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POSITIVE TOEPLITZ OPERATORS ON THE BERGMAN SPACE

NAMITA DAS[1] AND MADHUSMITA SAHOO[2*]

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ABSTRACT. In this paper we find conditions on the existence of bounded linear operators A on the Bergman space $L^2_a(\mathbb{D})$ such that $A^*T_\phi A \geq S_\psi$ and $A^*T_\phi A \geq T_\phi$ where T_ϕ is a positive Toeplitz operator on $L^2_a(\mathbb{D})$ and S_ψ is a self-adjoint little Hankel operator on $L^2_a(\mathbb{D})$ with symbols $\phi, \psi \in L^\infty(\mathbb{D})$ respectively. Also we show that if T_ϕ is a non-negative Toeplitz operator then there exists a rank one operator R_1 on $L^2_a(\mathbb{D})$ such that $\widetilde{\phi}(z) \geq \alpha^2 \widetilde{R_1}(z)$ for some constant $\alpha \geq 0$ and for all $z \in \mathbb{D}$ where $\widetilde{\phi}$ is the Berezin transform of T_ϕ and $\widetilde{R_1}(z)$ is the Berezin transform of R_1.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $dA(z) = \frac{1}{\pi}dxdy$ be the normalized area measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ be the space of complex-valued, absolutely integrable, measurable functions on $\mathbb{D}$ with respect to the area measure dA and $L^2_a(\mathbb{D})$ be the Bergman space consisting of all analytic functions that are in $L^2(\mathbb{D}, dA)$. Here the norm $\|\cdot\|_2$ and the inner product are taken in the space $L^2(\mathbb{D}, dA)$. It is [4] not difficult to see that $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$. We denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ into $L^2_a(\mathbb{D})$ by P. Let $L^\infty(\mathbb{D})$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. For $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. The sequence $\{e_n\}_{n=0}^\infty$ forms an orthonormal basis of $L^2_a(\mathbb{D})$. Let $K(z,\overline{w}) = \overline{K_z(w)} = \dfrac{1}{(1-z\overline{w})^2} = \sum_{n=0}^\infty e_n(z)\overline{e_n(w)}$.

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* Corresponding author.
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The function $K(z, \overline{w})$ defined on $\mathbb{D} \times \mathbb{D}$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L_a^2(\mathbb{D})$. Let $k_z(w) = \dfrac{K(w, \bar{z})}{K(z, \bar{z})} = \dfrac{1 - |z|^2}{(1 - \bar{z}w)^2} = \dfrac{K_z(w)}{\|K_z\|_2}$. These functions k_z are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$ for each $z \in \mathbb{D}$. It is clear [10] that they are unit vectors in $L_a^2(\mathbb{D})$.

For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator from $L_a^2(\mathbb{D})$ into itself by $T_\phi f = P(\phi f)$ and the Hankel operator H_ϕ from $L_a^2(\mathbb{D})$ into $(L_a^2(\mathbb{D}))^\perp$ is defined by $H_\phi f = (I - P)(\phi f)$. The little Hankel operator S_ϕ from $L_a^2(\mathbb{D})$ into itself is defined as $S_\phi f = P(J(\phi f))$ where $J : L^2(\mathbb{D}, dA) \longrightarrow L^2(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\bar{z})$. These operators [10] are all bounded.

Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators from the Hilbert space H into itself. Let $\mathcal{LC}(H)$ denote the ideal of compact operators in $\mathcal{L}(H)$. A bounded linear operator $A \in \mathcal{L}(H)$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. The notation $A \geq 0$ will mean that A is positive. We say $A \geq B$ when $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for all $x \in H$. For arbitrary selfadjoint operators $A, B \in \mathcal{L}(H)$ we write $A \leq B$ if and only if $B - A \geq 0$. An operator $A \in \mathcal{L}(H)$ is called hyponormal if $A^*A \geq AA^*$ and the operator $A \in \mathcal{L}(H)$ is called power bounded if $\|A^n\| \leq K$ for a fixed $K > 0$ and $n = 1, 2, \ldots$. Let T be a bounded linear operator on a Hilbert space H. We denote $\dfrac{T + T^*}{2}$ by $\mathrm{Re}(T)$ and $\dfrac{T - T^*}{2i}$ by $\mathrm{Im}(T)$. Define the Berezin transform for operators $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ by the formula

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle, z \in \mathbb{D}.$$

The function $\widetilde{T}$ is called the Berezin transform of T. If $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ then $\widetilde{T} \in L^\infty(\mathbb{D})$ and $\|\widetilde{T}\|_\infty \leq \|T\|$ as $|\widetilde{T}(z)| = |\langle Tk_z, k_z\rangle| \leq \|T\|$ for all $z \in \mathbb{D}$. We shall write $\widetilde{T_\phi} = \widetilde{\phi}$ for $\phi \in L^\infty(\mathbb{D})$. That is, $\widetilde{\phi}(z) = \langle T_\phi k_z, k_z\rangle = \widetilde{T_\phi}(z)$ for all $z \in \mathbb{D}$.

In the set of bounded Hermitian operators from a Hilbert space H into itself, various types of ordering by means of the cones of non-negative, positive definite and positive invertible operators can be defined. In this paper we investigate whether it is possible to compare the Berezin transform of non-negative Toeplitz and little Hankel operators. In section 2, we prove a few preliminary lemmas. In section 3, we show that if T_ϕ is a positive Toeplitz operator on the Bergman space and S_ψ is a self-adjoint little Hankel operator then there exist bounded linear operators $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi$. Similarly, we show that there exists $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $A^*T_\phi A \geq T_\phi$. Further, one can find a sequence $\{A_n\} \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $A_n \xrightarrow{w} 0$ and $A_n^*T_\phi A_n \geq T_\phi$ for all n. In section 4, we prove that if T_ϕ is a non-negative Toeplitz operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ then there exists a rank one operator $R_1 \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\widetilde{\phi}(z) \geq \beta\widetilde{R_1}(z)$ for all $z \in \mathbb{D}$ and for some constant $\beta \geq 0$.

