Ann. Funct. Anal. 3 (2012), no. 1, 1–9
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

A NOTE ON THE EXPRESSION OF MODULUS OF CONVEXITY

XIAOLIN ZENG

Communicated by J. Esterle

ABSTRACT. It is well known that different kinds of expressions of modulus of convexity are essentially based on two geometrical propositions. For one of the propositions we first present a new proof by the Hahn–Banach theorem and intermediate value theorem, then give some corollaries to it which lead to some new expressions of modulus of convexity.

1. INTRODUCTION AND PRELIMINARIES

Recently, companying the deep and systematic development of random metric theory [3, 4, 5, 12], the geometry of random normed modules naturally began in [6]. A random version of modulus of convexity of a normed space, namely, modulus of random convexity, was introduced therein as a powerful tool for the study of the geometry of random normed modules. The rich and complicated stratification structure of a random normed module often causes many difficulties in the investigation into modulus of random convexity, which motivated us to analyze classical modulus of convexity in more details. In fact, this note is a by-product of such considerations.

The modulus of convexity of a normed space X was defined in [1] by

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = \|y\| = 1 \text{ and } \|x-y\| \geq \epsilon\right\} \ (\epsilon \in [0,2]).$$

We can, without loss of generality, suppose that the scalar field of every normed space occurring in this note is $\mathbb{R}$.

It is clear that $\delta(0) = 0$. Moreover, when $\dim(X) = 1$ we have $\delta(\epsilon) = 1$ for all $\epsilon \in (0, 2]$. When $\dim(X) \geq 2$, there have appeared in [2, 8] and [9] different kinds

Date: Received: 11 April 2011; Accepted: 18 November 2011.
2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 46A22, 54C30.
Key words and phrases. Hahn–Banach theorem, normed spaces, modulus of convexity.

of expressions of $\delta(\epsilon)$ as summarized in (1,1) and (1.2) below, respectively.

$$\begin{aligned}\delta(\epsilon) &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|=\|y\|=1 \text{ and } \|x-y\|=\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|=1, \|y\|\le 1 \text{ and } \|x-y\|=\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|\le 1, \|y\|\le 1 \text{ and } \|x-y\|=\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|=1, \|y\|\le 1 \text{ and } \|x-y\|\ge\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|\le 1, \|y\|\le 1 \text{ and } \|x-y\|\ge\epsilon\right\},\end{aligned} \tag{1.1}$$

for all $\epsilon\in[0,2]$ and

$$\begin{aligned}\delta(\epsilon) &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|=\|y\|=1 \text{ and } \|x-y\|>\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|=1, \|y\|\le 1 \text{ and } \|x-y\|>\epsilon\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X, \|x\|\le 1, \|y\|\le 1 \text{ and } \|x-y\|>\epsilon\right\},\end{aligned} \tag{1.2}$$

for all $\epsilon\in[0,2)$.

In the establishment of (1.1) and (1.2), Propositions 1.1 and 1.2 below have played important roles, respectively.

For the sake of conciseness, X always denotes a normed space with $\dim(X)\ge 2$, $S_X := \{x\in X \mid \|x\|=1\}$ the unit sphere of X and $U_X^\circ := \{x\in X \mid \|x\|<1\}$ the open unit ball of X in the sequel.

Proposition 1.1. [8] *For any two elements x_0 and y_0 in X such that $\|x_0\|=1$ and $\|y_0\|\le 1$ there exist $x_1,y_1\in X$ such that $\|x_1\|=\|y_1\|=1, x_1-y_1=x_0-y_0$ and $\|x_1+y_1\|\ge\|x_0+y_0\|$.*

Proposition 1.2. [9] *Suppose that $0<\epsilon<2$, x_0 and $y_0\in S_X$ such that $\|x_0-y_0\|=\epsilon$. Then there exist two sequences $\{x_n, n\in\mathbb{N}\}$ and $\{y_n, n\in\mathbb{N}\}$ in S_X such that $x_n\to x_0, y_n\to y_0$ as $n\to\infty$ and $\|x_n-y_n\|>\epsilon$ $(n\in\mathbb{N})$.*

There are two known approaches to the proofs of Propositions 1.1 and 1.2: one depends on the connectedness of the unit sphere and delicate geometrical arguments in $\mathbb{R}^2$ as demonstrated in [8, 9], and the other on the Hahn–Banach theorem and intermediate value theorem as given in [11] for the proof of Proposition 1.1. Either is feasible in classical analysis. But when we study the geometry of random normed modules we find that the difficulty caused by inherent stratification structure of a random normed module can hardly be overcome with the first method. However, as shown in [7], the second method is fairly suitable for the study in random normed modules. Thus it is meaningful to give a new proof of Proposition 1.2 by using the second method. It forms an interesting part of this note.

