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ON f-CONNECTIONS OF POSITIVE DEFINITE MATRICES

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. In this paper, by using Mond-Pečarić method we provide some inequalities for connections of positive definite matrices. Next, we discuss specifications of the obtained results for some special cases. In doing so, we use α-arithmetic, α-geometric and α-harmonic operator means.

1. Introduction

Throughout $M_n(\mathbb{C})$ denotes the C^*-algebra of $n \times n$ complex matrices. For matrices $X, Y \in \mathbb{M}_n(\mathbb{C})$, the notation $Y \leq X$ (resp., $Y < X$) means that $X - Y$ is positive semidefinite (resp., positive definite). A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is said to be *positive* if $0 \leq \Phi(X)$ for $0 \leq X \in \mathbb{M}_n(\mathbb{C})$. If in addition $0 < \Phi(X)$ for $0 < X \in \mathbb{M}_n(\mathbb{C})$ then Φ is said to be *strictly positive*.

A real function $h : J \to \mathbb{R}$ defined on interval $J \subset \mathbb{R}$ is called an *operator monotone function*, if for all Hermitian matrices A and B (of the same order) with spectra in J,

$$A \leq B \quad \text{implies} \quad h(A) \leq h(B)$$

(see [4, p. 112]).

For $\alpha \in [0, 1]$, the α*-arithmetic mean* of $n \times n$ positive definite matrices A and B is defined as follows

$$A \nabla_\alpha B = (1 - \alpha)A + \alpha B. \tag{1.1}$$

For $\alpha = \frac{1}{2}$ one obtains the *arithmetic mean* $A \nabla B = \frac{1}{2}(A + B)$.

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For $\alpha \in [0,1]$, the *α-geometric mean* of $n \times n$ positive definite matrices A and B is defined by

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2} \tag{1.2}$$

(see [9, 15]). In particular, for $\alpha = \frac{1}{2}$ equation (1.2) defines the *geometric mean* of A and B defined by

$$A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}$$

(see [2, 10, 15]).

For $\alpha \in [0,1]$, the *α-harmonic mean* of $n \times n$ positive definite matrices A and B is defined by

$$A \,!_\alpha B = ((1-\alpha)A^{-1} + \alpha B^{-1})^{-1}. \tag{1.3}$$

For $\alpha = \frac{1}{2}$ we obtain the *harmonic mean* of A and B given by

$$A \,! B = \left(\frac{1}{2}A^{-1} + \frac{1}{2}B^{-1}\right)^{-1}$$

(see [11]).

Ando's inequality [1] asserts that if $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a positive linear map and $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive definite then

$$\Phi(A \sharp_\alpha B) \le \Phi(A) \sharp_\alpha \Phi(B). \tag{1.4}$$

Lee [10] established the following reverse of inequality (1.4) with $\alpha = \frac{1}{2}$ (see also [12]).

Theorem A [10, Theorem 4] *Let A and B be $n \times n$ positive definite matrices. Assume $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a positive linear map.*

If $mA \le B \le MA$ with positive scalars m, M then

$$\Phi(A) \sharp \Phi(B) \le \frac{\sqrt{M} + \sqrt{m}}{2\sqrt[4]{mM}}\, \Phi(A \sharp B).$$

Recently, Seo [15] showed difference and ratio type reverses of Ando's inequality (1.4), as follows.

Theorem B [15, Theorem 1] *Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$ for some scalars $0 < m < M$ and let $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ be a positive linear map.*

Then for each $\alpha \in (0,1)$

$$\Phi(A) \sharp_\alpha \Phi(B) - \Phi(A \sharp_\alpha B) \le -C(m, M, \alpha)\Phi(A),$$

where the Kantorovich constant for the difference $C(m, M, \alpha)$ is defined by

$$C(m, M, \alpha) = (\alpha - 1)\left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{\alpha}{\alpha-1}} + \frac{Mm^\alpha - mM^\alpha}{M - m}.$$

Theorem C [15, Theorem 3] *Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$ for some scalars $0 < m < M$ and let $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ be a positive linear map.*

