Ann. Funct. Anal. 4 (2013), no. 2, 183–194
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

CONVEXITY, SUBADDITIVITY AND GENERALIZED JENSEN'S INEQUALITY

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Communicated by C. P. Niculescu

ABSTRACT. In this paper we extend some theorems published lately on the relationship between convexity/concavity, and subadditivity/superadditivity. We also generalize inequalities of compound functions that refine Minkowski inequality.

1. INTRODUCTION

In recent publications the relationships between convexity/concavity and subadditivity/superadditivity are discussed.

In this paper we use results that appeared in [1, 2, 3, 7, 8] to extend some theorems published about this subject in [4, 5, 6]. We also use the classical Jensen's inequality to generalize [9].

We start with some definitions needed in the sequel.

Definition 1.1. A convex cone is a subset $\mathbf{C}$ of a linear space X that satisfies
(i) $x, y \in \mathbf{C} \Longrightarrow x + y \in \mathbf{C}$,
(ii) $x \in \mathbf{C}, \quad \alpha > 0 \Longrightarrow \alpha x \in \mathbf{C}$.

Let $\mathbf{C}$ be a convex cone in a linear space. A functional $a : \mathbf{C} \to \mathbb{R}$ is called subadditive (superadditive, resp.) on $\mathbf{C}$ if $a(x) + a(y) \geq (\leq, \text{ resp.})\, a(x + y)$ for any $x, y \in \mathbf{C}$.

Definition 1.2. Let $f_i : I_i \to \mathbb{R}_+$, $I_i \subseteq (0, \infty)$, $i = 1, \cdots, m - 1$.

Date: Received: 4 February 2013; Accepted: 8 March 2013.
2010 *Mathematics Subject Classification.* Primary: 47A63; Secondary: 47A07, 26D15.
Key words and phrases. Convexity, subadditivity, Jensen's inequality, Hölder's inequality, Minkowski's inequality.

Denote for $r, s \in \{1, \cdots, m-1\}$ with $r \leq s$,

$$\begin{aligned} & G_{r,s}\left(x_r, x_{r+1}, \cdots, x_{s+1}\right) \\ & \quad = x_r f_r\left(\frac{x_{r+1}}{x_r} f_{r+1}\left(\frac{x_{r+2}}{x_{r+1}}, \cdots, f_{s-1}\left(\frac{x_s}{x_{s-1}} f_s\left(\frac{x_{s+1}}{x_s}\right)\right)\right)\right), \quad (1.1) \\ & G_{s+1,s}(x) = x, \end{aligned}$$

where

$$\frac{1}{x_i} G_{i+1,m-1}\left(x_{i+1}, \cdots, x_m\right) \in I_i, \qquad i = 1, \cdots, m-1.$$

In particular

$$G_{1,m-1}(\mathbf{x}) = G_{1,m-1}\left(x_1, x_2, \cdots, x_m\right) = x_1 f_1\left(\frac{x_2}{x_1} f_2\left(\frac{x_3}{x_2} \cdots f_{m-1}\left(\frac{x_m}{x_{m-1}}\right)\right)\right). \tag{1.2}$$

Definition 1.3. We say that a set of convex and concave functions f_i, $i = 1, \cdots, m-1$ satisfies the **Monotonicity Condition** (MC) if all the pairs of functions $\left(f_k, f_{k+1}\right)$, $k = 1, \cdots, m-2$ satisfy the following:

(i) when both functions f_k and f_{k+1} are either convex or concave, then f_k is increasing.

(ii) when either f_k is convex and f_{k+1} is concave or f_k is concave and f_{k+1} is convex, then f_k is decreasing.

In [8, theorems 3 and 4] the following assertions are proved:

If $f : \mathbb{R}_+^k \to \mathbb{R}_+$ *is bounded in the neighbourhood of* 0, $f(0) = 0$, *and* f *satisfies* $af(x) + bf(y) \geq f(ax + by)$, $x, y \in \mathbb{R}_+^k$, *where* a *and* b are positive real numbers, *then, for each* $i = 1, \cdots, k$, $h_i : \mathbb{R}_+^{k-1} \to \mathbb{R}_+$ *defined by*

$$h_i\left(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k\right) = f\left(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_k\right)$$

is convex, and

$$f\left(x_1, \cdots, x_k\right) = h_i\left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_k}{x_i}\right) x_i, \ x_i > 0.$$

Conversely, if $h : \mathbb{R}_+^{k-1} \to \mathbb{R}_+$ *is convex, then for each* $i = 1, \cdots, k$ *the function* $f : \mathbb{R}_+^k \to \mathbb{R}_+$ *given by* $f\left(x_1, \cdots, x_k\right) = h\left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_k}{x_i}\right) x_i$, $x_i > 0$ *satisfies* $af(x) + bf(y) \geq f(ax + by)$, $x, y \in \mathbb{R}_+^k$, *where* a *and* b *are positive real numbers.*

