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YOUNG TYPE INEQUALITIES FOR POSITIVE OPERATORS

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ABSTRACT. In this paper we present refinements and improvement of the Young inequality in the context of linear bounded operators.

1. Introduction

The most familiar form of the Young inequality, which is frequently used to prove the well-known Hölder inequality for L_p functions, is the following:

$$a^{\nu}b^{1-\nu} \leq \nu a + (1-\nu)b, \tag{1.1}$$

with $a, b \geq 0$ and $\nu \in [0,1]$, or equivalently

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.2}$$

where $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. A fundamental inequality between positive real numbers is the arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{a+b}{2}, \tag{1.3}$$

which is of interest herein. It is a particular case of (1.1) when $\nu = 1/2$. The Heinz mean is defined as

$$H_{\nu}(a,b) = \frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2}.$$

The function H_ν is symmetric about the point $\nu = \frac{1}{2}$. Note that $H_0(a,b) = H_1(a,b) = \frac{a+b}{2}$, $H_{1/2}(a,b) = \sqrt{ab}$ and

$$H_{1/2}(a,b) \leq H_{\nu}(a,b) \leq H_0(a,b) \tag{1.4}$$

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for $0 \le \nu \le 1$, i.e., the Heinz means interpolates between the geometric mean and the arithmetic mean. Recently, Kittaneh and Manasrah [14] obtained a refinement of (1.1)

$$a^{\nu}b^{1-\nu} + r_0(\sqrt{a}-\sqrt{b})^2 \le \nu a + (1-\nu)b \tag{1.5}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Along this work $\mathscr{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $(\mathbb{B}(\mathscr{H}), \|\cdot\|)$ be the C^*-algebra of all bounded linear operators acting on $(\mathscr{H}, \langle \cdot, \cdot \rangle)$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$.

A selfadjoint operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \ge 0$ for every $x \in \mathscr{H}$ and the cone of positive operators is denoted by $\mathbb{B}(\mathscr{H})_+$. A unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $\mathfrak{J}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ associated with it and has the property $|||UXV||| = |||X|||$, where $U, V \in \mathbb{B}(\mathscr{H})$ are unitaries and $X \in \mathfrak{J}_{|||\cdot|||}$. Whenever we write $|||X|||$, we mean that $X \in \mathfrak{J}_{|||\cdot|||}$

Now, we will explain historical background of the operator inequalities related to the previous classical inequalities. Heinz [6] proved that for operators A, B, X such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\nu \in [0,1]$

$$||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|| \le ||AX + XB||. \tag{1.6}$$

The proof is based on the complex analysis and is somewhat complicated. McIntosh [15] showed that the Heinz inequality is a consequence of the following inequality

$$2\,\|AXB\| \;\le \|A^*AX + XBB^*\|\,,$$

where $A, B, X \in \mathbb{B}(\mathscr{H})$. In the literature, the above inequality is called the arithmetic–geometric mean inequality in the context of bounded linear operators.

Bhatia and Davis [2] obtained the following double inequality

$$2|||A^{1/2}XB^{1/2}||| \le |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \le |||AX + XB|||,$$

for matrices, which of course remains valid for Hilbert space operators $A, B \ge 0$ and X by a standard approximation argument. Indeed, it has been proved that $F(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is a convex function of ν on $[0,1]$ with symmetry about $\nu = 1/2$, which attains its minimum at $\nu = 1/2$ and its maximum at $\nu = 0$ and $\nu = 1$.

On the other hand, T. Ando ([1]) showed that the Young inequality fails to hold for the operator norm, however he obtained that the following slightly weaker inequality holds

$$|||A^{1/p}XB^{1/q}||| \le \frac{1}{p}|||AX||| + \frac{1}{q}|||XB|||\,. \tag{1.7}$$

For a detailed study of these and associated norm inequalities along with their history of origin, refinements and applications, one may refer to [2, 11, 7, 8, 9, 3].

One of the purposes of the present article is to obtain a new refinement of (1.7) and new proofs of results previously obtained by Kittaneh and Manasrah [14].

2. Refinement of the Young inequality

An important result related to the improvement of the inequality between arithmetic and geometric means was obtained Kittaneh and Manasrah in [14] :

$$a^{\nu}b^{1-\nu} + r_0(\sqrt{a}-\sqrt{b})^2 \le \nu a + (1-\nu)b, \tag{2.1}$$

with $r_0 = \min\{\nu, 1-\nu\}$. The inequality (2.1) was previously obtained by Kober [10].

However, Minculete [16] gave a refinement for inequality (2.1), which improves the Young inequality.