Then for each $\alpha \in (0,1)$

$$\Phi(A) \sharp_\alpha \Phi(B) \le K(m, M, \alpha)^{-1} \Phi(A \sharp_\alpha B),$$

where the generalized Kantorovich constant $K(m,M,\alpha)$ *is defined by*

$$K(m,M,\alpha)=\frac{mM^\alpha-Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha-m^\alpha}{mM^\alpha-Mm^\alpha}\right)^\alpha.$$

Theorem D [8, Theorem 2.1] *Let A and B be $n\times n$ positive definite matrices such that $0<b_1\le A\le a_1$ and $0<b_2\le B\le a_2$ for some scalars $0<b_i<a_i$, $i=1,2$.*

If $\Phi:\mathbb{M}_n(\mathbb{C})\to\mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map, then for any operator mean σ with the representing function f, the following double inequality holds:

$$\omega^{1-\alpha}\left(\Phi(A)\sharp_\alpha\Phi(B)\right)\le(\omega\Phi(A))\nabla_\alpha\Phi(B)\le\frac{\alpha}{\mu}\Phi(A\sigma B),\tag{1.5}$$

where $\mu=\frac{a_1b_1(f(b_2a_1^{-1})-f(a_2b_1^{-1}))}{b_1b_2-a_1a_2}$, $\nu=\frac{a_1a_2f(b_2a_1^{-1})-b_1b_2f(a_2b_1^{-1})}{a_1a_2-b_1b_2}$, $\omega=\frac{\alpha\nu}{(1-\alpha)\mu}$ *and* $\alpha\in(0,1)$.

The purpose of this paper is to demonstrate a unified framework including Theorems **A**, **B**, **C** and **D** as special cases. Following the idea of Mond-Pečarić method [5, 11], in our approach we use a connection σ_f induced by a continuous function $f:J\to\mathbb{R}$. We focus on double inequalities as in (1.5) (cf. [6, Theorem 3.1]).

In Section 2, we formulate conditions for four functions f_1,f_2,g_1,g_2, under which the following double inequality holds (see Theorem 2.3):

$$c_{g_2}\,\Phi(A)\sigma_{f_2}\Phi(B)\le\Phi(A\sigma_{g_2}B)\le\Phi(A\sigma_{g_2g_1^{-1}}(A\sigma_{f_1}B)),\tag{1.6}$$

with suitable constant c_{g_2} (see (2.8)). Here the crucial key is the behaviour of the superposition $g_2g_1^{-1}$. By substituting $\alpha t+1-\alpha$, t^α and $(\alpha t^{-1}+1-\alpha)^{-1}$ in place of $g_2g_1^{-1}(t)$, we get variants of the above double inequality (1.6) for α-arithmetic, α-geometric and α-harmonic operator means, respectively. Also, some further substitutions for f_1,f_2,g_2 are possible. Thus we can obtain some old and new results as special cases of (1.6) (see Theorem 2.9 and Corollaries 2.6-2.18).

2. Results

Let $f:J\to\mathbb{R}$ be a continuous function on an interval $J\subset\mathbb{R}$. The *f-connection* of an $n\times n$ positive definite matrix A and an $n\times n$ hermitian matrix B such that the spectrum $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2})\subset J$, is defined by

$$A\sigma_fB=A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}\tag{2.1}$$

(cf. [7, p. 637], [9]).

Note that the operator means (1.1), (1.2) and (1.3) are of the form (2.1) with the functions $\alpha t+1-\alpha$, t^α and $(\alpha t^{-1}+1-\alpha)^{-1}$, respectively.

For a function $f:J\to\mathbb{R}_+$ defined on an interval $J=[m,M]$ with $m<M$, we define

$$a_f=\frac{f(M)-f(m)}{M-m}\quad,\quad b_f=\frac{Mf(m)-mf(M)}{M-m}\quad\text{and}\quad c_f=\min_{t\in J}\frac{a_ft+b_f}{f(t)}\tag{2.2}$$

(see [11]).