The other part of this note is devoted to the equivalent formulas for the modulus of convexity. Suppose that $0 < \epsilon < 2$, x_0 and $y_0 \in S_X$ such that $\|x_0 - y_0\| = \epsilon$. From Proposition 1.2 we can obtain the following corollaries, which together with some known expressions in (1.1) and (1.2) easily lead to Theorem 1.7 below.

Corollary 1.3. *There exist two sequences* $\{x_n, n \in \mathbb{N}\}$ *and* $\{y_n, n \in \mathbb{N}\}$ *in* U_X° *such that* $x_n \to x_0$, $y_n \to y_0$ *as* $n \to \infty$ *and* $\|x_n - y_n\| > \epsilon$ $(n \in \mathbb{N})$.

Corollary 1.4. *There exist two sequences* $\{x_n, n \in \mathbb{N}\}$ *and* $\{y_n, n \in \mathbb{N}\}$ *in* U_X° *such that* $x_n \to x_0$, $y_n \to y_0$ *as* $n \to \infty$ *and* $\|x_n - y_n\| = \epsilon$ $(n \in \mathbb{N})$.

Corollary 1.5. *There exist two sequences* $\{x'_n, n \in \mathbb{N}\}$ *in* U_X° *and* $\{y_n, n \in \mathbb{N}\}$ *in* S_X *such that* $x'_n \to x_0$, $y_n \to y_0$ *as* $n \to \infty$ *and* $\|x'_n - y_n\| > \epsilon$ $(n \in \mathbb{N})$.

Corollary 1.6. *There exist two sequences* $\{x''_n, n \in \mathbb{N}\}$ *in* U_X° *and* $\{y_n, n \in \mathbb{N}\}$ *in* S_X *such that* $x''_n \to x_0$, $y_n \to y_0$ *as* $n \to \infty$ *and* $\|x''_n - y_n\| = \epsilon$ $(n \in \mathbb{N})$.

Theorem 1.7.

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| < 1, \|y\| < 1 \text{ and } \|x-y\| > \epsilon\right\} \quad (1.3)$$

$$= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| < 1, \|y\| < 1 \text{ and } \|x-y\| = \epsilon\right\} \quad (1.4)$$

and

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x-y\| > \epsilon\right\} \quad (1.5)$$

$$= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x-y\| = \epsilon\right\}, \quad (1.6)$$

for all $\epsilon \in (0, 2)$.

Finally, as a consequence of Theorem 1.7, (1.1) and (1.2), we can express the modulus of convexity in the following way.

Corollary 1.8.

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\|(R_1)1, \|y\|(R_2)1 \text{ and } \|x-y\|(R_3)\epsilon\right\}, \quad (1.7)$$

where $(R_1), (R_2) \in \{\leq, <, =\}$ *and* $(R_3) \in \{\geq, >, =\}$. *When either* $<$ *or* $>$ *occurs, the corresponding expression is valid for any* $\epsilon \in (0, 2)$; *otherwise, for any* $\epsilon \in (0, 2]$.

With the concise proofs in this note, Corollaries from 1.3 to 1.6, Theorem 1.7 and Proposition 1.2 can all be generalized into random normed modules based on complicate analysis of stratification structure as in [7]. Since the work involves more, it will be discussed in a forthcoming paper.

The rest of this section consists of some simple facts in normed spaces which are useful in our new proof for Proposition 1.2.

Recall that the following proposition is deduced from the convexity of the unit ball of a normed space X.