2. Preliminary lemmas

In this section we prove a few preliminary lemmas which will be used in proving the main results of the paper.

For finite rank operators in $\mathcal{L}(L_a^2(\mathbb{D}))$ one can define a trace functional tr by $\operatorname{tr}(T) = \sum_{k=1}^{n} \langle f_k, g_k \rangle$ when $T = \sum_{k=1}^{n} f_k \otimes g_k$.

Lemma 2.1. *Let $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$. If $tr(ASA) = tr(ATA)$ for every rank one projection $A \in \mathcal{L}(L_a^2(\mathbb{D}))$, then $S = T$.*

Proof. Let $A = f \otimes f$, where f is a unit vector. Then A is a rank one projection and every rank one projection takes this form. By the assumption, we have

$$\begin{aligned} \langle Sf, f \rangle &= \operatorname{tr}(Sf \otimes f) \\ &= \operatorname{tr}(ASA) = \operatorname{tr}(ATA) \\ &= \operatorname{tr}(Tf \otimes f) \\ &= \langle Tf, f \rangle . \end{aligned}$$

Thus $\langle Sf, f \rangle = \langle Tf, f \rangle$ holds for every unit vector $f \in L_a^2(\mathbb{D})$. Therefore, $\langle Sk_z, k_z \rangle = \langle Tk_z, k_z \rangle$ for all $z \in \mathbb{D}$. Hence $S = T$. □

Lemma 2.2. *If T_ϕ is invertible and $\langle A^* T_\phi^{-1} Af, g \rangle \langle A^* T_\phi Af, g \rangle = \langle A^* Af, g \rangle^2$ for every $f, g \in L_a^2(\mathbb{D})$ and for some $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ whose range is dense in $L_a^2(\mathbb{D})$ then ϕ is a constant function.*

Proof. Since $\overline{\text{Range}A} = L_a^2(\mathbb{D})$ we have $\left\langle T_\phi^{-1} f, g \right\rangle \langle T_\phi f, g \rangle = \langle f, g \rangle^2$ for all $f, g \in L_a^2(\mathbb{D})$. Now fix a nonzero $f \in L_a^2(\mathbb{D})$. Then, for every $g \in (\text{Sp}\{f\})^\perp \subset L_a^2(\mathbb{D})$, we have $\langle T_\phi^{-1} f, g \rangle = 0$ or $\langle T_\phi f, g \rangle = 0$ since $\left\langle T_\phi^{-1} f, g \right\rangle \langle T_\phi f, g \rangle = \langle f, g \rangle^2 = 0$. Let $M_f = \left\{ g \in (\text{Sp}\{f\})^\perp : \langle T_\phi f, g \rangle = 0 \right\}$ and $N_f = \left\{ g \in (\text{Sp}\{f\})^\perp : \langle T_\phi^{-1} f, g \rangle = 0 \right\}$. Then $M_f \cup N_f = (\text{Sp}\{f\})^\perp$. Because $(\text{Sp}\{f\})^\perp, M_f$ and N_f are all closed linear subspaces, we must have $M_f \subseteq N_f = (\text{Sp}\{f\})^\perp$ or $N_f \subseteq M_f = (\text{Sp}\{f\})^\perp$. If $N_f = (\text{Sp}\{f\})^\perp$, then $T_\phi^{-1} f \in \text{Sp}\{f\}$. So there exists a $\lambda_f \in \mathbb{C}$ such that $T_\phi^{-1} f = \lambda_f f \neq 0$, that is, $T_\phi f = \lambda_f^{-1} f$. If $M_f = (\text{Sp}\{f\})^\perp$, then $T_\phi f \in \text{Sp}\{f\}$, that is, $T_\phi f = \lambda_f f$ for some scalar λ_f. Since f is arbitrary, we see that for every $f \in L_a^2(\mathbb{D})$, there is a scalar λ_f such that $T_\phi f = \lambda_f f$. This implies that there exists a $\lambda \in \mathbb{C}$ such that $\phi \equiv \lambda$. □

Corollary 2.3. *Suppose that T_ϕ is invertible and $\langle S_{\psi^+} T_\phi^{-1} S_\psi f, g \rangle \langle S_{\psi^+} T_\phi S_\psi f, g \rangle = \langle S_{\psi^+} S_\psi f, g \rangle^2$ for every $f, g \in L_a^2(\mathbb{D})$ and $\ker S_\psi = \{0\}$. Then $\phi \equiv C$, a constant function.*

Proof. We need only to observe that $\overline{\text{Range}S_\psi} = L_a^2(\mathbb{D})$ if and only if $\ker S_\psi = \{0\}$. □

Lemma 2.4. *Let A be a nonnegative operator in $\mathcal{L}(L_a^2(\mathbb{D}))$. Then $\ker A = \ker A^{1/2}$ and $\overline{\text{Range}A} = \overline{\text{Range}A^{1/2}}$. If RangeA is closed then Range$A^{1/2}$ is closed and Range$A$ = Range$A^{1/2}$ and $A = A^{1/2} B$, for some invertible $B \in \mathcal{L}(L_a^2(\mathbb{D}))$.*

Proof. Since $\langle Af, f \rangle = \langle A^{1/2} f, A^{1/2} f \rangle, f \in L_a^2(\mathbb{D})$, it follows that $\ker A \subseteq \ker A^{1/2}$. Conversely, if $f \in \ker A^{1/2}$, we obtain $Af = A^{1/2} A^{1/2} f = 0$. Thus $\ker A = \ker A^{1/2}$. Also, observe that $\overline{\text{Range}A} = (\ker A)^\perp = (\ker A^{1/2})^\perp = \overline{\text{Range}A^{1/2}}$. The lemma follows from [7]. □

Lemma 2.5. *Let $\psi \in C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$ and $\|\psi\|_\infty \leq 1$. Let T_ϕ be a positive Toeplitz operator on $L^2_a(\mathbb{D})$ such that $T_\phi \leq S_{\psi^+}T_\phi S_\psi$ where $\psi^+(z) = \overline{\psi(\bar{z})}$. Then $T_\phi = S_{\psi^+}T_\phi S_\psi$. Further $\overline{RangeT_\phi}$ reduces S_ψ and $S_\psi|_{\overline{(RangeT_\phi)}}$ is unitary.*

