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ON THE CESÁRO OPERATOR IN WEIGHTED ℓ^2-SEQUENCE SPACES AND THE GENERALIZED CONCEPT OF NORMALITY

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ABSTRACT. The weighted Cesáro operator C_h in $\ell^2(h)$-spaces is investigated in terms of several concepts of normality, where h denotes a positive discrete measure on $\mathbb{N}_0$. We classify exactly those h for which C_h is hyponormal. Two examples related to the Haar measures of orthogonal polynomials are discussed. We show that the Cesáro operator is not always paranormal. Furthermore, we prove that the Cesáro operator is not quasinormal for any choice of h.

1. Introduction and preliminaries

In this paper we discuss the Cesáro operator in weighted ℓ^2-spaces. For a sequence $h = (h(n))_{n\in\mathbb{N}_0}$ of positive numbers, called weights and a sequence $a = (a(n))_{n\in\mathbb{N}_0}$ of complex numbers the discrete weighted Cesáro operator C_h is defined by

$$(C_h a)(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k)a(k), \quad \text{with} \quad H(n) = \sum_{k=0}^{n} h(k). \tag{1.1}$$

Let $1 < p < \infty$ and

$$\ell^p(h) = \{a = (a(n))_{n\in\mathbb{N}_0} : \ a(n) \in \mathbb{C}, \|a\|_{p,h}^p := \sum_{n=0}^{\infty} h(n)\,|a(n)|^p < \infty\}. \tag{1.2}$$

It is well known that the Cesáro operator in $\ell^p(h)$ is bounded by $\|C_h\| \le \frac{p}{p-1}$, see [3, 7, 8, 9]. An easy computation shows that the dual operator C_h^* of C_h in $l^q(h)$,

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$\frac{1}{p} + \frac{1}{q} = 1$, is

$$(C_h^* a)(n) = \sum_{k=n}^{\infty} \frac{h(k)a(k)}{H(k)}. \tag{1.3}$$

In the Hilbert space $\ell^2(h)$ the inner product is defined by

$$\langle a, b\rangle_h = \sum_{n=0}^{\infty} h(n)a(n)\overline{b(n)}, \quad a, b \in \ell^2(h). \tag{1.4}$$

We will focus our attention on the weighted Cesáro operator in $\ell^2(h)$ and the property of normality in Hilbert spaces. One obtains the classical sequence space ℓ^2 when choosing $h = (1, 1, 1, \cdots)$. By weakening the conditions of normality in various ways, the following classes of not necessarily normal operators are obtained, see [5] and [6, Problems 137, 195, 203, 216]:

Definition 1.1 (generalised concept of normality)**.** Let H be a Hilbert space and T be a bounded linear operator in H, symbolically $T \in B(H)$. Then, T is called

(1) **normal**, if and only if $T^*T = TT^*$.
(2) **quasinormal**, if and only if $T^*TT = TT^*T$.
(3) **subnormal**, if and only if T has a normal extension, i.e. there exists a Hilbert space K, H can be embedded in K, and a normal operator $N \in B(K)$, which has the shape $N = \begin{pmatrix} T & B \\ 0 & A \end{pmatrix}$, where A, B are bounded operators.
(4) **hyponormal**, if and only if $T^*T \geq TT^*$, i.e. $T^*T - TT^*$ is positive.
(5) **paranormal**, if and only if $\|T^2x\| \geq \|Tx\|^2$ for all $x \in H$ with $\|x\| = 1$.

As shown in [5], the following inclusion relations hold for the operator classes and all of them are proper.

normal operators $\subset$ quasinormal operators $\subset$ subnormal operators
$\subset$ hyponormal operators $\subset$ paranormal operators.

In their 1965 paper, Brown, Halmos and Shields showed that the Cesáro operator in ℓ^2 is hyponormal, see [1]. Later on, Kriete, Trutt [10] and Cowen [4] proved the subnormality of the Cesáro operator in ℓ^2. Here we investigate the properties of the weighted Cesáro operator C_h in $\ell^2(h)$. To which class of operators from Definition 1.1 the operator C_h belongs, depends on the sequence h.
The remaining part of the paper is organised as follows: First, we study necessary and sufficient conditions for the hyponormality of the Cesáro operator. Then, the Haar measures of Jacobi polynomials and polynomials related to homogeneous trees are discussed as examples of weights for which C_h becomes hyponormal. Afterwards, we analyse a sequence of weights for which C_h is not paranormal. Last but not least, we show that C_h never satisfies the conditions of quasinormality, independently of the choice of h.