Proposition 1.9. *Let x and y be two elements in S_X. If there exists some $\alpha_0 \in \mathbb{R}$ with $0 < \alpha_0 < 1$ such that $\|\alpha_0 \cdot x + (1-\alpha_0)\cdot y\| = 1$, then $\|\alpha \cdot x + (1-\alpha)\cdot y\| = 1$ for any $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$.*

Proposition 1.10 below shows that the dimension of the normed space discussed in Proposition 1.2 is not less than 2.

Proposition 1.10. *Let x and y be two elements in S_X. Then x and y are linearly independent if $0 < \|x-y\| < 2$.*

Proof. Otherwise, without loss of generality, suppose that there exists $a \in \mathbb{R}$ such that $y = ax$, then $\|y\| = |a|\|x\|$, i.e., $|a| = 1$, so that we have $\|x-y\| = 0$ or 2, which is a contradiction. □

Definition 1.11. ([9]) Let x and y be two elements in X. Then $\overline{xy} := \{\alpha x + (1-\alpha)y \mid \alpha \in \mathbb{R})\}$ is called the straight line through the points x and y.

Lemma 1.12. *Let $(X, \|\cdot\|)$ be a normed space and $x, y, u, v \in X$. If $\overline{xy} \cap \overline{uv} = \emptyset$ and $u = k_1x + l_1y$, $v = k_2x + l_2y$, where $k_i, l_i \in \mathbb{R}, i = 1, 2$. Then $k_1 + l_1 = k_2 + l_2$, which implies that $u - v = \xi(x-y)$, where $\xi = k_1 - k_2$.*

Proof. Since $\lambda u + (1-\lambda)v \notin \overline{xy}$ $(\lambda \in \mathbb{R})$, namely, $(\lambda k_1 + (1-\lambda)k_2)x + (\lambda l_1 + (1-\lambda)l_2)y \notin \overline{xy}$, we have

$$\lambda(k_1 + l_1) + (1-\lambda)(k_2 + l_2) \neq 1 \ (\lambda \in \mathbb{R})\,. \tag{1.8}$$

Suppose that $k_1 + l_1 \neq k_2 + l_2$, let $\lambda' = (k_1 + l_1 - k_2 - l_2)^{-1}(1 - k_2 - l_2)$, then it is easy to see that $\lambda'(k_1 + l_1) + (1-\lambda')(k_2 + l_2) = 1$, which contradicts to (1.8). Thus $k_1 + l_1 = k_2 + l_2$. □

Lemma 1.13. *Let x and y be two elements in X which are linearly independent. Then there exists $x^* \in X^*$ such that $x^*(x) = 0$ and $x^*(y) = 1$.*

Proof. Let $M = \{kx + ly \mid k, l \in \mathbb{R}\}$, then M is a two-dimensional subspace of X. Define a mapping $f : M \to \mathbb{R}$ by

$$f(kx + ly) = l \ (k, l \in \mathbb{R}).$$

Since x and y are linearly independent, f is a linear functional, so that f is bounded. Obviously, $f(x) = 0$ and $f(y) = 1$, then by Hahn–Banach theorem there exists $x^* \in X^*$ such that $x^*(x) = 0$ and $x^*(y) = 1$. □

The geometrical meaning of Lemma 1.14 below is very clear. Meanwhile, its proof is, in some sense, interesting.

Lemma 1.14. *Let $(X, \|\cdot\|)$ be a real normed space and $x, y \in S_X$. If $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta < 0$, $z = \alpha x + (1-\alpha)y$ and $w = \beta x + (1-\beta)y$, then $\|z\| \geq \|w\| \geq 1$.*

Proof. Clearly, $\|w\| \geq \|(1-\beta)y\| - \|\beta x\| = (1-\beta) - (-\beta) = 1$. Applying Hahn–Banach theorem to w we can obtain an $f \in X^*$ such that $f(w) = \|w\|$ and $\|f\|^* = 1$. Thus $\beta f(x) + (1-\beta)f(y) = \|w\| \geq 1$, namely, $\beta(f(x) - f(y)) + f(y) \geq 1$. Since $f(y) \leq \|f\|^*\|y\| = 1$, then $f(x) \leq f(y)$. Finally, from $\|z\| - \|w\| = \|z\| - f(w) \geq f(z) - f(w) = (\alpha - \beta)(f(x) - f(y)) \geq 0$ it follows that $\|z\| \geq \|w\|$. □

2. Proofs

The new proof of Proposition 1.2 together with the proofs of Corollaries from 1.3 to 1.6, Theorem 1.7 and Corollary 1.8 are presented in this section.