Proof. Let $T_\phi^{1/2}S_\psi = L$. The operator L is compact [10] as $\psi \in C(\overline{\mathbb{D}})$ and S_ψ is a contraction as $\|\psi\|_\infty \leq 1$. Further, $LL^* = T_\phi^{1/2}S_\psi S_{\psi^+}T_\phi^{1/2} \leq T_\phi$. This is so since $S_\psi^* = S_{\psi^+}$. Hence $0 \leq S_{\psi^+}T_\phi S_\psi - T_\phi \leq S_{\psi^+}T_\phi S_\psi - T_\phi^{1/2}S_\psi S_{\psi^+}T_\phi^{1/2} = L^*L - LL^*$. Hence the operator L is hyponormal. Since L is compact, L is normal. The normality of L implies that $T_\phi = S_{\psi^+}T_\phi S_\psi = T_\phi^{1/2}S_\psi S_{\psi^+}T_\phi^{1/2}$, and hence it follows that S_{ψ^+} is an isometry on $\overline{\text{Range}T_\phi}$ and T_ϕ commutes with S_ψ (and so also with S_{ψ^+}). Consequently, $S_{\psi^+}S_\psi T_\phi = S_{\psi^+}T_\phi S_\psi = T_\phi = T_\phi S_\psi S_{\psi^+}$. Hence $\overline{\text{Range}T_\phi}$ reduces S_ψ and $S_\psi|_{\overline{(\text{Range}T_\phi)}}$ is unitary. □

3. Non-negative Toeplitz operators

In this section we show that if T_ϕ is a positive Toeplitz operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ and $\psi \in L^\infty(\mathbb{D})$ can be expressed as a linear combination of Bergman kernels and some of its derivative then there exist bounded linear operators $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi^* S_\psi$. If in addition $\psi(z) = \overline{\psi(\bar{z})}$ then we can find $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi$. Further, we find conditions for the existence of $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq T_\phi$. It is also possible to find sequences $\{A_n\}$ of operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ such that $A_n \xrightarrow{w} 0$ and $A_n^*T_\phi A_n \geq T_\phi$ for all n.

Theorem 3.1. *Let T_ϕ be a positive Toeplitz operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ with symbol $\phi \in L^\infty(\mathbb{D})$ and S_ψ be a little Hankel operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ where*

$$\overline{\psi(z)} = \sum_{j=1}^{N}\sum_{\gamma=0}^{m_j-1} c_{j\gamma}\frac{\partial^\gamma}{\partial\overline{b_j}^\gamma}K_{b_j}(z)$$

where $\boldsymbol{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$, $c_{j\gamma} \neq 0$ for all j,γ and m_j is the number of times b_j appears in $\boldsymbol{b}$. Then there exists an operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi^*S_\psi$ and $\|A^*T_\phi A\| \geq \|S_\psi^*S_\psi\|$. Further, in addition if $\psi(z) = \overline{\psi(\bar{z})}$ then it is also possible to find $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi$ and $(\widetilde{A^*T_\phi A})(z) \geq \widetilde{S_\psi}(z)$ where $\widetilde{H}$ denotes the Berezin transform of $H \in \mathcal{L}(L^2_a(\mathbb{D}))$, and $\|A^*T_\phi A\| \geq \|S_\psi\|$. In case A is positive, then there exists an invertible $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^{1/2}T_\phi A^{1/2} \geq (T^*)^{-1}S_\psi T^{-1}$.

Proof. From [5] it follows that S_ψ is a finite rank operator on $L^2_a(\mathbb{D})$ and therefore $S_\psi^*S_\psi$ is a finite rank operator and Range $S_\psi^*S_\psi$ is closed in $L^2_a(\mathbb{D})$. Notice also that

$$\dim\left(\overline{\bigcup_{\lambda>0} E^{T_\phi}[\lambda,\infty)(L^2_a(\mathbb{D}))}\right) = \infty. \tag{3.1}$$

This is so as $E^{T_\phi}(0,\infty)(L^2_a(\mathbb{D})) = \overline{\text{Range}T_\phi}$ and from [9] it follows that $\overline{\text{Range}T_\phi}$ is infinite dimensional. Let $M = \left\{Y \in \mathcal{L}(L^2_a(\mathbb{D})) \mid Y^*T_\phi Y \geq S_\psi^*S_\psi\right\}$. We first claim that 0 is in the WOT-closure of M. To show this suppose 0 is not in the WOT-closure of M. Then there is a WOT-neighborhood

$$V = \{B \in \mathcal{L}(L^2_a(\mathbb{D})) : |\langle Bf_i, g_i\rangle| \leq \epsilon, i = 1, \cdots, n\}$$

of 0 (for some $\epsilon > 0$) which does not intersect M where $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^2_a(\mathbb{D})$. Let K be the linear span of $g_1, g_2, \ldots, g_n$. From (3.1), it follows that there exists $\lambda > 0$ such that $\dim E^{T_\phi}[\lambda, \infty)(L^2_a(\mathbb{D})) > n + \operatorname{rank}(S^*_\psi S_\psi)$. It thus follows that $\dim \left(E^{T_\phi}[\lambda, \infty)(L^2_a(\mathbb{D})) \bigcap K^\perp\right) \geq \operatorname{rank}(S^*_\psi S_\psi)$. Since $S^*_\psi S_\psi$ is a self adjoint operator of finite rank, there exist real numbers $\{\theta_i\}_{i=1}^k$ and an orthonormal basis $\{\delta_i\}_{i=1}^k$ for $\operatorname{Range} S^*_\psi S_\psi$ such that $S^*_\psi S_\psi f = \sum_{i=1}^k \theta_i \langle f, \delta_i\rangle \delta_i$ and $|\theta_i| > 0$ for all $i = 1, \ldots, k$. Let $B \in \mathcal{L}(L^2_a(\mathbb{D}))$ be such that $B|_{(\operatorname{Range} S^*_\psi S_\psi)^\perp} = 0$ and $B\delta_i = u_i$ where $\{u_i\}_{i=1}^k$ is an orthonormal set in $E^{T_\phi}[\lambda, \infty)(L^2_a(\mathbb{D})) \bigcap K^\perp$.