From these results and also independently for somewhat different conditions in [2], the following theorem is obtained which is crucial for our ivestigation:

Theorem 1.4. [2, Theorem 1] *Let* $f_i : I_i \to \mathbb{R}_+$, $I_i \subseteq (0, \infty)$, $i = 1, \cdots, m-1$ *be a set of functions with the MC property.*

a) Let p *and* q *be positive real numbers. If* f_r *is a concave function, then for* $\mathbf{x} = \left(x_r, \cdots, x_{s+1}\right)$ *and* $\mathbf{y} = \left(y_r, \cdots, y_{s+1}\right)$ *we have for any* s, $s \in \{r, \cdots, m-1\}$

$$pG_{r,s}(\mathbf{x}) + qG_{r,s}(\mathbf{y}) \leq G_{r,s}(p\mathbf{x} + q\mathbf{y}).$$

If f_r *is a convex function then the reversed inequality holds.*

b) *If f_r is a concave function, then $G_{r,s}$ is a superadditive and concave function. If f_r is a convex function, then $G_{r,s}$ is a subadditive and convex.*

Remark 1.5. Theorem 1.4 still holds when $f_1 : I_1 \to \mathbb{R}$ instead of $f_1 : I_1 \to \mathbb{R}_+$ as explained in [3, Remark 1].

The main results of this paper are presented and proved in Section 2. There we start with the proof of Lemma 2.1 which deals with inequalities related to compound monotone functions.

The results of Theorem 2.4 in which we get inequalities involving convex/concave functions and subadditive/superadditive functionals that extend results of [1, 4, 5] and [6], are obtained by using Lemma 2.1 and Theorem 1.4.

In Section 3 some examples of Theorem 2.4 and Lemma 2.1 are demonstrated.

2. Main Results

In the following Theorem 2.4 we present and prove inequalities related to $F\left(G_{1,m-1}\left(\mathbf{a}\right)\right)$, $\mathbf{a}=\left(a_1,\cdots,a_m\right)$. These inequalities involve monotone convex and concave functions f_i, $i=1,\cdots,m-1$ that compose $G_{1,m-1}$ as defined in (1.2), subadditive and superadditive functionals a_i, $i=1,\cdots,m$ that replace the x_i-th in (1.2), in addition to a subadditive/superadditive monotone function F. These inequalities are associated with generalized Jensen and Hölder inequalities presented in [1, 7, 2, 3].

To prove Theorem 2.4 below we first prove the following lemma

Lemma 2.1. *Let functions $f_i : I_i \to \mathbb{R}_+$, $I_i \subseteq (0,\infty)$, $i=1,\cdots,m-1$ be such that*

a) *the functions f_{k_j}, $j=1,\cdots,l$, $0\le k_l\le m-1$ are decreasing on I_{k_j} and f_i, $i=1,\cdots,m-1$, $i\neq k_j$, $j=1,\cdots,l$ are increasing on I_i.*

b) *for A_i denoted as*

$$A_i = G_{i+1,m-1}\left(x_{i+1},\cdots,x_m\right), \qquad i=1,\cdots,m-2$$

the

$$range\left(\frac{1}{x_i}A_i\right)\subseteq I_i, \qquad i=1,\cdots,m-2$$

is satisfied.

c) $(-1)^{d_i} g_i(x)$ *is increasing when* $\frac{A_i}{x}\subseteq I_i$ *for fixed integers d_i where*

$$g_i(x) = x f_i\left(\frac{A_i}{x}\right), \qquad i=1,\cdots,m-1. \tag{2.1}$$

If

$$(-1)^{d_i+j}\left(z_i-y_i\right)\ge 0, \qquad i=k_j+1,\cdots,k_{j+1}, \qquad j=0,\cdots,l,$$
$$k_0=0, \qquad k_{l+1}=m-1, \qquad (-1)^l\left(z_m-y_m\right)\ge 0,$$

then

$$G_{1,m-1}\left(z_1,\cdots,z_m\right)\ge G_{1,m-1}\left(y_1,\cdots,y_m\right). \tag{2.2}$$

In particular, when $m=2$ and f is increasing (decreasing) $(-1)^d g$ is increasing, if $(-1)^d\left(z_1-y_1\right)\ge 0$, $z_2-y_2\ge(\le)\,0$, *then*

$$z_1 f\left(\frac{z_2}{z_1}\right) \geq y_1 f\left(\frac{y_2}{y_1}\right).$$

Proof. Let us replace in $G_{1,m-1}(z_1,..,z_r,..,z_m)$ a specific term z_r with y_r, $1 \leq r \leq m-1$, for which there is a specific j_0 such that $k_{j_0}+1 \leq r \leq k_{j_0+1}$where k_{j_0} is the j_0-th decreasing f_i.