2. Weighted Cesáro Operator and the Generalised Concept of Normality

Let e_j be the j^{th} unit sequence, $e_j(i) = \delta_{ij},\ i,j \in \mathbb{N}_0$. As C_h and C_h^* are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j\in\mathbb{N}_0}$ of $l^2(h)$ (in the following also denoted by C_h and C_h^*, respectively). From (1.1) and (1.3) we can infer that

$$C_h = \begin{pmatrix} \frac{h(0)}{H(0)} & & 0 & \\ \frac{h(0)}{H(1)} & \frac{h(1)}{H(1)} & & \\ \frac{h(0)}{H(2)} & \frac{h(1)}{H(2)} & \frac{h(2)}{H(2)} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad {C_h}^* = \begin{pmatrix} \frac{h(0)}{H(0)} & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ 0 & & \frac{h(2)}{H(2)} & \cdots \\ & & & \ddots \end{pmatrix}.$$

Direct computation yields the matrix representations of $C_hC_h^*$ and $C_h^*C_h$ with respect to $(e_j)_{j\in\mathbb{N}_0}$:

$$C_hC_h^* = \begin{pmatrix} h(0)\alpha_0 & h(1)\alpha_1 & h(2)\alpha_2 & \cdots \\ h(0)\alpha_1 & h(1)\alpha_1 & h(2)\alpha_2 & \cdots \\ h(0)\alpha_2 & h(1)\alpha_2 & h(2)\alpha_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{with } \alpha_n = \frac{1}{H(n)} \tag{2.1}$$

and

$$C_h^*C_h = \begin{pmatrix} h(0)\beta_0 & h(1)\beta_1 & h(2)\beta_2 & \cdots \\ h(0)\beta_1 & h(1)\beta_1 & h(2)\beta_2 & \cdots \\ h(0)\beta_2 & h(1)\beta_2 & h(2)\beta_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{with } \beta_n = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2}. \tag{2.2}$$

The associated matrices of $C_hC_h^*$ and $C_h^*C_h$ in (2.1) and (2.2), respectively, have the same shape. Despite the prefactor $h(j)$ in the j^{th} column, the above matrices are "L-shaped" as analyzed in [1].

Theorem 2.1. *The weighted Cesáro operator C_h in $\ell^2(h)$ is hyponormal (i.e. $C_h^*C_h - C_hC_h^*$ is positive), if and only if*

(1)

$$\forall n \in \mathbb{N}_0 : \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq 0 \quad \textit{and}$$

(2)

$$\forall n \in \mathbb{N}_0 :\ H(n)^2 \geq H(n-1)H(n+1) \quad (H(-1) := 0).$$

Proof. Let

$$T := C_h^*C_h - C_hC_h^*,\ \gamma_n := \beta_n - \alpha_n = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)}.$$

With (2.1) and (2.2) it follows that

$$\begin{pmatrix} h(0)\gamma_0 & h(1)\gamma_1 & h(2)\gamma_2 & \cdots \\ h(0)\gamma_1 & h(1)\gamma_1 & h(2)\gamma_2 & \cdots \\ h(0)\gamma_2 & h(1)\gamma_2 & h(2)\gamma_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the associated matrix of T.
In [1] the positivity of the matrix T acting on ℓ^2 was proved by considering the determinants of its finite sections. In order to include the case when the matrix T is positive semidefinite, we give a more detailed proof for the positivity of the operator T here.
The bilinear form $\langle ., T. \rangle_h$ is defined for all sequences $a, b \in \ell^2(h)$. Using the vector representations for a and b, the matrix representation for T and the inner product as defined in (1.4), we obtain