We first give a new proof of Proposition 1.2 with the above preparations.

Proof of Proposition 1.2. Since x_0 and y_0 are linearly independent, so are $x_0 - y_0$ and y_0. By Lemma 1.13, there exists $x^* \in X^*$ such that $x^*(x_0 - y_0) = 0$ and $x^*(y_0) = 1$, namely

$$x^*(x_0) = x^*(y_0) = 1. \tag{2.1}$$

Let $z_n = (1 + \frac{1}{n})x_0 - \frac{1}{n}y_0$, then

$$x^*(z_n) = 1 \ (n \in \mathbb{N}), \tag{2.2}$$

$$z_n \to x_0 \text{ as } n \to \infty. \tag{2.3}$$

By Lemma 1.14 we have

$$\|z_n\| \geq \|z_m\| \geq 1 \ (n, m \in \mathbb{N} \text{ and } n \leq m). \tag{2.4}$$

Let $\delta_n = \|z_n\|^{-1}$ $(n \in \mathbb{N})$, then $0 < \delta_n \leq 1$ and $\delta_n^{-1} = \|z_n\|$ $(n \in \mathbb{N})$. Combining (2.3) and (2.4) we can see that $\delta_n \nearrow 1$ as $n \to \infty$.

Define a real function f as follows:

$$f(\lambda) = \|\lambda x_0 + (1 - \lambda)y_0\| \ (\lambda \in \mathbb{R}).$$

The continuity of f is obvious.

When $\lambda < 0$, $f(\lambda) = \|\lambda(x_0 - y_0) + y_0\| \geq -\lambda\epsilon - 1$, certainly, there exists $\Lambda \in \mathbb{R}$ such that $\Lambda \leq 0$ and $f(\Lambda) > \delta_1^{-1} = \|z_1\|$.

Notice that $f(0) = 1 \leq \delta_1^{-1} < f(\Lambda)$, by the intermediate value theorem there exists $\lambda_1 \in \mathbb{R}$ such that $\Lambda \leq \lambda_1 \leq 0$ and $f(\lambda_1) = \delta_1^{-1}$.

Then notice that $f(0) \leq \delta_2^{-1} \leq \delta_1^{-1} = f(\lambda_1)$, by the intermediate value theorem there exists $\lambda_2 \in \mathbb{R}$ such that $\lambda_1 \leq \lambda_2 \leq 0$ and $f(\lambda_2) = \delta_2^{-1}$.

Repeating the process, we can obtain a nondecreasing sequence $\{\lambda_n, n \in \mathbb{N}\}$ in $\mathbb{R}$ such that $\lambda_n \leq 0$ and $f(\lambda_n) = \delta_n^{-1}$ $(n \in \mathbb{N})$. Observe that

$$f(\lambda_n) = \delta_n^{-1} = \|z_n\| \searrow 1 \tag{2.5}$$

as $n \to \infty$. Let $\lambda' = \lim_{k\to\infty} \lambda_k$, then $\lambda' \leq 0$. By the continuity of f we can see that $f(\lambda_k) \to f(\lambda')(k \to \infty)$, which together with (2.5) implies that $f(\lambda') = 1$. The following proof is divided into three cases.

(Case 1). When there exists some $n_0 \in \mathbb{N}$ such that $\|z_{n_0}\| = 1$, then $\|z_n\| = 1$ $(n \geq n_0)$. Let

$$\begin{aligned} x_n &= z_{n+n_0}, \\ y_n &= y_0 \ (n \in \mathbb{N}). \end{aligned}$$

Then it is easy to check that $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ are just desired.

(Case 2). When $\|z_n\| > 1$ $(n \in \mathbb{N})$ and $\lambda' < 0$, let

$$\begin{aligned} x_n &= x_0, \\ y_n &= \frac{\lambda'}{n+1}(x_0 - y_0) + y_0 \ (n \in \mathbb{N}). \end{aligned}$$

Observe that $0 > \frac{\lambda'}{n+1} > \lambda'$ and $f(\lambda') = \|\lambda' x_0 + (1-\lambda')y_0\| = 1$, by Lemma 1.14 we have $\|y_n\| = 1$ $(n \in \mathbb{N})$. Consequently, it is direct to check that $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ are desired.