According to (1.1), (1.2) and (2.1)

$$\begin{aligned} G_{1,m-1}(z_1,\cdots,z_r,\cdots,z_m) &= G_{1,r-1}(z_1,z_2,\cdots,z_{r-1},G_{r,m-1}(z_r,\cdots,z_m)) \\ &= G_{1,r-1}\left(z_1,\cdots,z_{r-1},z_r f_r\left(\frac{A_r}{z_r}\right)\right) \\ &= G_{1,r-1}(z_1,\cdots,z_{r-1},g_r(z_r)). \end{aligned}$$

Therefore, if the compound function $f_1 \circ f_2 \circ \cdots \circ f_{r-1} \circ g_r$ is increasing and $z_r - y_r \geq 0$ we get that

$$G_{1,m-1}(z_1,\cdots,z_r,\cdots,z_m) \geq G_{1,m-1}(z_1,\cdots,y_r,\cdots,z_m). \tag{2.3}$$

If the compound function $f_1 \circ f_2 \circ \cdots \circ f_{r-1} \circ g_r$ is decreasing and $z_r - y_r \leq 0$ then inequality (2.3) holds too.

Both these possibilities are combined in the condition $(-1)^{d_r+j_0}(z_r - y_r) \geq 0$, when $k_{j_0}+1 \leq r \leq k_{j_0+1}$, where $f_{k_{j_0}}$, is the j_0-th decreasing f_i . For $r = m$, a similar reasoning leads to inequality (2.3) when $(-1)^l(z_m - y_m) \geq 0$.

Going over all $r = 1,\cdots,m$ we get inequality (2.2). □

Remark 2.2. From Remark 1.5 it is obvious that Lemma 2.1 holds if we relax the condition on the range of the function f_1, so that $f_1 : I_1 \to \mathbb{R}$, $I_1 \subseteq (0,\infty)$.

Corollary 2.3. *Let $a(x)$, $b(x)$ and $F(x)$ be positive concave functions, $F(x)$ and $xF\left(\frac{1}{x}\right)$ be increasing functions on $(0,\infty)$, and let $y(x)$, $z(x)$, $u(x)$, $v(x)$ be positive functions. Then*

$$\begin{aligned} &\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) F\left(\frac{u(x)\, b\left(\frac{v(x)}{u(x)}\right)}{y(x)\, a\left(\frac{z(x)}{y(x)}\right)}\right) dx \\ \leq\ &\left(\int y(x)\,dx\right) a\left(\frac{\int z(x)\,dx}{\int y(x)\,dx}\right) F\left(\frac{\left(\int u(x)\,dx\right) b\left(\frac{\int v(x)dx}{\int u(x)dx}\right)}{\left(\int y(x)\,dx\right) a\left(\frac{\int z(x)dx}{\int y(x)dx}\right)}\right). \end{aligned} \tag{2.4}$$

Indeed, from the concavity of F we get by Jensen's inequality that

$$\begin{aligned} &\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) F\left(\frac{u(x)\, b\left(\frac{v(x)}{u(x)}\right)}{y(x)\, a\left(\frac{z(x)}{y(x)}\right)}\right) dx \\ \leq\ &\left(\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx\right) F\left(\frac{\int u(x)\, b\left(\frac{v(x)}{u(x)}\right) dx}{\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx}\right). \end{aligned} \tag{2.5}$$

By choosing

$$y_1 = \int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx, \qquad y_2 = \int u(x)\, b\left(\frac{v(x)}{u(x)}\right) dx$$

and

$$z_1 = \left(\int y(x)\, dx\right) a\left(\frac{\int z(x)\, dx}{\int y(x)\, dx}\right), \qquad z_2 = \left(\int u(x)\, dx\right) b\left(\frac{\int v(x)\, dx}{\int u(x)\, dx}\right),$$

we get that

$$y_1 = \int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx \le \left(\int y(x)\, dx\right) a\left(\frac{\int z(x)\, dx}{\int y(x)\, dx}\right) = z_1$$

and

$$y_2 = \int u(x)\, b\left(\frac{v(x)}{u(x)}\right) dx \le \left(\int u(x)\, dx\right) b\left(\frac{\int v(x)\, dx}{\int u(x)\, dx}\right) = z_2$$

hold.

Now, applying Lemma 2.1

$$\left(\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx\right) F\left(\frac{\int u(x)\, b\left(\frac{v(x)}{u(x)}\right) dx}{\int y(x)\, a\left(\frac{z(x)}{y(x)}\right) dx}\right)$$
$$\le \left(\int y(x)\, dx\right) a\left(\frac{\int z(x)\, dx}{\int y(x)\, dx}\right) F\left(\frac{\left(\int u(x)\, dx\right) b\left(\frac{\int v(x)dx}{\int u(x)dx}\right)}{\left(\int y(x)\, dx\right) a\left(\frac{\int z(x)dx}{\int y(x)dx}\right)}\right) \tag{2.6}$$

holds. Hence from (2.5) and (2.6) we get (2.4).

A special case of (2.4) was proved in [9] which will be discussed in Section 3.