Proposition 2.1. *For any $a, b > 0$ and any $\nu \in (0,1)$,*

$$r_0(\sqrt{a}-\sqrt{b})^2 + A(\nu)\log^2\left(\frac{a}{b}\right) \leq \nu a + (1-\nu)b - a^\nu b^{1-\nu} \tag{2.2}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $A(\nu) = [\frac{\nu(1-\nu)}{2} - \frac{r_0}{4}]\min\{a, b, ab, 1\}$.

Now, we try to obtain a new refinement of (1.7). To achieve this, we need the following well-known result.

Lemma 2.2. *If $A, B \in \mathbb{B}(\mathscr{H})_+$ and $X \in \mathbb{B}(\mathscr{H})$, then*

$$|||A^\nu X B^{1-\nu}||| \leq |||AX|||^\nu |||XB|||^{1-\nu} \tag{2.3}$$

for every unitarily norm $|||.|||$ and every $\nu \in [0,1]$.

The proof of this lemma can be found in [12, Theorem 2]. In addition, Kittaneh showed that (2.3) is equivalent to the following generalization of the classical Heinz inequality

$$|||A^\nu X B^\nu||| \leq |||AXB|||^\nu |||X|||^{1-\nu}.$$

On the other hand, Kosaki [11] gave a new proof of (2.3) using the well-known Poisson integral formula. Also, Yamazaki [19] used the previous inequality to characterize the chaotic order relation and to study Aluthge transformations.

In view of inequalities (2.2) and (2.3), we can improve the following inequality.

Theorem 2.3. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\nu \in (0,1)$ and $|||\cdot|||$ be a unitarily invariant norm. Then*

$$\begin{aligned} |||A^\nu X B^{1-\nu}||| &+ r_0(\sqrt{|||AX|||} - \sqrt{|||XB|||})^2 + A(\nu)\log^2\left(\frac{|||AX|||}{|||XB|||}\right) \\ &\leq \nu|||AX||| + (1-\nu)|||XB|||, \end{aligned} \tag{2.4}$$

where $A(\nu) = [\frac{\nu(1-\nu)}{2} - \frac{r_0}{4}]\min\{|||AX|||, |||XB|||, |||AX|||\ |||XB|||, 1\}$ and $r_0 = \min\{\nu, 1-\nu\}$ and

We continue this section with the following technical lemma.

Lemma 2.4. *Let $a, b \geq 0$ and $\nu \in [0,1]$. Then*

$$(2a^\nu b^{1-\nu} + (a^\nu - b^{1-\nu})^2)^2 = a^{4\nu} + b^{4(1-\nu)} + 2a^{2\nu}b^{2(1-\nu)} = (a^{2\nu} + b^{2(1-\nu)})^2. \tag{2.5}$$

Taking $\nu = 1/2$ in the double inequality (2.5), we obtain

$$\sqrt{ab} + \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 = \frac{1}{2}\sqrt{a^2+b^2+2ab} = \frac{1}{2}(a+b),$$

which is a refinement of the scalar arithmetic-geometric mean inequality. Furthermore, we remark that the previous equality characterize the case when the arithmetic and geometric mean coincide. The first inequality that we can obtain immediately using the previous relationship is the following:

Proposition 2.5. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then*

$$2|||A^\nu XB^{1-\nu}||| + \left(|||AX|||^\nu - |||XB|||^{1-\nu}\right)^2 \leq |||AX|||^{2\nu} + |||XB|||^{2(1-\nu)} \quad (2.6)$$

for every $\nu \in [0,1]$ and every unitarily invariant norm $|||\cdot|||$.

Proof. It follows from inequality (2.3) that

$$\begin{aligned}\left(|||AX|||^\nu - |||XB|||^{1-\nu}\right)^2 &= |||AX|||^{2\nu} + |||XB|||^{2(1-\nu)} - 2|||AX|||^\nu |||XB|||^{1-\nu} \\ &\leq |||AX|||^{2\nu} + |||XB|||^{2(1-\nu)} - 2|||A^\nu XB^{1-\nu}|||.\end{aligned}$$

□

Now, we are ready to state our operator version of inequality (2.5)