(Case 3). When $\|z_n\| > 1$ $(n \in \mathbb{N})$ and $\lambda' = 0$, let

$$\begin{aligned} x_n &= \delta_n z_n, \\ y_n &= \delta_n(\lambda_n x_0 + (1-\lambda_n)y_0) \ (n \in \mathbb{N}). \end{aligned}$$

It is a straightforward verification that $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$ and $\|x_n\| = \|y_n\| = 1$ $(n \in \mathbb{N})$. We only need to show that $\|x_n - y_n\| > \epsilon$ $(n \in \mathbb{N})$.

Notice that

$$x_n - y_n = \delta_n(1 + \frac{1}{n} - \lambda_n)(x_0 - y_0), \tag{2.6}$$

which certainly implies that

$$\|x_n - y_n\| > 0. \tag{2.7}$$

We claim that $\overline{x_0(-y_0)} \cap \overline{x_n y_n} \neq \emptyset$. Otherwise, by Lemma 1.12 there exists $\xi_n \in \mathbb{R}$ such that $x_n - y_n = \xi_n(x_0 + y_0)$ for each $n \in \mathbb{N}$. Since x_0 and y_0 are linearly independent, we have

$$\begin{cases} \delta_n(1 + \frac{1}{n} - \lambda_n) = \xi_n, \\ -\delta_n(1 + \frac{1}{n} - \lambda_n) = \xi_n, \end{cases}$$

which leads to $\xi_n = 0$, so that $x_n - y_n = \theta$, a contradition to (2.7).

Similarly, $\overline{y_0(-x_0)} \cap \overline{x_n y_n} \neq \emptyset$.

For each $n \in \mathbb{N}$, take $x_n' \in \overline{x_0(-y_0)} \cap \overline{x_n y_n}$ and $y_n' \in \overline{y_0(-x_0)} \cap \overline{x_n y_n}$, and suppose that

$$x_n' = a_n x_n + (1-a_n)y_n = b_n x_0 + (1-b_n)(-y_0), \tag{2.8}$$

$$y_n' = c_n x_n + (1-c_n)y_n = d_n y_0 + (1-d_n)(-x_0), \tag{2.9}$$

where $a_n, b_n, c_n, d_n \in \mathbb{R}$.

Consider (2.8). Recalling (2.1), (2.2) and letting x^* act on each side of (2.8), by the definition of x_n and y_n we have $x^*(x_n') = \delta_n = 2b_n - 1$, thus $b_n = \frac{\delta_n + 1}{2}$, further, $\|x_n'\| \leq 1$. We will show that $0 < a_n < 1$ as follows.

If $a_n \leq 0$ (or $a_n \geq 1$) for some $n \in \mathbb{N}$, then $\|x_n'\| \geq 1$, so that $\|x_n'\| = 1$. Since $0 < \delta_n < 1$, then $0 < b_n < 1$. By Proposition 1.9 we derive that $\|\frac{x_0 - y_0}{2}\| = 1$, which is a contradiction to $\|x_0 - y_0\| = \epsilon$. Therefore, $0 < a_n < 1$ $(n \in \mathbb{N})$. Similarly, we have $0 < c_n < 1$ $(n \in \mathbb{N})$ by (2.9).

Noticing (2.6), (2.9) − (2.8) yields

$$y_n' - x_n' = (c_n - a_n)(x_n - y_n) = (c_n - a_n)\delta_n(1 + \frac{1}{n} - \lambda_n)(x_0 - y_0), \tag{2.10}$$

and

$$y_n' - x_n' = (d_n - b_n + 1)y_0 + (d_n - b_n - 1)x_0, \tag{2.11}$$

which together with the linear independence of x_0 and y_0 imply that

$$\begin{cases} (d_n - b_n + 1) + (c_n - a_n)\delta_n(1 + \frac{1}{n} - \lambda_n) = 0, \\ (d_n - b_n - 1) - (c_n - a_n)\delta_n(1 + \frac{1}{n} - \lambda_n) = 0, \end{cases}$$

for each $n \in \mathbb{N}$. Thus $d_n = b_n$ $(n \in \mathbb{N})$, so that $y'_n - x'_n = y_0 - x_0$ by (2.11). Then by (2.10) $y_0 - x_0 = (c_n - a_n)(x_n - y_n)$, so that $\epsilon = \|y_0 - x_0\| = |c_n - a_n| \|x_n - y_n\|$, which yields $|c_n - a_n| > 0$ $(n \in \mathbb{N})$.