Now we are ready to prove the following theorem by using Theorem 1.4 and Lemma 2.1.

Theorem 2.4. *Let $f_i : I_i \to \mathbb{R}_+$, $I_i \subseteq (0,\infty)$, $i = 1, \cdots, m-1$, be a set of functions with the MC property where conditions a), b), and c) of Lemma 2.1 are satisfied.*

Let $a_i : \mathbf{C}_i \to \mathbb{R}_+$, where $\mathbf{C}_i$ are convex cones in the linear spaces X_i, and let a_i, $i = 1, \cdots, m$ be either subadditive functionals on $\mathbf{C}_i$ or superadditive functionals on $\mathbf{C}_i$ satisfying

$$\text{range}\left\{\frac{1}{a_i} G_{i+1,m-1}(a_{i+1}, \cdots, a_m)\right\} \subseteq I_i.$$

Let $F : I_0 \to \mathbb{R}$, $I_0 \subseteq \mathbb{R}_+$ be monotone and either subadditive on I_0 or superadditive on I_0 and range $\{G_{i,m-1}(a_1, \cdots, a_m)\} \subseteq I_0$.

A) If F is increasing and subadditive (superadditive), f_1 is convex (concave), $(-1)^{d_i+j} a_i$ are subadditive (superadditive) for $i = k_j + 1, \cdots, k_{j+1}$, $j = 0, \cdots, l$, $k_0 = 0$, $k_{l+1} = m-1$ and $(-1)^l a_m$ is subadditive (superadditive), then the compound functional

$$H = F \circ G_{1,m-1} : \mathbf{C}_1 \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_m \to \mathbb{R}$$

is subadditive (superadditive). That is

$$F\left(G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)\right)+F\left(G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)\right)$$
$$\geq\ (\leq)\,F\left(G_{1,m-1}\left(a_1\left(t_1+s_1\right),\cdots,a_m\left(t_m+s_m\right)\right)\right). \tag{2.7}$$

B) If F is decreasing and subadditive (superadditive) and f_1 concave (convex) $(-1)^{d_i+j}\,a_i$ are superadditive (subadditive), $i=k_j+1,\cdots,k_{j+1}$, $j=0,\cdots,l$ and $(-1)^l\,a_m$ is superadditive (subadditive). Then the compound functional

$$H=F\circ G_{1,m-1}:\mathbf{C}_1\times\mathbf{C}_2\times\cdots\times\mathbf{C}_m\to\mathbb{R}$$

is subadditive (superadditive) that is (2.7) holds.

Proof. We will prove here case B) of the theorem where F is decreasing and subadditive. The other cases follow similarly.

From case b in Theorem 1.4 it follows that when f_1 is concave and f_i, $i=1,\cdots,m-1$ satisfy the MC condition,

$$G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)+G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)$$
$$\leq\ G_{1,m-1}\left(a_1\left(t_1\right)+a_1\left(s_1\right),\cdots,a_m\left(t_m\right)+a_m\left(s_m\right)\right) \tag{2.8}$$

holds. In our case it is given that $(-1)^{d_i+j}\,a_i$, $i=k_j+1,\cdots,k_{j+1}$, $j=0,\cdots,l$, are superadditive, which means that

$$(-1)^{d_i+j}\left(a_i\left(t_i+s_i\right)-\left(a_i\left(t_i\right)+a_i\left(s_i\right)\right)\right)\geq 0,$$
$$i=k_j+1,\cdots,k_{j+1},\qquad j=0,\cdots,l,$$

and

$$(-1)^l\left(a_m\left(t_m+s_m\right)-\left(a_m\left(t_m\right)+a_m\left(s_m\right)\right)\right)\geq 0.$$

Using Lemma 2.1 for

$$a_i\left(t_i+s_i\right)=z_i,\qquad a_i\left(t_i\right)+a_i\left(s_i\right)=y_i,$$

we get from (2.2) that

$$G_{1,m-1}\left(a_1\left(t_1\right)+a_1\left(s_1\right),\cdots,a_m\left(t_m\right)+a_m\left(s_m\right)\right)$$
$$\leq\ G_{1,m-1}\left(a_1\left(t_1+s_1\right),\cdots,a_m\left(t_m+s_m\right)\right). \tag{2.9}$$

Inequalities (2.8) and (2.9) lead to

$$G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)+G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)$$
$$\leq\ G_{1,m-1}\left(a_1\left(t_1+s_1\right),\cdots,a_m\left(t_m+s_m\right)\right). \tag{2.10}$$

Now as F is subadditive we get that

$$F\left(G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)\right)+F\left(G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)\right)$$
$$\geq\ F\left(G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)+G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)\right). \tag{2.11}$$

Because F is decreasing, inequalities (2.10) and (2.11) yield

$$F\left(G_{1,m-1}\left(a_1\left(t_1\right),\cdots,a_m\left(t_m\right)\right)\right)+F\left(G_{1,m-1}\left(a_1\left(s_1\right),\cdots,a_m\left(s_m\right)\right)\right)$$
$$\geq\ F\left(G_{1,m-1}\left(a_1\left(t_1+s_1\right),\cdots,a_m\left(t_m+s_m\right)\right)\right),$$

hence $H=F\circ G_{1,m-1}:\mathbf{C}_1\times\mathbf{C}_2\times\cdots\times\mathbf{C}_m\to\mathbb{R}$ is subadditive.