Theorem 2.6. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then for every $\nu \in [0,1]$ and every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}2|||A^\nu XB^{1-\nu}||| &+ 2r_0\left(|||AX|||^\nu - |||XB|||^{1-\nu}\right)^2 \\ &\leq 2|||A^\nu XB^{1-\nu}||| + \left(|||AX|||^\nu - |||XB|||^{1-\nu}\right)^2 \\ &\leq \sqrt{|||AX|||^{4\nu} + |||XB|||^{4(1-\nu)} + 2|||A^\nu XB^{1-\nu}|||^2} \\ &\leq |||AX|||^{2\nu} + |||XB||^{2(1-\nu)}, \quad (2.7)\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Proof. Let $\beta = (|||AX|||^\nu - |||XB|||^{1-\nu})^2$ and $\alpha = (2|||A^\nu XB^{1-\nu}||| + \beta)^2$. It follows from (2.3) that

$$\begin{aligned}\alpha &\leq \alpha + 4\beta\left(|||AX|||^\nu |||XB|||^{1-\nu} - |||A^\nu XB^{1-\nu}|||\right) \\ &\leq 4|||A^\nu XB^{1-\nu}|||^2 + \beta[\beta + 4|||AX|||^\nu |||XB|||^{1-\nu}] \\ &= 4|||A^\nu XB^{1-\nu}|||^2 + \beta[\,|||AX|||^{2\nu} + |||XB|||^{2(1-\nu)} + 2|||AX|||^\nu |||XB|||^{1-\nu}] \\ &= 4|||A^\nu XB^{1-\nu}|||^2 + \beta[\,|||AX|||^\nu + |||XB|||^{(1-\nu)}]^2 \\ &= 4|||A^\nu XB^{1-\nu}|||^2 + [\,|||AX|||^{2\nu} - |||XB|||^{2(1-\nu)}]^2 \\ &= 4|||A^\nu XB^{1-\nu}|||^2 + |||AX|||^{4\nu} + |||XB|||^{4(1-\nu)} - 2|||AX|||^{2\nu}|||XB|||^{2(1-\nu)} \\ &\leq 4|||A^\nu XB^{1-\nu}|||^2 + |||AX|||^{4\nu} + |||XB|||^{4(1-\nu)} - 2|||A^\nu XB^{1-\nu}|||^2 \\ &\leq |||AX|||^{4\nu} + |||XB|||^{4(1-\nu)} + 2|||A^\nu XB^{1-\nu}|||^2 \\ &\leq |||AX|||^{4\nu} + |||XB|||^{4(1-\nu)} + 2|||AX|||^{2\nu}|||XB|||^{2(1-\nu)} \\ &= \left(|||AX|||^{2\nu} + |||XB|||^{2(1-\nu)}\right)^2\end{aligned}$$

□

If we consider $\nu = 1/2$ in Theorem 2.6, we get the following result, which is a refinement of (1.7).

Corollary 2.7. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned} 2|||A^{1/2}XB^{1/2}||| &+ \left(\sqrt{|||AX|||} - \sqrt{|||XB|||}\right)^2 \\ &\leq \sqrt{|||AX|||^2 + |||XB|||^2 + 2|||A^{1/2}XB^{1/2}|||^2} \\ &\leq |||AX||| + |||XB|| \end{aligned} \tag{2.8}$$

In particular, for the Hilbert-Schmidt norm $\|\cdot\|_2$ it holds that

$$\begin{aligned} 2||A^{1/2}XB^{1/2}||_2 + \left(\sqrt{||AX||_2} - \sqrt{||XB||_2}\right)^2 &\leq ||AX + XB||_2 \\ &\leq ||AX||_2 + ||XB||_2 \end{aligned} \tag{2.9}$$

Proof. We only note that the following equality holds:

$$||AX + XB||_2^2 = ||AX||_2^2 + ||XB||_2^2 + 2||A^{1/2}XB^{1/2}||_2^2. \tag{2.10}$$

□

We remark that inequality (2.9) has been obtained in [14, Theorem 3.3] by using a different technique and it is a refinement of the arithmetic-geometric mean inequality for the Hilbert-Schmidt norm.

3. Refinements of the Heinz inequality

Applying the triangle inequality and Theorem 2.3 we reach the following result.

Corollary 3.1. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\nu \in (0,1)$. Then*

$$\begin{aligned} |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| + 2r_0(\sqrt{|||AX|||} - \sqrt{|||XB|||})^2 + 2A(\nu)\log^2\left(\frac{|||AX|||}{|||XB|||}\right) \\ \leq |||AX||| + |||XB||| \end{aligned} \tag{3.1}$$

where $A(\nu) = [\frac{\nu(1-\nu)}{2} - \frac{r_0}{4}]\min\{|||AX|||, |||XB|||, |||AX||||||XB|||, 1\}$ and $r_0 = \min\{\nu, 1-\nu\}$ and

It is shown in [13, Corollary 3], utilizing the convexity of $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$ for $\nu \in [0,1]$ and a basic property of convex functions, that

$$|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leq 4r_0|||A^{1/2}XB^{1/2}||| + (1-2r_0)|||AX + XB|||, \tag{3.2}$$

where $A, B \in \mathbb{B}(\mathscr{H})_+$ and $r_0 = \min\{\nu, 1-\nu\}$.