Finally, by $0 < |c_n - a_n| < 1$, we have

$$\|x_n - y_n\| = \frac{\epsilon}{|c_n - a_n|} > \epsilon \ (n \in \mathbb{N}).$$

□

Remark 2.1. The above proof for Proposition 1.2 as well as the one in [11] for Proposition 1.1 relies on the Hahn–Banach theorem and intermediate value theorem in a considerably different way. Such technique of transforming geometrical questions of normed spaces into the ones of continuous linear functionals has their root in the earlier literature [10].

Now, let us give the respective proofs of the four corollaries to Proposition 1.2, even if they are not very difficult.

Proof of Corollary 1.3. Let $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences obtained for x_0 and y_0 as in Proposition 1.2. Take arbitrarily c_n such that $1 < c_n < \frac{\|x_n - y_n\|}{\epsilon}$ for each $n \in \mathbb{N}$, and let $x'_n = \frac{x_n}{c_n}$ and $y'_n = \frac{y_n}{c_n}$ for each $n \in \mathbb{N}$. Then it is easy to check that $\{x'_n, n \in \mathbb{N}\}$ and $\{y'_n, n \in \mathbb{N}\}$ are just required sequences. □

Proof of Corollary 1.4. Let $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences obtained for x_0 and y_0 as in Proposition 1.2. Take $d_n = \frac{\|x_n - y_n\|}{\epsilon}$ for each $n \in \mathbb{N}$, and let $x''_n = \frac{x_n}{d_n}$ and $y''_n = \frac{y_n}{d_n}$ for each $n \in \mathbb{N}$. Then it is easy to check that $\{x''_n, n \in \mathbb{N}\}$ and $\{y''_n, n \in \mathbb{N}\}$ are just required sequences. □

Proof of Corollary 1.5. Let $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences obtained for x_0 and y_0 as in Proposition 1.2. Take a real number $\alpha_n \in (0, 1)$ such that $1 - \alpha_n < \|x_n - y_n\| - \epsilon$ for each $n \in \mathbb{N}$, then $\alpha_n \to 1$ as $n \to \infty$. Let $x'_n = \alpha_n x_n$ $(n \in \mathbb{N})$. It is easy to see that $\|y_n - x'_n\| = \|y_n - x_n + (1 - \alpha_n)x_n\| \geq \|y_n - x_n\| - (1 - \alpha_n) > \epsilon, \|x'_n\| < 1, \|y_n\| = 1$ $(n \in \mathbb{N}), x'_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$. □

Proof of Corollary 1.6. Let $\{x'_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences obtained for x_0 and y_0 as in Corollary 1.5. Let $\lambda_n = \frac{\epsilon}{\|x'_n - y_n\|}$ $(n \in \mathbb{N})$, and $x''_n = \lambda_n x'_n + (1 - \lambda_n)y_n$. Then it is easy to see that $\lambda_n \to 1$, so that $x''_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$. It is a straightforward verification that $\|x''_n\| < 1, \|y_n\| = 1$ and $\|x''_n - y_n\| = \epsilon$ $(n \in \mathbb{N})$. □

Then, we can prove Theorem 1.7.

Proof of Theorem 1.7. In order to prove (1.3), by the third expression in (1.2) we only need to check that

$$\delta(\epsilon) \geq \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in X, \|x\| < 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}.$$

For any $x_0, y_0 \in S_X$ such that $\|x_0 - y_0\| = \epsilon$, let $\{x_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences as obtained in Corollary 1.3. Then

$$\begin{aligned} 1 - \left\| \frac{x_0 + y_0}{2} \right\| &= \lim_{n\to\infty} \left(1 - \left\| \frac{x_n + y_n}{2} \right\| \right) \\ &\geq \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| < 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}. \end{aligned}$$

Consequently, the desired inequality can be verified by the first expression in (1.1).