This completes the proof of the theorem. □

All the notations in the following Corollary 2.5 are as in Theorem 1.4, Theorem 2.4 and Lemma 2.1.

Corollary 2.5. *Let* $\mathbf{C}_i$, *be convex cones in linear spaces* X_i, $i = 1, 2$.

(i) If $a_i : \mathbf{C}_i \to (0, \infty)$, *are superadditive (subadditive) functionals on* $\mathbf{C}_i$, $i = 1, 2$, $f : [0, \infty) \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$ *are concave (convex) and monotonic nondecreasing, where* $g(x) = xf\left(\frac{1}{x}\right)$, *then* $H : C_1 \times C_2 \to \mathbb{R}$, $H = a_1(t) f\left(\frac{a_2(s)}{a_1(t)}\right)$ *is a supereadditive (subadditive) functional.*

(ii) If $a_1 : \mathbf{C}_1 \to (0, \infty)$ *is a subadditive (superadditive) functional on* $\mathbf{C}_1$, $a_2 : \mathbf{C}_2 \to (0, \infty)$ *is a superadditive (subadditive) functional on* $\mathbf{C}_2$, $f : [0, \infty) \to \mathbb{R}$ *is convex (concave) and monotonic nonincreasing, and* $g(x)$ *is nondecreasing, then* $H = a_1(t) f\left(\frac{a_2(s)}{a_1(t)}\right)$ *is a subadditive (superadditive) functional on* $\mathbf{C}$.

In particular the same results on H *are obtained also when* a_1 *is additive and in this case the conditions on* g *are redundant. This special case was proved in* [5, Theorem 5]. *To see Corollary 2.5(i) take in Theorem 2.4* $F(x) = x$, $m = 2$, $l = 0$, $d = 0$, *and to see Corollary 2.5(ii), take* $m = 2$, $l = 1$, $d = 1$ *in Theorem 2.4,*

Corollary 2.6. *Let* $\mathbf{C}_i$ *be convex cones in linear spaces* X_i , $i = 1, \cdots, m$

a) If f_i *and* g_i, $i = 1, \cdots, m-1$ *are non-negative concave increasing functions on* $(0, \infty)$ *where* $g_i(x) = xf_i\left(\frac{1}{x}\right)$ *and* $a_i : \mathbf{C}_i \to \mathbb{R}_+$, $i = 1, \cdots, m$ *are superadditive then* $G_{1,m-1}(a_1(t_1), \cdots, a_m(t_m))$ *is superadditive in* $(t_1, \cdots, t_m)$. *In particular this case holds when* f_i, $i = 1, \cdots, n-1$ *are differentiable nonnegative concave increasing functions on* $[0, \infty)$ *satisfying* $f_i(0) = \lim_{z \to 0^+} zf_i'(z) = 0$ *because then* g_i *are nonnegative increasing too. For example let* $f_i(x) = x^{\alpha_i}$, $x \geq 0$, $0 < \alpha_i < 1$, $i = 1, \cdots, m-1$.

b) If f_i *are non-negative convex increasing functions on* $(0, \infty)$, g_i *are non-negative decreasing on* $(0, \infty)$, $i = 1, \cdots, m-1$, $a_i : \mathbf{C}_i \to \mathbb{R}_+$, $i = 1, \cdots, m-1$ *are superadditive and* a_m *is subadditive, then* $G_{1,m-1}(a_1(t_1), \cdots, a_m(t_m))$ *is subadditive in* $(t_1, \cdots, t_m)$. *In particular this case holds when* f_i, $i = 1, \cdots, n-1$ *are differentiable nonnegative convex increasing functions on* $[0, \infty)$ *satisfying* $f_i(0) = \lim_{z \to 0^+} zf_i'(z) = 0$ *because then* g_i *are decreasing.*

For example let $f_i(x) = x^{\alpha_i}$, $x \geq 0$, $\alpha_i > 1$, $i = 1, \cdots, m-1$.

3. Examples and Comments

Example 3.1. A special case of Corollary 2.3 was proved in [9] by choosing for $x > 0$, $p, s, t > 1$, $0 < \frac{s-p}{s-t} \leq 1$, $F(x) = x^{\frac{s-p}{s-t}}$, $a(x) = \left(1 + x^{\frac{1}{s}}\right)^s$, $b(x) = \left(1 + x^{\frac{1}{t}}\right)^t$, $y(x) = f^s(x)$, $z(x) = g^s(x)$, $u(x) = f^t(x)$, $v(x) = g^t(x)$, where $f(x), g(x) \geq 0$.