$$\begin{aligned} \langle a, Tb \rangle_h &= \begin{pmatrix} a(0) \\ a(1) \\ \vdots \end{pmatrix}^T \begin{pmatrix} h(0) & 0 & \\ 0 & h(1) & \\ & & \ddots \end{pmatrix} \\ &\quad \times \begin{pmatrix} h(0)\gamma_0 & h(1)\gamma_1 & \cdots \\ h(0)\gamma_1 & h(1)\gamma_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b(0) \\ b(1) \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \end{pmatrix}^T T_h \begin{pmatrix} b(0) \\ b(1) \\ b(2) \\ \vdots \end{pmatrix}, \end{aligned}$$

with

$$T_h = \begin{pmatrix} h(0)h(0)\gamma_0 & h(0)h(1)\gamma_1 & h(0)h(2)\gamma_2 & \cdots \\ h(1)h(0)\gamma_1 & h(1)h(1)\gamma_1 & h(1)h(2)\gamma_2 & \cdots \\ h(2)h(0)\gamma_2 & h(2)h(1)\gamma_2 & h(2)h(2)\gamma_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore,

$$\langle e_i, Te_j \rangle_h = T_h(i,j) = h(i)h(j)\gamma_{\max(i,j)}, \quad \text{for all } i, j \in \mathbb{N}_0. \tag{2.3}$$

For $n \in \mathbb{N}_0$, let us define

$$c_n := \frac{1}{h(n)} e_n - \frac{1}{h(n+1)} e_{n+1} = (0, \cdots, 0, \frac{1}{h(n)}, -\frac{1}{h(n+1)}, 0, \cdots).$$

Now, $(c_n)_{n\in\mathbb{N}_0}$ is a basis for $l^2(h)$. Using (2.3), we conclude for all $i,j\in\mathbb{N}_0$, that

$$\begin{aligned}
\langle c_i, Tc_j\rangle_h &= \left\langle \frac{1}{h(i)}e_i - \frac{1}{h(i+1)}e_{i+1}, T\left(\frac{1}{h(j)}e_j - \frac{1}{h(j+1)}e_{j+1}\right)\right\rangle_h \\
&= \frac{\langle e_i, Te_j\rangle_h}{h(i)h(j)} - \frac{\langle e_i, Te_{j+1}\rangle_h}{h(i)h(j+1)} - \frac{\langle e_{i+1}, Te_j\rangle_h}{h(i+1)h(j)} + \frac{\langle e_{i+1}, Te_{j+1}\rangle_h}{h(i+1)h(j+1)} \\
&= \gamma_{\max(i,j)} - \gamma_{\max(i,j+1)} - \gamma_{\max(i+1,j)} + \gamma_{\max(i+1,j+1)} \\
&= \begin{cases} \gamma_i - \gamma_i - \gamma_{i+1} + \gamma_{i+1} = 0 & \text{for } i > j, \\ \gamma_i - \gamma_{i+1} - \gamma_{i+1} + \gamma_{i+1} = \gamma_i - \gamma_{i+1} & \text{for } i = j, \\ \gamma_j - \gamma_{j+1} - \gamma_j + \gamma_{j+1} = 0 & \text{for } i < j \end{cases} \\
&= (\gamma_i - \gamma_{i+1})\delta_{ij}. \qquad (2.4)
\end{aligned}$$

Assume first that T is positive, then $\langle a, Ta\rangle_h \geq 0$ for all $a\in\ell^2(h)$.
In particular, for all $n\in\mathbb{N}_0$, by (2.3) and (2.4) we have

$$0 \leq \langle e_n, Te_n\rangle_h = h(n)^2\gamma_n \Leftrightarrow \sum_{k=n}^{\infty}\frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq 0$$

and

$$\begin{aligned}
&0 \leq \langle c_n, Tc_n\rangle_h = \gamma_n - \gamma_{n+1} \\
&\Leftrightarrow \sum_{k=n}^{\infty}\frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq \sum_{k=n+1}^{\infty}\frac{h(k)}{H(k)^2} - \frac{1}{H(n+1)} \\
&\Leftrightarrow \frac{h(n)}{H(n)^2} - \frac{1}{H(n)} \geq -\frac{1}{H(n+1)} \\
&\Leftrightarrow \frac{H(n-1)}{H(n)^2} \leq \frac{1}{H(n+1)} \\
&\Leftrightarrow H(n)^2 \geq H(n+1)H(n-1),
\end{aligned}$$