(1.4) is justified in the same way by Corollary 1.4 together with the first and third expressions in (1.1).

Now, let us prove (1.5). By the last expression in (1.1) it is clear that

$$\delta(\epsilon) \leq \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}.$$

Thus we only need to check that

$$\delta(\epsilon) \geq \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}.$$

For any $x_0, y_0 \in S_X$ such that $\|x_0 - y_0\| = \epsilon$, let $\{x'_n, n \in \mathbb{N}\}$ and $\{y_n, n \in \mathbb{N}\}$ be the two sequences as obtained in Corollary 1.5. Then we have

$$\begin{aligned} 1 - \left\| \frac{x_0 + y_0}{2} \right\| &= \lim_{n\to\infty} \left(1 - \left\| \frac{y_n + x'_n}{2} \right\| \right) \\ &\geq \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}. \end{aligned}$$

Combining the first expression in (1.1) we complete the verification of (1.5). Similarly, (1.6) can be verified by Corollary 1.6 and the first expression in (1.1).□

Finally, we conclude this note with the proof of Corollary 1.8.

Proof of Corollary 1.8. Combining (1.3) and the last expression in (1.1) we have

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| < 1, \|y\| < 1 \text{ and } \|x - y\| \geq \epsilon \right\}, \quad (2.12)$$

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| \leq 1, \|y\| < 1 \text{ and } \|x - y\| \geq \epsilon \right\}, \quad (2.13)$$

and

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| \leq 1, \|y\| < 1 \text{ and } \|x - y\| > \epsilon \right\}. \quad (2.14)$$

From (1.4) and the third expression in (1.1) it follows that

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| \leq 1, \|y\| < 1 \text{ and } \|x - y\| = \epsilon \right\}. \quad (2.15)$$

From (1.6) and the last expression in (1.1) it follows that

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = 1, \|y\| < 1 \text{ and } \|x-y\| \geq \epsilon\right\}. \quad (2.16)$$

Summarizing the expressions in (1.1) and (1.2), from (1.3) to (1.6), and from (2.12) to (2.16), we obtain (1.7). □

Acknowledgement. The author would like to thank the referee and Professor Guo TieXin for many invaluable suggestions which considerably improves the readability of the paper.

References

1. M.M. Day, *Uniform convexity in factor and conjugate spaces*, Ann. of Math. **45** (1944), 375–385.
2. K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, 28, Cambridge university press, Cambridge, 1990.
3. T.X. Guo, *Relations between some basic results derived from two kinds of topologies for a random locally convex module*, J. Funct. Anal., **258** (2010), 3024–3047.
4. T.X. Guo, *Recent progress in random metric theory and its applications to conditional risk measures*, Sci China Ser A, **54** (2011), 633–660.
5. T.X. Guo and G. Shi, *The algebraic structure of finitely generated $L^0(\mathcal{F}, \mathbb{K})$-modules and the Helly theorem in random normed modules*, J. Math. Anal. Appl. **381** (2011), 833–842.
6. T.X. Guo and X.L. Zeng, *Random strict convexity and random uniform convexity in random normed modules*, Nonlinear Anal. **73** (2010), 1239–1263.
7. T.X. Guo and X.L. Zeng, *An $L^0(\mathcal{F}, \mathbb{R})$-valued function's intermediate value theorem and its applications to random uniform convexity*, Acta Math. Sin. (Engl. Ser.) (to appear), DOI: 10.1007/s10114-011-0367-2.
8. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin, 1979.
9. R.E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.
10. A.F. Ruston, *A note on convexity in Banach spaces*, Math. Proc. Cambridge Philos. **45** (1949), 157–159.
11. C.S. Yang and H.L. Zuo, *An application of the Hahn–Banach theorem to modulus of convexity*, Acta Math. Sci. Ser. A Chin. Ed. **21** (2001), no. 1, 133–137.
12. S.E. Zhao and T.X. Guo, *The random subreflevivity of complete random normed modules*, Internat. J. Math. (to appear), DOI: 10.1142/S0129167X12500474.

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing, 400067, PR. China.

E-mail address: xlinzeng@163.com