Now, for each $g \in \operatorname{Range} S^*_\psi S_\psi$, we have $\|Bg\| = \|g\|$ and $Bg \in E^{T_\phi}[\lambda, \infty)(L^2_a(\mathbb{D}))$. Thus $\langle B^*T_\phi Bg, g\rangle = \langle T_\phi Bg, Bg\rangle \geq \lambda\|Bg\|^2 = \lambda\|g\|^2$. Let $f \in L^2_a(\mathbb{D})$. Then $f = g + h$, where $g \in \operatorname{Range} S^*_\psi S_\psi$ and $h \in (\operatorname{Range} S^*_\psi S_\psi)^\perp$. Hence

$$\langle S^*_\psi S_\psi f, f\rangle = \sum_{i=1}^k \theta_i |\langle f, \delta_i\rangle|^2 \leq \max_i |\theta_i| \|g\|^2$$

and

$$\langle A^*T_\phi Af, f\rangle = \langle T_\phi Af, Af\rangle = \langle T_\phi Ag, Ag\rangle \geq \lambda\|g\|^2 \geq \frac{1}{t^2}\langle S^*_\psi S_\psi f, f\rangle,$$

where $\frac{1}{t^2} = \frac{\lambda}{\max_i |\theta_i|}$. Thus $t^2 B^*T_\phi B \geq S^*_\psi S_\psi$ and $tB \in M$. Further since $B(L^2_a(\mathbb{D})) \subset K^\perp$, we have $tB \in V$. Hence $V \bigcap M \neq \phi$. This is a contradiction. Thus there exists operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S^*_\psi S_\psi$ and therefore $\|A^*T_\phi A\| \geq \|S^*_\psi S_\psi\|$. In case $\psi(z) = \overline{\psi(\overline{z})}$, the operator S_ψ is self-adjoint. Proceeding similarly as above, one can show that there exists $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A^*T_\phi A \geq S_\psi$ and therefore $\widetilde{(A^*T_\phi A)}(z) \geq \widetilde{S_\psi}(z)$ and $\|A^*T_\phi A\| \geq \|S_\psi\|$. If A is positive then by Lemma 2.4 there exists an invertible operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $A = A^{1/2}T$. Hence $A^*T_\phi A \geq S_\psi$ implies $A^{1/2}T_\phi A^{1/2} \geq (T^*)^{-1}S_\psi T^{-1}$. □

If $f(z) = \sum_{n=0}^\infty a_n z^n$ is holomorphic on $\mathbb{D}$, a simple calculation shows that

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) = \sum_{n=0}^\infty \frac{|a_n|^2}{n+1}.$$

Consequently, $f \in L_a^2(\mathbb{D})$ if and only if the last expression is finite. The scalar product of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $f, g \in L_a^2(\mathbb{D})$, is given by

$$\langle f, g\rangle_{L_a^2(\mathbb{D})} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

The truncation projections on $L_a^2(\mathbb{D})$ will be denoted by $P_n, 0 \leq n < \infty$, and it is defined by

$$P_n f = P_n(a_0, a_1, a_2, \cdots, a_n, a_{n+1}, \cdots) = (a_0, a_1, \cdots, a_n, 0, 0, \cdots).$$

These are, of course, orthogonal projections on $L_a^2(\mathbb{D})$ which converges strongly to the identity I on $L_a^2(\mathbb{D})$.

Theorem 3.2. *Let T_ϕ be a non-negative nonzero Toeplitz operator on $L_a^2(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$. Then*

- **(i):** *For each $\epsilon > 0$, there exists an operator $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\|P_n A P_n\| \leq \epsilon$ and $A^* T_\phi A \geq T_\phi$. If $tr(BA^*T_\phi AB) = tr(BT_\phi B)$ for every rank one projection operator $B \in \mathcal{L}(L_a^2(\mathbb{D}))$, then $A^* T_\phi A = T_\phi$.*
- **(ii):** *If $T_\phi \leq Re(A^* T_\phi)$ for some $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ then $T_\phi \leq A^* T_\phi A$. That is, $\widetilde{\phi}(z) \leq \widetilde{A^* T_\phi A}(z)$ for all $z \in \mathbb{D}$. Furthermore if $T_\phi \leq Re(A^* T_\phi)$ and $T_\phi = A^* T_\phi A$ for some $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ then $A^* T_\phi = T_\phi$.*
- **(iii):** *If $K = A^* T_\phi$ for some $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $T_\phi \leq Re(K)$ and A^* is power bounded then $K = T_\phi$.*
- **(iv):** *If for some $A \in \mathcal{L}(L_a^2(\mathbb{D})), \|A\| \leq 1, A^* T_\phi A \geq T_\phi$ then $T_\phi^{1/2} A$ is a hyponormal operator.*
- **(v):** *Let T_ϕ be invertible and E be a nonzero projection and $\lambda \in \mathbb{R}, \lambda > 0$ such that $ET_\phi E = \lambda E$ and $ET_\phi^{-1} E = \frac{1}{\lambda} E$. Then RangeE is a subspace of the eigenspace of T_ϕ corresponding to the eigenvalue λ.*
- **(vi):** *If T_ϕ is invertible and $\langle A^* T_\phi^{-1} Af, g\rangle \langle A^* T_\phi Af, g\rangle = \langle A^* Af, g\rangle^2$ for every $f, g \in L_a^2(\mathbb{D})$ and for some $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\overline{RangeA} = L_a^2(\mathbb{D})$ then ϕ is a constant function.*
- **(vii):** *If $\psi \in C(\overline{\mathbb{D}})$, $\|\psi\|_\infty \leq 1$, $S_\psi^* T_\phi S_\psi \geq T_\phi$ then $T_\phi = S_\psi^* T_\phi S_\psi$ and $T_\phi^{1/2} S_\psi$ is a hyponormal operator.*