From this choice of the concave increasing functions $F(x)$, $a(x)$, $b(x)$, and $xF\left(\frac{1}{x}\right)$ we get inequality 3 in [9], that refines Minkowski's inequality:

$$\int (f(x) + g(x))^p \, dx \tag{3.1}$$

$$= \int f^s(x)\left(1+\left(\frac{g^s(x)}{f^s(x)}\right)^{\frac{1}{s}}\right)^s \left(\frac{f^t(x)\left(1+\left(\frac{g^t(x)}{f^t(x)}\right)^{\frac{1}{t}}\right)^t}{f^s(x)\left(1+\left(\frac{g^s(x)}{f^s(x)}\right)^{\frac{1}{s}}\right)^s}\right)^{\frac{s-p}{s-t}} dx$$

$$\leq \int f^s(x)\,dx\left(1+\left(\frac{\int g^s(x)\,dx}{\int f^s(x)\,dx}\right)^{\frac{1}{s}}\right)^s \left(\frac{\int f^t(x)\,dx\left(1+\left(\frac{\int g^t(x)dx}{\int f^t(x)dx}\right)^{\frac{1}{t}}\right)^t}{\int f^s(x)\,dx\left(1+\left(\frac{\int g^s(x)dx}{\int f^s(x)dx}\right)^{\frac{1}{s}}\right)^s}\right)^{\frac{s-p}{s-t}}$$

$$= \left(\left(\int f^s(x)\,dx\right)^{\frac{1}{s}}+\left(\int g^s(x)\,dx\right)^{\frac{1}{s}}\right)^{s\left(\frac{p-t}{s-t}\right)}$$

$$\times\left(\left(\int f^t(x)\,dx\right)^{\frac{1}{t}}+\left(\int g^t(x)\,dx\right)^{\frac{1}{t}}\right)^{t\left(\frac{s-p}{s-t}\right)}.$$

From Corollary 2.3 we get the reverse of inequality (3.1) when $\frac{s-p}{s-t} > 1$, $s > 1$, $0 < t < 1$ and also when $\frac{s-p}{s-t} < 0$, $0 < s < 1$, $t > 1$.

The four examples below can be derived from Theorem 2.4 as special cases.

Example 3.2. [6, Theorem 6] Let $\mathbf{C}$ be a convex cone in a linear space X. If $a_1 : \mathbf{C} \to (0,\infty)$ is a subadditive functional on $\mathbf{C}$ and $a_2 : \mathbf{C} \to (0,\infty)$ is a superadditive functional then

$$H(x) = \frac{a_1^2(x)}{a_2(x)}$$

is a subadditive functional on $\mathbf{C}$.

This follows from Theorem 2.4 and from Corollary 2.5 by observing that $\frac{a_1^2(x)}{a_2(x)} = a_1(x)\left(\frac{a_2(x)}{a_1(x)}\right)^{-1}$, $m = 2$, $F(x) = x$, $f_1(x) = x^{-1}$, is a convex decreasing function on $(0,\infty)$, and $xf\left(\frac{1}{x}\right) = x^2$ is increasing on $(0,\infty)$.

All other results quoted below from [4, 5] deal only with additive a_1.

The following example appears in [4, Theorem 2.1, Corollary 2.2]:

Example 3.3. Let $\mathbf{C}$ be a convex cone in a linear space X and $a_1 : \mathbf{C} \to (0,\infty)$ be an additive functional on $\mathbf{C}$. If $h : \mathbf{C} \to [0,\infty)$ is a superadditive (subadditive) functional on $\mathbf{C}$ and $p,q \geq 1$ $(0 < p,q < 1)$, then $H_{p,q}(x) = a_1^{q\left(1-\frac{1}{p}\right)}(x)\,h^q(x)$ is a superadditive (subadditive) functional on $\mathbf{C}$.

Take $F(x) = x^q$ and $f(x) = x^{\frac{1}{p}}$, $a_2(x) = h^p(x)$ and observe that if $p > 1$, $(0 < p < 1)$ and h is superadditive (subadditive) then $a_2(x)$ is also superadditive (subadditive). Also observe that $f(x) = x^{\frac{1}{p}}$, $p \geq 1$, $(0 < p < 1)$ is concave (convex) and increasing, and $F(x) = x^q$, $q \geq 1$ is superadditive (subadditive) and increasing.

From these observations, we conclude that:

$$H_{p,q}(x) = a_1^{q\left(1-\frac{1}{p}\right)}(x)\, h^q(x) = \left(a_1(x)\left(\frac{a_2(x)}{a_1(x)}\right)^{\frac{1}{p}}\right)^q, \qquad a_2(x) = h^p(x).$$

is superadditive (subadditive) on $\mathbf{C}$.