which shows that the conditions (1) and (2) hold for hyponormal T.
Conversely, let us assume that $\gamma_n \geq \gamma_{n+1} \geq 0$ for all $n\in\mathbb{N}_0$.
As $(c_n)_{n\in\mathbb{N}_0}$ is a basis for $\ell^2(h)$, each sequence a in $\ell^2(h)$ has a unique representation $a = \sum_{k=0}^{\infty} a_c(k)c_k$, and in particular, $e_n = \sum_{k=n}^{\infty} h(n)c_k$ for the n^{th} unit sequence. It follows that

$$\langle a, Ta\rangle_h = \sum_{i,j=0}^{\infty} a_c(i)\overline{a_c(j)}\,\langle c_i, Tc_j\rangle_h = \sum_{k=0}^{\infty}|a_c(k)|^2(\gamma_k - \gamma_{k+1}) \geq 0,$$

and therefore T is positive. □

Before we discuss several examples, the next theorem will give equivalent conditions for the hyponormality of the Cesáro operator.

Theorem 2.2. *The weighted Cesáro operator C_h in $\ell^2(h)$ is hyponormal, if and only if*

(1)'

$$H := \lim_{n\to\infty} H(n) = \infty.$$

(2)'

$$\frac{h(n)}{H(n)} \geq \frac{h(n+1)}{H(n+1)} \quad \forall n \in \mathbb{N}_0.$$

Proof. First note, that the conditions (2)' and (2) are equivalent, because for all $n \in \mathbb{N}_0$ and $H(-1) := 0$, we have

$$\begin{aligned}
\frac{h(n)}{H(n)} \geq \frac{h(n+1)}{H(n+1)} \quad &\Leftrightarrow \quad \frac{H(n)-H(n-1)}{H(n)} \geq \frac{H(n+1)-H(n)}{H(n+1)} \\
&\Leftrightarrow \quad \frac{H(n)}{H(n+1)} \geq \frac{H(n-1)}{H(n)} \\
&\Leftrightarrow \quad H(n)^2 \geq H(n-1)H(n+1).
\end{aligned}$$

If additionally condition (1)' is satisfied, we obtain

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} &= \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \sum_{k=n}^{\infty} \left(\frac{1}{H(k)} - \frac{1}{H(k+1)} \right) \\
&= \sum_{k=n}^{\infty} \frac{1}{H(k)} \left(\frac{h(k)}{H(k)} - \frac{h(k+1)}{H(k+1)} \right) \overset{(2)'}{\geq} 0,
\end{aligned}$$

which is (1). On the other hand, if $H < \infty$, we have

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} &\leq \int_{H(n)}^{H} \frac{1}{x^2} dx + \frac{h(n)}{H(n)^2} \\
&= -\frac{1}{H} + \frac{1}{H(n)} + \frac{h(n)}{H(n)^2} \overset{n\to\infty}{\to} 0.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \left(\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \right) = -\frac{1}{H} < 0,$$

and (1) is not satisfied. □

In the following two examples we will consider the case, when h is the Haar measure of certain orthogonal polynomial sequences. First, we want to recall some basic facts about orthogonal polynomials, see [2] and [12]. Let μ be a probability measure on the real line and denote the support of μ by S and assume $\text{card} S = \infty$. Let $(P_n)_{n\in\mathbb{N}_0}$ denote an orthogonal polynomial sequence with respect to μ. Then $(P_n)_{n\in\mathbb{N}_0}$ satisfies a three term recurrence relation

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1, \tag{2.5}$$

with $P_0(x) = 1$ and $P_1 = (x - b_0)/a_0$. The coefficients are real numbers with $c_n a_{n-1} > 0,\ n > 0$. Conversely, if we define $(P_n)_{n\in\mathbb{N}_0}$ by (2.5), there is a measure μ with the assumed properties. We consider those orthogonal polynomial sequences, where we can additionally assume $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1,\ n > 0$. Then, the Haar measure h satisfies

$$h(n)^{-1} = \int_S P_n^2(x) d\mu(x) \quad n \in \mathbb{N}_0,$$

and

$$h(n+1) = \frac{a_n}{c_{n+1}} h(n), \quad \text{for} \quad n \in \mathbb{N}_0,$$

see also [12].