Proof. We shall assume first that T_ϕ is one-one. For $\lambda > 0$, let E_λ be the spectral measure of the interval $[\lambda, \infty)$. Since T_ϕ is one-one and non-negative, hence $E_\lambda \longrightarrow I$, the identity operator, in the strong operator topology. Thus there exists $\lambda = \lambda(\epsilon) > 0$ such that the orthogonal projection $E_\lambda \in \mathcal{L}(L_a^2(\mathbb{D}))$ satisfies

$$T_\phi E_\lambda = E_\lambda T_\phi, \|(I - E_\lambda) P_n\| \leq \sqrt{\epsilon}$$

and $\dim(\text{Range}E_\lambda) \geq 2\dim(\text{Range}P_n)$. Also the spectral measure E_λ satisfies

$$\langle T_\phi f, f\rangle \geq \lambda \|f\|^2 \tag{3.2}$$

for all $f \in \text{Range}E_\lambda$, From [9], it follows that RangeT_ϕ is infinite dimensional. Thus there exists an unitary operator U on RangeE_λ such that

$$(\text{Range}UE_\lambda P_n) \perp (\text{Range}E_\lambda P_n).$$

Define $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ as $Af = \alpha U E_\lambda f + (I - E_\lambda)f$, where $\alpha > 0$ is chosen in such a way that $A^* T_\phi A \geq T_\phi$. We shall now verify that such α exists. Since T_ϕ commutes with E_λ we have

$$\begin{aligned}\langle A^* T_\phi A f, f\rangle &= \langle T_\phi A f, Af\rangle \\ &= \langle \alpha T_\phi U E_\lambda f + T_\phi (I - E_\lambda) f, \alpha U E_\lambda f + (I - E_\lambda) f\rangle \\ &= \alpha^2 \langle T_\phi U E_\lambda f, U E_\lambda f\rangle + \langle T_\phi (I - E_\lambda) f, (I - E_\lambda) f\rangle .\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle T_\phi f, f\rangle &= \langle T_\phi E_\lambda f, f\rangle + \langle T_\phi (I - E_\lambda) f, f\rangle \\ &= \langle T_\phi E_\lambda f, E_\lambda f\rangle + \langle T_\phi E_\lambda f, (I - E_\lambda) f\rangle \\ &+ \langle T_\phi (I - E_\lambda) f, E_\lambda f\rangle + \langle T_\phi (I - E_\lambda) f, (I - E_\lambda) f\rangle \\ &= \langle T_\phi E_\lambda f, E_\lambda f\rangle + \langle T_\phi (I - E_\lambda) f, (I - E_\lambda) f\rangle.\end{aligned}$$

Hence the only condition which has to be satisfied by α is

$$\alpha^2 \langle T_\phi U E_\lambda f, U E_\lambda f\rangle \geq \langle T_\phi E_\lambda f, E_\lambda f\rangle .$$

The condition is satisfied by sufficiently large α because of (3.2) and because RangeT_ϕ is an infinite dimensional subspace of $L^2_a(\mathbb{D})$. To show that $\|P_n A P_n\| \leq \epsilon$, observe that $\|P_n A P_n\| = \sup\{|\langle P_n A P_n f, g\rangle| : f, g \in L^2_a(\mathbb{D}), \|f\| = \|g\| = 1\}$. Let $\|f\| = \|g\| = 1$. We have

$$\begin{aligned}|\langle P_n A P_n f, g\rangle| &= |\langle A P_n f, P_n g\rangle| \\ &= |\langle E_\lambda A P_n f, E_\lambda P_n g\rangle + \langle (I - E_\lambda) A P_n f, (I - E_\lambda) P_n g\rangle| \\ &= |\langle \alpha U E_\lambda P_n f, E_\lambda P_n g\rangle + \langle (I - E_\lambda) P_n f, (I - E_\lambda) P_n g\rangle| \\ &= |0 + \langle (I - E_\lambda) P_n f, (I - E_\lambda) P_n g\rangle| \\ &\leq \|(I - E_\lambda) P_n f\| \|(I - E_\lambda) P_n g\| \\ &\leq \|(I - E_\lambda) P_n\| \|(I - E_\lambda) P_n\| \leq \epsilon.\end{aligned}$$

To prove the general case, let $M = \ker T_\phi$. Decompose $L^2_a(\mathbb{D})$ into an orthogonal direct sum $L^2_a(\mathbb{D}) = (\ker T_\phi)^\perp \oplus \ker T_\phi = M^\perp \oplus M$ and let Q be the orthogonal projection onto $M^\perp$. Let $T^{M^\perp}_\phi = T_\phi|_{M^\perp}$ be the restriction of T to $M^\perp$. Let $N = Q P_n L^2_a(\mathbb{D})$ and let Q_1 be the orthogonal projection from $M^\perp$ onto N. Applying the first of the proof to the operator $T^{M^\perp}_\phi$ and the projection Q_1 we find an operator $A_1 \in \mathcal{L}(M^\perp)$ with $\|Q_1 A_1 Q_1\| \leq \dfrac{\epsilon}{\|P_n\|^2}$ and $A_1^* T^{M^\perp}_\phi A_1 \geq T^{M^\perp}_\phi$. Let $A = A_1 \oplus 0$, so $A_1 = QAQ$. Then $A^* T_\phi A \geq T_\phi$. It remains to show that

$\|P_nAP_n\| \leq \epsilon$. Since Q and Q_1 are self-adjoint we have

$$\begin{aligned}
\|P_nAP_n\| &= \sup_{\|f\|=\|g\|=1} |\langle P_nAP_nf, g\rangle| \\
&= \sup_{\|f\|=\|g\|=1} |\langle AP_nf, P_ng\rangle| \\
&= \sup_{\|f\|=\|g\|=1} |\langle QAQP_nf, P_ng\rangle| \\
&= \sup_{\|f\|=\|g\|=1} |\langle AQP_nf, QP_ng\rangle| \\
&\leq \sup_{\substack{\|f\|\leq\|P_n\| \\ \|g\|\leq\|P_n\|}} |\langle A_1f, g\rangle| \\
&\leq \|P_n\|^2 \sup_{\substack{\|f\|=\|g\|=1 \\ f,g\in M^{\perp}}} |\langle A_1Q_1f, Q_1g\rangle| \\
&\leq \|P_n\|^2\|Q_1AQ_1\| \leq \epsilon.
\end{aligned}$$