Theorem 2.5 and Corollary 2.6 in [4] are also special cases of Theorem 2.4. It says:

Example 3.4. Let X, $\mathbf{C}$ and a_1 be as in Example 3.3. If $h : \mathbf{C} \to (0,\infty)$ is a superadditive functional on $\mathbf{C}$ and $0 < p, q < 1$ then the functional $H : \mathbf{C} \to (0,\infty)$, $H(x) = \frac{1}{a_1^{q\left(\frac{1}{p}-1\right)}(x) h^q(x)}$ is subadditive on $\mathbf{C}$.

This follows from Theorem 2.4 by observing that

$$\frac{1}{a_1^{q\left(\frac{1}{p}-1\right)}(x)\, h^q(x)} = \left(a_1(x)\left(\frac{h^{-p}(x)}{a_1(x)}\right)^{\frac{1}{p}}\right)^q,$$

and by observing also that when h is superadditive, $a_2 = h^{-p}$ is subadditive for $p > 0$, and that $f(x) = x^{\frac{1}{p}}$, $x > 0$ is convex increasing and that $F(x) = x^q$ is subadditive increasing.

More special cases of Theorem 2.4 are the following (demonstrated in [5, Proposition 1]):

Example 3.5. Let $\mathbf{C}$ be a convex cone in a linear space X and $a_1 : \mathbf{C} \to (0,\infty)$ be an additive functional on $\mathbf{C}$.

(i) If $a_2 : \mathbf{C} \to (0,\infty)$ is a superadditive functional on $\mathbf{C}$ and $r > 0$ then $H(x) := \frac{(a_1(x))^{1+r}}{(a_2(x))^r}$ is subadditive on $\mathbf{C}$. In particular $\frac{a_1^2(x)}{a_2^2(x)}$ is subadditive.

(ii) If $a_2 : \mathbf{C} \to (0,\infty)$ is a superadditive functional on $\mathbf{C}$ $q \in (0,1)$ then $H := a_1^{1-q}(x)\, a_2^q$ is superadditive on $\mathbf{C}$. In particular $\sqrt{a_1(x)\, a_2(x)}$ is superadditive.

(iii) If $a_2 : \mathbf{C} \to (0,\infty)$ is a subadditive functional on $\mathbf{C}$ and $p \geq 1$, then $H(x) := \frac{(a_2(x))^p}{(a_1(x))^{p-1}}$ is subadditive on $\mathbf{C}$. In particular $\frac{a_2^2(x)}{a_1(x)}$ is subadditive.

We see that these three cases hold as special cases of Theorem 2.4 and Corollary 2.5 by rewriting

$$\begin{aligned}
\tfrac{(a_1(x))^{1+r}}{(a_1(x))^r} &= a_1(x)\left(\tfrac{a_2(x)}{a_1(x)}\right)^{-r}, \quad f_1(x) = x^{-r}, \quad x > 0\\
a_1^{1-q} \cdot a_2^q &= a_1(x)\left(\tfrac{a_2(x)}{a_1(x)}\right)^{q}, \quad f_1(x) = x^{q}, \quad x > 0\\
\tfrac{(a_2(x))^p}{(a_1(x))^{p-1}} &= a_1(x)\left(\tfrac{a_2(x)}{a_1(x)}\right)^{p}, \quad f_1(x) = x^{p}, \quad x > 0,
\end{aligned}$$

and taking in Theorem 2.4 $F(x) = x$, $m = 2$.

Theorem 6 in [5] deals with log-convex (log-concave) functions that means a function f for which $\log f$ is convex (concave). The results there follow from

Theorem 2.4 for $F(x) = x$, $m = 2$ and the convex (concave) function $f_1(x) = logf$.

In the next comments we deal with subadditive/superadditive functionals related to the Minkowski and Hölder inequalities. Although the results may be obtained by a direct and simple way it is interesting to see how they are special cases of Theorem 2.4.

In [2] the functions

$$G_{1,m-1}(x_1, \cdots, x_m) = \left(x_1^{\frac{1}{p}} + \cdots + x_m^{\frac{1}{p}}\right)^p$$

and

$$G_{1,m-1}(x_1, \cdots, x_m) = \left(w_1 x_1^{\frac{1}{p}} + \cdots + w_m x_m^{\frac{1}{p}}\right)^p,$$

$$x_i, w_i > 0, \qquad i = 1, \cdots, m, \qquad \sum_{i=1}^{m} w_i = 1$$

are dealt with. Here we add conditions on a_i, $i = 1, \cdots, m$, and get comments (i) and (ii):

Comment (i). Let $p > 1$ be a real number and f be the real function defined by

$$f(x) = \left(1 + x^{\frac{1}{p}}\right)^p, \qquad x > 0.$$

Let $f_1 = \cdots = f_{m-1} = f$. Let $\mathbf{C}$ be a convex cone in a linear space X, and $a_i(t) : \mathbf{C} \to \mathbb{R}_+$ be superadditive on $\mathbf{C}$, $i = 1, \cdots, m$.

