) = \widehat{f}(\lambda), \quad \text{for all} \quad \lambda \in \mathbb{R}^*,\ f \in L^2(\mathbb{H}^n).$$

□

**Acknowledgement.** This work is supported by J. C. Bose Fellowship from the Department of Science and Technology (DST). The first author is thankful to her advisor Prof. K. Parthasarathy for useful discussions and to the National Board for Higher Mathematics, India, for the financial support.

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[1] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai-600 005, India.

*E-mail address*: rlakshmilavanya@gmail.com

[2] Department of Mathematics, Indian Institute of Science, Bangalore-560 012, India.

*E-mail address*: veluma@math.iisc.ernet.in