If further $\text{tr}(BA^*T_\phi AB) = \text{tr}(BT_\phi B)$ for every rank one projection $B \in \mathcal{L}(L^2_a(\mathbb{D}))$ then from Lemma 2.1 it follows that $A^*T_\phi A = T_\phi$. This proves (i). We shall now prove (ii). By applying Schwarz inequality [1] to the positive semi-definite form $\langle f, g\rangle \longrightarrow \langle T_\phi f, g\rangle$, $f, g \in L^2_a(\mathbb{D})$ we obtain

$$\begin{aligned}
\langle T_\phi f, f\rangle &\leq \langle \text{Re}(A^*T_\phi)f, f\rangle \\
&= \text{Re}\langle A^*T_\phi f, f\rangle \\
&\leq |\langle A^*T_\phi f, f\rangle| \\
&\leq \langle T_\phi f, f\rangle^{\frac{1}{2}}\langle T_\phi Af, Af\rangle^{\frac{1}{2}}
\end{aligned}$$

for all $f \in L^2_a(\mathbb{D})$. Hence $\langle T_\phi f, f\rangle \leq \langle A^*T_\phi Af, f\rangle$ for all $f \in L^2_a(\mathbb{D})$. That is, $T_\phi \leq A^*T_\phi A$. In addition to $T_\phi \leq \text{Re}(A^*T_\phi)$, if $T_\phi = A^*T_\phi A$ is assumed, then we obtain $\langle T_\phi f, f\rangle = \text{Re}\langle A^*T_\phi f, f\rangle = |\langle A^*T_\phi f, f\rangle| = \langle A^*T_\phi f, f\rangle$ for all $f \in L^2_a(\mathbb{D})$ and hence $T_\phi = A^*T_\phi$. Now we shall prove (iii). Since $A^*T_\phi A - T_\phi \geq 0$, it follows that $A^*(A^*T_\phi A - T_\phi)A \geq 0$. That is, ${A^*}^2T_\phi A^2 \geq A^*T_\phi A$. Repeating the process n times, we have ${A^*}^{n+1}T_\phi A^{n+1} \geq {A^*}^nT_\phi A^n$. Thus, $\{{A^*}^nT_\phi A^n \mid n = 1, 2, \ldots\}$ is an increasing sequence of positive operators. This sequence is bounded, since A^* is power bounded. Therefore, it converges to a positive operator on $L^2_a(\mathbb{D})$, say B, in the strong operator topology. Notice that

$$\begin{aligned}
A^*BA &= A^*(\lim_{n\to\infty} {A^*}^nT_\phi A^n)A \\
&= \lim_{n\to\infty} {A^*}^{n+1}T_\phi A^{n+1} \\
&= B.
\end{aligned}$$

From the operator inequality $T_\phi \le \dfrac{(A^*T_\phi + T_\phi A)}{2}$, we have

$$A^{*^n}T_\phi A^n \le \frac{[A^{*^n}(A^*T_\phi + T_\phi A)A^n]}{2} = \frac{[A^*(A^{*^n}T_\phi A^n) + (A^{*^n}T_\phi A^n)A]}{2}.$$

By letting n tend to ∞, we have $B \le \frac{(A^*B+BA)}{2} = \mathrm{Re}(A^*B)$. Thus $B = A^*B$. Since $T_\phi \le B$, it follows that the range of T_ϕ is contained in the range of B, and hence [6], we have $T_\phi = A^*T_\phi = K$. To prove (iv) suppose $\|A\| \le 1$ and $A^*T_\phi A \ge T_\phi$. Now

$$\begin{aligned}(T_\phi^{1/2}A)^*(T_\phi^{1/2}A) - (T_\phi^{1/2}A)(T_\phi^{1/2}A)^* &= A^*T_\phi A - T_\phi^{1/2}AA^*T_\phi^{1/2} \\ &= A^*T_\phi A - T_\phi^{1/2}AA^*T_\phi^{1/2} \\ &\ge T_\phi - T_\phi^{1/2}AA^*T_\phi^{1/2} \\ &= T_\phi^{1/2}(I - AA^*)T_\phi^{1/2} \\ &\ge 0\end{aligned}$$

and therefore $T_\phi^{1/2}A$ is a hyponormal operator. To prove (v), we can assume without loss of generality that $\lambda = 1$. Let h be any unit vector from the range of E. Multiplying the equations $ET_\phi E = E$ and $ET_\phi^{-1}E = E$ by $F_h = h \otimes h$ from the left and also from the right we obtain $(h \otimes h)T_\phi(h \otimes h) = h \otimes h$ and $(h\otimes h)T_\phi^{-1}(h\otimes h) = h\otimes h$. These imply $\langle T_\phi h, h\rangle = 1$ and $\left\langle T_\phi^{-1}h, h\right\rangle = 1$. Consider the Cauchy-Schwarz inequality for the new inner product

$$(f, g) = \left\langle T_\phi^{-1}f, g\right\rangle, f, g \in L^2_a(\mathbb{D}).$$

Insert $f = T_\phi h$ and $g = h$. As h is a unit vector, we see that there is equality in the corresponding inequality

$$\left|\left\langle T_\phi^{-1}T_\phi h, h\right\rangle\right|^2 \le \left\langle T_\phi^{-1}T_\phi h, T_\phi h\right\rangle \left\langle T_\phi^{-1}h, h\right\rangle.$$