Remark 3.2. A natural generalization (or refinement) of (3.2) would be

$$|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leq |||4r_0A^{1/2}XB^{1/2} + (1-2r_0)(AX + XB)|||$$

which in fact is not true, in general. The following counterexample justifies this:

Take $X = \begin{pmatrix} 52.39 & 38.71 & 12.36 \\ 32.86 & 35.38 & 64.82 \\ 91.79 & 99.45 & 66.10 \end{pmatrix}$, $A = \begin{pmatrix} 92.315 & 87.791 & 71.090 \\ 87.791 & 120.130 & 83.340 \\ 71.090 & 83.340 & 103.610 \end{pmatrix}$,
$B = \begin{pmatrix} 118.482 & 23.249 & 112.676 \\ 23.249 & 10.343 & 38.224 \\ 112.676 & 38.224 & 156.551 \end{pmatrix}$ and $\nu = 0.4680$.

Then $\mathrm{tr}|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu| = 78135.5$, while $\mathrm{tr}|4r_0A^{1/2}XB^{1/2} + (1-2r_0)(AX + XB)| = 78125.4$.

However, we [4] obtained another result, which is a possible generalization of (3.2) for matrices.

Theorem 3.3. *Let $A, B, X \in \mathcal{M}_n$ and A, B positive definite matrices. Then for $\nu \in [0, 1]$ and for every unitarily invariant norm $|||\cdot|||$, it holds that*

$$|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \le |||4r_1A^{1/2}XB^{1/2} + (1-2r_1)(AX + XB)|||\,, \tag{3.3}$$

where $r_1(\nu) = \min\{\nu, |\frac{1}{2} - \nu|, 1-\nu\}$.

Now, we present new proofs and some refinements of various Young type inequalities obtained by Kittaneh and Manasrah [14]. The key tools of this approach are inequalities (2.9) and (3.2).

Corollary 3.4. [14, Theorem 3.4 and 3.8] *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$. Then*

$$\begin{aligned} &|||\ A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| + 2r_0\left(\sqrt{|||AX|||} - \sqrt{|||XB|||}\right)^2 \\ \le\ & 2r_0\sqrt{|||AX|||^2 + |||XB|||^2 + 2|||A^{1/2}XB^{1/2}|||^2} + (1-2r_0)|||AX + XB||| \\ \le\ & |||AX||| + |||XB||| \end{aligned} \tag{3.4}$$

In particular, for the Hilbert-Schmidt norm $\|\cdot\|_2$,

$$\begin{aligned} ||A^\nu XB^{1-\nu} \ + \ & A^{1-\nu}XB^\nu||_2 + 2r_0\left(\sqrt{||AX||_2} - \sqrt{||XB||_2}\right)^2 \le ||AX + XB||_2 \\ \le\ & ||AX||_2 + ||XB||_2 \end{aligned} \tag{3.5}$$

holds.

Proof. Let $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$. Theorem 2.6 yields that

$$\begin{aligned} F(\nu) &+ 2r_0\left(\sqrt{|||AX|||} - \sqrt{|||XB|||}\right)^2 \\ \le\ & 2r_0\left(2|||A^{1/2}XB^{1/2}||| + \left(\sqrt{|||AX|||} - \sqrt{|||XB|||}\right)^2\right) + (1-2r_0)|||AX + XB||| \\ \le\ & 2r_0\sqrt{|||AX|||^2 + |||XB|||^2 + 2|||A^{1/2}XB^{1/2}|||^2} + (1-2r_0)|||AX + XB||| \\ \le\ & |||AX||| + |||XB|||\,. \end{aligned} \tag{3.6}$$

□

Mimicking the previous proof we obtain the following inequality for matrices. We remark that both inequalities are refinements of Theorem 3.4 and Theorem 3.8 of [14].