Example 2.3 (Haar weights of the normalised Jacobi polynomials)**.** Let $\alpha, \beta > -1$ and $(P_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ be defined by (2.5), where

$$\begin{aligned} a_n &= \frac{2(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ b_n &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\ c_n &= \frac{2n(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \end{aligned}$$

see [11]. Then, we obtain for the Haar weights

$$h(n) = \frac{(\alpha+1)_n(\alpha+\beta+1)_n(2n+\alpha+\beta+1)}{(\beta+1)_n n!(\alpha+\beta+1)},$$

where we denote by $(a)_n$ the Pochhammer symbol for $a \in \mathbb{R}$ and $n \in \mathbb{N}_0$, which is

$$(a)_n = \begin{cases} 1 & \text{for} \quad n = 0, \\ a(a+1)\dots(a+n-1) & \text{for} \quad n \geq 1. \end{cases}$$

Inductively, one has

$$H(n) = \frac{(\alpha+\beta+1)_{n+1}(\alpha+2)_n}{(\beta+1)_n n!(\alpha+\beta+1)}.$$

We want to check, whether the conditions in Theorem 2.2 are satisfied. As

$$\alpha+\beta+2 > \beta+1 > 0 \quad \text{and} \quad \alpha+2 > 1,$$

we obtain

$$H(n) = \frac{(\alpha+\beta+2)_n}{(\beta+1)_n} \cdot \frac{(\alpha+2)_n}{(1)_n} \overset{n\to\infty}{\to} \infty,$$

which is condition (1)'. To verify condition (2)', observe that for $n \geq 1$

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} = \frac{(\alpha+1)(2n+\alpha+\beta+1)}{(n+\alpha+\beta+1)(n+\alpha+1)} - \frac{(\alpha+1)(2n+\alpha+\beta+3)}{(n+\alpha+\beta+2)(n+\alpha+2)}.$$

First note, that $1 = \frac{h(0)}{H(0)} > \frac{h(1)}{H(1)}$ by definition. Thus, we have to check whether

$$\begin{aligned} &(2n+\alpha+\beta+1)(n+\alpha+\beta+2)(n+\alpha+2) \\ &\geq (2n+\alpha+\beta+3)(n+\alpha+\beta+1)(n+\alpha+1) \end{aligned} \tag{2.6}$$

holds for all $n \geq 1$.

$$\begin{aligned}
(2.6) \quad \Leftrightarrow \quad & (2n+\alpha+\beta+1) \\
& \quad \times((n+\alpha+\beta+1)(n+\alpha+1)+n+\alpha+\beta+1+n+\alpha+2) \\
& \geq ((2n+\alpha+\beta+1)+2)(n+\alpha+\beta+1)(n+\alpha+1) \\
\Leftrightarrow \quad & (2n+\alpha+\beta+1)(2n+2\alpha+\beta+3) \\
& \geq (2n+2\alpha+2\beta+2)(n+\alpha+1) \\
\Leftrightarrow \quad & ((n+\beta)+(n+\alpha+1))((2n+2\alpha+\beta+2)+1) \\
& \geq ((2n+2\alpha+\beta+2)+\beta)(n+\alpha+1) \\
\Leftrightarrow \quad & (n+\beta)(2n+2\alpha+\beta+2)+(n+\beta)+(n+\alpha+1) \geq \beta(n+\alpha+1) \\
\Leftrightarrow \quad & n(2n+2\alpha+\beta+2)+\beta(n+\alpha+\beta+1) \\
& +(n+\beta)+(n+\alpha+1) \geq 0 \\
\Leftrightarrow \quad & n(2n+2\alpha+\beta+3)+(\beta+1)(n+\alpha+\beta+1) \geq 0,
\end{aligned}$$

which is satisfied, since for $n \geq 1$ and $\alpha, \beta > -1$ both summands are positive. Therefore, the weights of the normalised Jacobi polynomials define a hyponormal Cesàro operator.