This gives us that $T_\phi h$ is a nonzero scalar multiple of h. It is clear that this scalar is necessarily 1. So we have $T_\phi h = h$ for any unit vector h from the range of E. This proves our claim. The proof of (vi) follows from Lemma 2.2. To prove (vii), observe that $S_\psi^* = S_{\psi^+}$ where $\psi^+(z) = \overline{\psi(\overline{z})}$. From Lemma 2.5, it follows that $S_{\psi^+}T_\phi S_\psi = T_\phi$ and from (iv) we obtain $T_\phi^{1/2}S_\psi$ is a hyponormal operator. □

Theorem 3.3. *Let T_ϕ be a positive Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$. Then there exists a sequence $\{A_n\}$ of operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ such that $A_n \longrightarrow 0$ in weak operator topology and $A_n^*T_\phi A_n \ge T_\phi$ for all n. Thus $\widetilde{A_n^*T_\phi A_n}(z) \ge \widetilde{\phi}(z)$ for all $z \in \mathbb{D}$.*

Proof. We take an index set I for the set of all pairs $n_\epsilon = (P_n, \epsilon)$ where P_n is the finite dimensional projection on $L^2_a(\mathbb{D}), \epsilon > 0$. Set $(P_m, \epsilon_1) \prec (P_r, \epsilon_2)$ if $m \le r$ and $\epsilon_1 > \epsilon_2$. By Theorem 3.2, for each n_ϵ there exists $A_n \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\|P_nAP_n\| \le \epsilon$ and $A_n^*T_\phi A_n \ge T_\phi$. Let $U_{n_\epsilon} = \{A \in \mathcal{L}(L^2_a(\mathbb{D})) : \|P_nAP_n\| < \epsilon\}$. It is not difficult to see that each $n_\epsilon \in I$ defines a WOT-neighbourhood U_{n_ϵ} of 0. It is also clear that in this way we obtain a basis of the weak operator topology neighbourhoods of 0. Furthermore notice that for each n_ϵ, we have $A_m \in U_{n_\epsilon}$ for all $m > n_\epsilon$. Hence $A_n \longrightarrow 0$ in the weak operator topology. □

4. Berezin transform of positive Toeplitz operators

In this section we show that if T_ϕ is a non-negative Toeplitz operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ then there exists a rank one operator $R_1 \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\widetilde{\phi}(z) \geq \alpha^2\widetilde{R_1}(z)$ for all $z \in \mathbb{D}$ and for some constant $\alpha \geq 0$. Here $\widetilde{\phi}$ is the Berezin transform of T_ϕ and $\widetilde{R_1}$ is the Berezin transform of R_1.

Let H and K be Hilbert spaces and let $T \in \mathcal{L}(H, K)$. A maximizing vector for T is a non-zero vector $x \in H$ such that $\|Tx\| = \|T\|\|x\|$. Thus a maximizing vector for T is one at which T attains its norm. On a Banach space, even rank one operators need not have maximizing vectors [8]. The operator $(Hx)(t) = tx(t), 0 < t < 1$, is bounded on $L^2(0,1)$ but has no maximizing vector. However, compact operators on Hilbert spaces do have maximizing vectors [8].

Theorem 4.1. *Let T_ϕ be a non-negative Toeplitz operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ with symbol $\phi \in L^\infty(\mathbb{D})$ and $\epsilon > 0$. Then there exists a non-negative operator $C \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\|C - T_\phi\| < \epsilon, R_\epsilon = C - T_\phi = \epsilon(h \otimes h)$ for some $h \in L_a^2(\mathbb{D})$ and the operator C has a maximizing vector. Further, $\widetilde{\phi}(z) \geq \alpha^2\widetilde{R_1}(z)$ for all $z \in \mathbb{D}$ and for some constant $\alpha \geq 0$.*

Proof. Let T_ϕ be a non-negative Toeplitz operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ and $\epsilon > 0$. Now

$$\|T_\phi\| = \sup_{\substack{g \in L_a^2(\mathbb{D}) \\ \|g\|=1}} \langle T_\phi g, g\rangle = \sup\{\langle T_\phi g, g\rangle : \|g\| = 1, g \in (\ker T_\phi)^\perp\}.$$

Hence there exists a unit vector $h \in (\ker T_\phi)^\perp$ such that $\|T_\phi\| - \frac{\epsilon}{2} \leq \langle T_\phi h, h\rangle$. Define $R_\epsilon k = \epsilon\langle k, h\rangle h = \epsilon(h \otimes h)k$. Then R_ϵ is a non-negative operator of rank one and $\|R_\epsilon\| = \epsilon$. Moreover,

$$\begin{aligned} \|T_\phi + R_\epsilon\| &= \sup_{\|f\|=1} \langle (T_\phi + R_\epsilon)f, f\rangle \\ &\geq \langle (T_\phi + R_\epsilon)h, h\rangle \\ &\geq \|T_\phi\| + \frac{\epsilon}{2}. \end{aligned}$$

Now $T_\phi + R_\epsilon$ is non-negative, and so $\|T_\phi + R_\epsilon\|$ lies in the spectrum of $T_\phi + R_\epsilon$. Since R_ϵ is compact, Weyl's theorem implies essential spectrum of $T_\phi + R_\epsilon$ is equal to the essential spectrum of T_ϕ. But the spectrum of T_ϕ is bounded by $\|T_\phi\|$ and hence $\|T_\phi + R_\epsilon\|$ must lie in the discrete spectrum of $T_\phi + R_\epsilon$. In other words, there exists a unit vector $f \in L_a^2(\mathbb{D})$ such that $(T_\phi + R_\epsilon)f = \|T_\phi + R_\epsilon\|f$. Finally, we can assume without loss of generality that $f \in (\ker T_\phi)^\perp$. This is so, since $L_a^2(\mathbb{D}) = \ker T_\phi \oplus (\ker T_\phi)^\perp$ and if $f = f_1 + f_2, f_1 \in \ker T_\phi, f_2 \in (\ker T_\phi)^\perp$ then

$$(T_\phi + R_\epsilon)f_1 = \langle f_1, h\rangle h = 0.$$

Thus if we write $C = T_\phi + R_\epsilon$ then C is non-negative, $\|C - T_\phi\| = \|R_\epsilon\| = \epsilon$ and $\|Cf\| = \|C\|\|f\|$. That is, f is a maximizing vector of C. Now let $\epsilon = 1$. Then $R_1 = (h \otimes h), \|h\| = 1$. Let

$$E = \left\{X \in \mathcal{L}(L_a^2(\mathbb{D})) : X \geq 0, |\langle Xf, g\rangle|^2 \leq \langle T_\phi f, f\rangle \langle R_1 g, g\rangle \text{ for all} f, g \in L_a^2(\mathbb{D})\right\}.$$