Then

$$G_{1,m-1}(a_1(t), \cdots, a_m(t)) = \left(a_1^{\frac{1}{p}}(t) + \cdots + a_m^{\frac{1}{p}}(t)\right)^p$$

is superadditive in t.

This result follows from Corollary 2.6a because f_i are concave and f_i and $g_i(x) = x f_i\left(\frac{1}{x}\right) = f_i(x)$, $i = 1, \cdots, m-1$ are increasing.

Comment (ii). Given functions f_i as

$$f_i(x) = \left(1 + \frac{w_{i+1}}{w_i} x^r\right)^{1/r}, \qquad x > 0, \qquad i = 2, \ldots, m-1,$$

$$f_1(x) = (w_1 + w_2 x^r)^{1/r}, \qquad x > 0,$$

where $w_i > 0$, $i = 1, \cdots, m$, $\sum_{i=1}^{m} w_i = 1$, $r \leq 1$, $r \neq 0$.

As shown in ([2]) the function $G_{1,m-1}$ for these special functions f_i, $i = 1, \cdots, m$ have the form

$$G_{1,m-1}(x_1, \ldots, x_m) = (w_1 x_1^r + \ldots + w_m x_m^r)^{1/r},$$

which is exactly the power mean of order r of a sequence $x = (x_1, \ldots, x_m)$ with weights $w = (w_1, \ldots, w_m)$.

Let $\mathbf{C}$ be a convex cone of a linear space X, $a_i : \mathbf{C} \to \mathbb{R}_+$, $i = 1, \ldots, m$ be superadditive functionals and the functions f_i, and $g_i(x) = xf\left(\frac{A}{x}\right)$, $A > 0$, $i = 1, \ldots, m-1$, be increasing and concave. Therefore Corollary 2.6a holds, that is, $G_{1,m-1}(a_1(t), \ldots, a_m(t)) = (w_1 a_1^r(t) + \ldots + w_m a_m^r(t))^{1/r}$ is superadditive.

Comment (iii). Let $\mathbf{C}$ be a convex cone in a linear space X, and $a_i(t) : \mathbf{C} \to \mathbb{R}_+, i = 1, \cdots, m$ be superadditive functional on $\mathbf{C}$. Let also $p_i > 0$, $i = 1, \cdots, m$, $\sum_{i=1}^m \frac{1}{p_i} = 1$. Then $H = \prod_{i=1}^{m} a_i^{\frac{1}{p}}(t)$ is superadditive on $\mathbf{C}$.

Comment (iv). Let $\mathbf{C}$ be a convex cone in a linear space X, $a_i(t) : \mathbf{C} \to \mathbb{R}_+, i = 1, \cdots, m-1$ be superadditive functionals on $\mathbf{C}$ and $a_m(t) : \mathbf{C} \to \mathbb{R}_+$ be subadditive. Let $p_i < 0$, $i = 1, \cdots, m-1$, $p_m > 1$, $\sum_{i=1}^m \frac{1}{p_i} = 1$.

Then $H = \prod_{i=1}^{m} a_i^{\frac{1}{p}}(t)$ is subadditive on $\mathbf{C}$.

As in [1, 2, 3] it can be verified that

$$\prod_{i=1}^{m} a_i^{\frac{1}{p}} = a_1 \left(\frac{a_2}{a_1} \left(\frac{a_3}{a_2} \cdots \left(\frac{a_m}{a_{m-1}} \right)^{\frac{1}{q_{m-1}}} \right)^{\frac{1}{q_2}} \right)^{\frac{1}{q_1}},$$

where

$$\frac{1}{q_1} = 1 - \frac{1}{p_1}, \qquad \frac{1}{q_i} = 1 - \frac{q_1 \cdots q_{i-1}}{p_i}, \qquad i = 2, \cdots, m-1,$$
$$q_1 q_2 \cdots q_{m-1} = p_m \qquad \frac{1}{q_1 q_2 \cdots q_i} = 1 - \sum_{j=1}^{i} \frac{1}{p_j}, \qquad i = 1, \cdots, m-1.$$

It is easy to see that when $p_i > 0, i = 1, \cdots, m$, $\sum_{i=1}^m \frac{1}{p_i} = 1, q_i > 1$, $f_i(x) = x^{\frac{1}{q_i}}$, $x > 0$, $i = 1, \cdots, m-1$ are concave increasing and so are $g_i(x) = xf_i\left(\frac{A}{x}\right)$, $A > 0$, and Corollary 2.6a) holds, and therefore Comment (iii) holds too.

In Comment (iv) it is easy to see that $\frac{1}{q_i} > 1$, $f_i(x) = x^{\frac{1}{q_i}}$, $x > 0$, are convex increasing and g_i, $i = 1, \cdots, m-1$ are decreasing and Corollary 2.6b) holds and therefore Comment (iv) holds too.

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