Example 2.4 (Haar weights of polynomials connected with homogeneous trees)**.** Let $a \geq 2$ and $(P_n^a)_{n\in\mathbb{N}_0}$ be defined by (2.5), where $a_0 = 1$ and $a_n = \frac{a-1}{a}$, $c_n = \frac{1}{a}$, $n \geq 1$. We obtain $h(0) = 1$ and, by using $h(n+1) = \frac{a_n}{c_{n+1}} h(n)$,

$$h(n) = a(a-1)^{n-1} \quad \text{for} \quad n \geq 1,$$

see [11]. For $a = 2$ these are the weights for the Tschebysheff polynomials of first kind, which are in the class of the Jacobi polynomials. Now let $a \neq 2$ and observe that

$$H(n) = 1 + a\sum_{k=0}^{n-1}(a-1)^k = 1 + a\frac{(a-1)^n - 1}{(a-1)-1} = \frac{h(n+1)-2}{a-2}, \quad n \in \mathbb{N}_0.$$

Thus, a necessary condition for the hyponormality of the corresponding Cesáro operator is

$$\lim_{n\to\infty} H(n) = \infty \Leftrightarrow a - 1 > 1 \Leftrightarrow a > 2.$$

We show that in this case condition (2)' is satisfied either. By definition, we have

$$\frac{h(0)}{H(0)} = 1 > \frac{h(n)}{H(n)} = 1 - \frac{H(n-1)}{H(n)} \quad \text{for all} \quad n \geq 1.$$

Furthermore, for $n \geq 1$, we obtain

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} = \frac{1}{H(n)H(n+1)}(h(n)H(n+1) - h(n+1)H(n))$$
$$= \frac{h(n)}{H(n)H(n+1)}(H(n+1) - (a-1)H(n))$$
$$= \frac{h(n)}{H(n)H(n+1)}\left(\frac{h(n+2)-2}{a-2} - (a-1)\frac{h(n+1)-2}{a-2}\right)$$
$$= \frac{h(n)}{H(n)H(n+1)}\left(\frac{h(n+1)(a-1) - 2 - (a-1)h(n+1) + 2(a-1)}{a-2}\right)$$
$$= \frac{2h(n)}{H(n)H(n+1)}.$$

Therefore, for all $n \in \mathbb{N}_0$,

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} \geq 0.$$

As the conditions of Theorem 2.1 and Theorem 2.2, respectively, are not always satisfied, there must be some h for which C_h is not hyponormal. The following example exhibits some weights for which the weaker condition for paranormality is not satisfied either.

Example 2.5. Let $h(0) = h(1) = 1$, $h(2) = 8$ and $h(n) = 0,99 \cdot 10^{2n-3}$ for $n \geq 3$. Then, we have $H(0) = 1,\ H(1) = 2$ and $H(n) = 10^{2n-3}$ for $n \geq 2$. Using (1.1) and (1.2), we obtain

$$\left\|C_h^2 a\right\|_{2,h}^2 = \sum_{n=0}^{\infty} h(n) \left| \frac{1}{H(n)} \sum_{k=0}^{n} \frac{1}{H(k)} \sum_{m=0}^{k} h(m)a(m) \right|^2$$

and

$$\|C_h a\|_{2,h}^2 = \sum_{n=0}^{\infty} h(n) \left| \frac{1}{H(n)} \sum_{k=0}^{n} h(k)a(k) \right|^2.$$