Proposition 3.5. *Let $A, B, X \in \mathcal{M}_n$ and A, B be positive definite matrices. Then for $\nu \in [0,1]$ and for every unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned} &||| \ A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu ||| + 2r_1 \left(\sqrt{|||AX|||} - \sqrt{|||XB|||} \right)^2 \\ \leq & \ 2r_1 \sqrt{|||AX|||^2 + |||XB|||^2 + 2|||A^{1/2} X B^{1/2}|||^2} + (1-2r_1)|||AX + XB||| \\ \leq & \ |||AX||| + |||XB||| \end{aligned} \tag{3.7}$$

where $r_1(\nu) = \min\{\nu, |\frac{1}{2} - \nu|, 1-\nu\}$. *In particular,*

$$\begin{aligned} ||A^\nu X B^{1-\nu} \ \ &+ \ \ A^{1-\nu} X B^\nu||_2 + 2r_1 \left(\sqrt{||AX||_2} - \sqrt{||XB||_2} \right)^2 \leq ||AX + XB||_2 \\ &\leq \ \ ||AX||_2 + ||XB||_2 \, . \end{aligned} \tag{3.8}$$

Theorem 3.6. [14, Theorem 3.5] *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\nu \in [0,1]$ and $|||\cdot|||$ be a unitarily invariant norm. Then*

$$||A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu||_2^2 + 2r_0 ||AX - XB||_2^2 \leq ||AX + XB||_2^2 \, . \tag{3.9}$$

Proof. Since $(B_2(\mathscr{H}), \|.\|_2)$ is a Hilbert space, by the parallelogram law (see also [17]), we have

$$\|AX + XB\|_2^2 + \|AX - XB\|_2^2 = 2(\|AX\|_2^2 + \|XB\|_2^2) \, . \tag{3.10}$$

Combining this equality with (2.10) and (3.2), we get

$$\begin{aligned} &|| \ A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu||_2^2 + 2r_0||AX - XB||_2^2 \\ \leq & \ \ 2r_0(2||A^{1/2} X B^{1/2}||_2^2 + \|AX\|_2^2 + ||XB||_2^2) + (1-2r_0)\|AX + XB\|_2^2 \\ = & \ \ \|AX + XB\|_2^2 \, . \end{aligned} \tag{3.11}$$

□

Now, we obtain a refinement of (3.2). To do this we need the following basic property of convex functions.

Lemma 3.7. *([5], [18, Theorem 1.3.1]) Let f be a real function defined on an interval I and $a \in I$. Then f is convex (respectively, strictly convex) if and only if the associated functions s_a are nondecreasing (respectively, increasing), where $s_a(x) = \frac{f(x)-f(a)}{x-a}$.*

As a consequence of this statement, it follows that

$$s_x(x_1) = \frac{f(x_1) - f(x)}{x_1 - x} \leq \frac{f(x_2) - f(x)}{x_2 - x} = s_x(x_2),$$

or equivalently,

$$f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}. \tag{3.12}$$

for $a \leq x_1 \leq x \leq x_2 \leq b$ and f convex.

Theorem 3.8. *Let A, B, X be operators such that $A, B \in \mathbb{B}(\mathscr{H})_+$ and $\nu \in [0,1]$ and $|||\cdot|||$ be a unitarily invariant norm. Then*

$$F(\nu) \leq \begin{cases} (1-4r_0)F(0) + 4r_0 F\left(\frac{1}{4}\right) & \text{if } \quad \nu \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \\ (4r_0 - 1)F\left(\frac{1}{2}\right) + 2(1-2r_0)F\left(\frac{1}{4}\right) & \text{if } \quad \nu \in [\frac{1}{4}, \frac{3}{4}] \end{cases} \tag{3.13}$$

where $F(\nu) = |||A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu|||$ and $r_0 = \min\{\nu, 1-\nu\}$.

Proof. It is an immediate consequence from the convexity of F and Lemma 3.7. □

Remark 3.9. If $\nu \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, we have

$$\begin{aligned} 2r_0 F\left(\frac{1}{2}\right) \quad &+ \quad (1-2r_0)F(0) - \left[(1-4r_0)F(0) + 4r_0 F\left(\frac{1}{4}\right)\right] \\ &= \quad 2r_0 \left[F\left(\frac{1}{2}\right) + F(0) - 2F\left(\frac{1}{4}\right)\right]. \end{aligned}$$

Analogously, if $\nu \in [\frac{1}{4}, \frac{3}{4}]$, then

$$\begin{aligned} 2r_0 F\left(\frac{1}{2}\right) \quad &+ \quad (1-2r_0)F(0) - \left[(4r_0-1)F\left(\frac{1}{2}\right) + 2(1-2r_0)F\left(\frac{1}{4}\right)\right] \\ &= \quad (1-2r_0)\left[F\left(\frac{1}{2}\right) + F(0) - 2F\left(\frac{1}{4}\right)\right]. \end{aligned}$$

Using Lemma 3.7 for the function F, we infer that $s_{1/4}(1/2) \geq s_{1/4}(0)$ and this is equivalent to the inequality

$$\left[F\left(\frac{1}{2}\right) + F(0) - 2F\left(\frac{1}{4}\right)\right] \geq 0.$$

So, inequality (3.13) is a refinement of (3.2).

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