Now suppose $X \in E$. Then for $f, g \in L_a^2(\mathbb{D})$,

$$\begin{aligned}\left\langle \begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle &= \langle T_\phi f, f\rangle + \langle Xg, f\rangle + \langle Xf, g\rangle + \langle R_1 g, g\rangle \\ &= \langle T_\phi f, f\rangle + \langle R_1 g, g\rangle + 2\mathrm{Re}\,\langle Xf, g\rangle \\ &\geq 2\,\langle T_\phi f, f\rangle^{1/2} \langle R_1 g, g\rangle^{1/2} + 2\mathrm{Re}\,\langle Xf, g\rangle \\ &\geq 2|\,\langle Xf, g\rangle\,| + 2\mathrm{Re}\,\langle Xf, g\rangle \\ &\geq 2|\langle Xf, g\rangle| - 2|\langle Xf, g\rangle| = 0.\end{aligned}$$

Conversely, if $X \geq 0$ and $\begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix}$ is a positive operator in $\mathcal{L}(L_a^2 \oplus L_a^2)$ then

$$\begin{aligned}\left|\left\langle \begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\rangle\right|^2 \leq &\left\langle \begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle \\ &\left\langle \begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\rangle\end{aligned}$$

for all $f, g \in L_a^2(\mathbb{D})$. A simplification of these inner products yields

$$|\langle Xf, g\rangle|^2 \leq \langle T_\phi f, f\rangle\langle R_1 g, g\rangle \text{ for all } f, g \in L_a^2(\mathbb{D}).$$

Hence $X \in E$. Thus

$$E = \left\{X \in \mathcal{L}(L_a^2(\mathbb{D})) : X \geq 0 \text{ and} \begin{pmatrix} T_\phi & X \\ X & R_1 \end{pmatrix} \text{is a positive operator in } \mathcal{L}(L_a^2 \oplus L_a^2)\right\}.$$

We shall now verify that $\max_{X \in E} X = \alpha R_1 = \alpha(h \otimes h)$ for some constant $\alpha \geq 0$. Suppose T_ϕ is a positive invertible operator in $\mathcal{L}(L_a^2(\mathbb{D}))$. Then from [2], [3] it follows that $\max_{X \in E} X = \frac{1}{\|T_\phi^{-\frac{1}{2}} h\|} h \otimes h = \frac{1}{\|T_\phi^{-\frac{1}{2}} h\|} R_1$, a scalar multiple of R_1. If T_ϕ is an arbitrary positive operator then it follows from [2] that $\max_{X \in E} X$ is again a scalar multiple of R_1, and

$$\max_{X \in E} X = \max\left\{rR_1 : r \geq 0, \begin{pmatrix} T_\phi & rR_1 \\ rR_1 & R_1 \end{pmatrix} \geq 0\right\}.$$

The inequality $\left\langle \begin{pmatrix} T_\phi & rR_1 \\ rR_1 & R_1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \geq 0$ is equivalent to

$$\langle T_\phi f, f\rangle + r\langle R_1 g, f\rangle + r\langle R_1 f, g\rangle + \langle R_1 g, g\rangle \geq 0 \text{ for all } f, g \in L_a^2(\mathbb{D}).$$

This can be rewritten as

$$\langle T_\phi f, f\rangle + \|R_1(g + rf)\|^2 - r^2\|R_1 f\|^2 \geq 0$$

which holds for all $f, g \in L_a^2(\mathbb{D})$ if and only if $\langle T_\phi f, f\rangle - r^2\|R_1 f\|^2 \geq 0$ or equivalently, $r^2 R_1 \leq T_\phi$. Thus from [3], it follows that $\max_{X\in E} X = \max\{rR_1 : r \geq 0, r^2 R_1 \leq T_\phi\} = \sqrt{\lambda(T_\phi, R_1)}R_1$ where

$$\lambda(T_\phi, R_1) = \begin{cases} \|T_\phi^{-\frac{1}{2}} h\|^{-2}, \text{if } h \in \text{Range}(T_\phi^{\frac{1}{2}}), \\ 0, \text{otherwise.} \end{cases}$$

Thus $\max_{X\in E} X = \alpha R_1$, for some $\alpha \geq 0$. Hence $\begin{pmatrix} T_\phi & \alpha R_1 \\ \alpha R_1 & R_1 \end{pmatrix} \geq 0$. That is, $|\langle \alpha R_1 k_z, k_w\rangle|^2 \leq \langle T_\phi k_z, k_z\rangle\langle R_1 k_w, k_w\rangle$ for all $z, w \in \mathbb{D}$. Hence

$$|\alpha|^2 |\langle k_z, h\rangle\langle h, k_w\rangle|^2 \leq \widetilde{\phi}(z)|\langle h, k_w\rangle|^2 \text{ for all } z, w \in \mathbb{D}.$$

If $h \neq 0$ then there exists $w \in \mathbb{D}$ such that $\langle h, k_w\rangle \neq 0$. Thus $\widetilde{\phi}(z) \geq |\alpha|^2|\langle h, k_z\rangle|^2 = \alpha^2\widetilde{R_1}(z)$ for all $z \in \mathbb{D}$. □

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[1] P.G.Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar-751004, Odisha, India.

E-mail address: namitadas440@yahoo.co.in

[2] School of Applied Sciences (Mathematics), KIIT University, Campus-3(Kathajori Campus), Bhubaneswar-751024, Odisha, India.

E-mail address: smita_782006@yahoo.co.in