Let us consider the sequence $e = h(3)^{-\frac{1}{2}} e_3 = (0,0,0,h(3)^{-\frac{1}{2}},0,\cdots)$, with $\|e\|_{2,h} = 1$. Then,

$$\begin{aligned}\left\|C_h^2 e\right\|_{2,h}^2 &= h(3) \sum_{n=3}^{\infty} \frac{h(n)}{H(n)^2} \left(\sum_{k=3}^{n} \frac{1}{H(k)} \right)^2 \\ &< \left(h(3) \sum_{n=3}^{\infty} \frac{h(n)}{H(n)^2} \right) \left(\sum_{k=3}^{\infty} \frac{1}{H(k)} \right)^2\end{aligned}$$

and

$$\|C_h e\|_{2,h}^2 = h(3) \sum_{n=3}^{\infty} \frac{h(n)}{H(n)^2}.$$

Substituting $h(3) = 990$, $\sum_{n=3}^{\infty} \frac{h(n)}{H(n)^2} = 10^{-3}$ and $\sum_{n=3}^{\infty} \frac{1}{H(n)} = \frac{1}{990}$ yields

$$\left\|C_h^2 e\right\|_{2,h} < (990 \cdot 10^{-3})^{\frac{1}{2}} \cdot 990^{-1} \quad \text{and} \quad \|C_h e\|_{2,h}^2 = 990 \cdot 10^{-3},$$

which implies $\|C_h^2 e\|_{2,h} < \|C_h e\|_{2,h}^2$ and contradicts paranormality.

On the one hand, the weights of Example 2.5 do not satisfy any weak normality condition. But on the other hand, the Cesáro operator is subnormal in the unweighted case, see [10] and [4]. Here, the question arises, whether quasinormality can be satisfied. The next theorem answers this question in the negative and also implies, that subnormality is the strongest property (in terms of the generalised concept of normality) C_h can have.

Theorem 2.6. *The weighted Cesáro operator C_h in $\ell^2(h)$ is not quasinormal independently of the choice of weights.*

Proof. By definition, C_h is quasinormal if and only if $(C_h^* C_h - C_h C_h^*) C_h = 0$. Let us define T and γ_n as in the proof of Theorem 2.1. Considering the matrix representations for the operators yields

$$\begin{aligned} TC_h(i,j) &= (h(0)\gamma_i, \cdots, h(i)\gamma_i, h(i+1)\gamma_{i+1}, \cdots) \\ &\quad \times (\underbrace{0, \cdots, 0}_{j \text{ times}}, \frac{h(j)}{H(j)}, \frac{h(j)}{H(j+1)}, \frac{h(j)}{H(j+2)}, \cdots)^T \\ &= \sum_{k=j}^{\infty} h(k)\gamma_{\max(i,k)} \frac{h(j)}{H(k)} \\ &= h(j) \begin{cases} \gamma_i \sum_{k=j}^{i-1} \frac{h(k)}{H(k)} + \sum_{k=i}^{\infty} \frac{h(k)}{H(k)} \gamma_k & \text{for } i > j, \\ \sum_{k=j}^{\infty} \frac{h(k)}{H(k)} \gamma_k & \text{for } i \leq j, \end{cases} \end{aligned}$$

for all $i, j \in \mathbb{N}_0$. Assume that C_h is quasinormal for some sequence h. Then, by definition, $\sum_{k=n}^{\infty} \frac{h(k)}{H(k)} \gamma_k = 0$ for all $n \in \mathbb{N}_0$. Hence,

$$\gamma_n = \frac{H(n)}{h(n)} \left(\sum_{k=n}^{\infty} \frac{h(k)}{H(k)} \gamma_k - \sum_{k=n+1}^{\infty} \frac{h(k)}{H(k)} \gamma_k \right) = 0 \quad \text{for all } n \in \mathbb{N}_0, \tag{2.7}$$

which is the condition for normality, i.e. $T = 0$. The definition of γ_n and (2.7) imply

$$\frac{1}{H(n)} = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} = \frac{h(n)}{H(n)^2} + \sum_{k=n+1}^{\infty} \frac{h(k)}{H(k)^2} = \frac{h(n)}{H(n)^2} - \frac{1}{H(n+1)},$$

or, equivalently,

$$\frac{1}{H(n+1)} = \frac{1}{H(n)} \left(\frac{h(n)}{H(n)} - 1 \right) \quad \text{for all } n \in \mathbb{N}_0.$$

Then, in particular for $n = 0$, we have

$$0 \neq \frac{1}{H(1)} = \frac{1}{H(0)} \left(\frac{h(0)}{H(0)} - 1 \right) = 0,$$

which is a contradiction. Thus, there exists no sequence h of weights for which C_h is quasinormal or normal. □

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