Lemma 2.1. *(See* [7, Theorem 1]*, cf. also* [11, Corollary 3.4]*.) Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$ with $0 < m < M$.*

If σ_f is a connection with operator monotone concave function $f > 0$ and Φ is a strictly positive linear map, then

$$c_f \, \Phi(A)\sigma_f \Phi(B) \le \Phi(A\sigma_f B), \tag{2.3}$$

where c_f is defined by (2.2).

Remark 2.2. **(i):** For *all* positive linear maps Φ, the equality

$$\Phi(A)\sigma_f \Phi(B) = \Phi(A\sigma_f B) \tag{2.4}$$

holds for the arithmetic operator mean $\sigma_f = \nabla_\alpha$, $\alpha \in [0,1]$.

(ii): In general, for other connections σ_f, (2.4) can hold for some *specific* Φ. For example, taking $\sigma_f = \sharp_\alpha$, $\alpha \in [0,1]$, and $\Phi(\cdot) = U^*(\cdot)U$ with unitary U, we have

$$U^*(A\sharp_\alpha B)U = (U^*AU)\sharp_\alpha(U^*BU),$$

which is of form (2.4).

(iii): Clearly, if the equality (2.4) is met (e.g., if f is affine), then (2.3) holds with $c_f = 1$ (see (2.20), (2.30)-(2.31)).

Our first result is motivated by [8, Theorem 2.1] (see Theorem **D** in Section 1).

Theorem 2.3. *Let f_1, f_2, g_1, g_2 be continuous real functions defined on an interval $J = [m, M] \subset \mathbb{R}$. Assume that $g_2 > 0$ and $g_2 g_1^{-1}$ are operator monotone on intervals J and $J' = g_1(J)$, respectively, with invertible g_1 and concave g_2. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$ with $0 < m < M$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and

$$g_1(t) \le f_1(t) \quad \text{and} \quad f_2(t) \le g_2(t) \quad \text{for } t \in J, \tag{2.5}$$

$$\max_{t\in J} g_1(t) = \max_{t\in J} f_1(t), \tag{2.6}$$

then

$$c_{g_2} \, \Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_{g_2}B) \le \Phi(A\sigma_{g_2 g_1^{-1}}(A\sigma_{f_1}B)), \tag{2.7}$$

where c_{g_2} is defined by

$$a_{g_2} = \tfrac{g_2(M)-g_2(m)}{M-m} \quad , \quad b_{g_2} = \tfrac{Mg_2(m)-mg_2(M)}{M-m} \quad \text{and} \quad c_{g_2} = \min_{t\in J} \tfrac{a_{g_2}t+b_{g_2}}{g_2(t)}. \tag{2.8}$$

Proof. Since $mA \le B \le MA$, we obtain $m\Phi(A) \le \Phi(B) \le M\Phi(A)$ by the positivity of Φ. In consequence, by the strict positivity of Φ, we get $m \le W \le M$ and $\mathrm{Sp}\,(W) \subset [m, M]$ for $W = \Phi(A)^{-1/2}\Phi(B)\Phi(A)^{-1/2}$.

It follows from the second inequality of (2.5) that

$$f_2((\Phi(A))^{-1/2}\Phi(B)(\Phi(A))^{-1/2}) \le g_2((\Phi(A))^{-1/2}\Phi(B)(\Phi(A))^{-1/2}),$$

and further

$$\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A)\sigma_{g_2}\Phi(B). \tag{2.9}$$

According to Lemma 2.1 applied to operator monotone function g_2, we have

$$c_{g_2} \, \Phi(A)\sigma_{g_2}\Phi(B) \le \Phi(A\sigma_{g_2}B).$$

This and (2.9) imply

$$c_{g_2}\,\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_{g_2}B), \tag{2.10}$$

proving the left-hand side inequality of (2.7).

It follows that for $h = g_2 \circ g_1^{-1}$,

$$A\sigma_{g_2}B = A\sigma_{h\circ g_1}B = A\sigma_h(A\sigma_{g_1}B), \tag{2.11}$$

where $\circ$ means superposition. In fact, we have

$$\begin{aligned} A\sigma_{h\circ g_1}B &= A^{1/2}(h\circ g_1)(A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}h(g_1(A^{-1/2}BA^{-1/2}))A^{1/2} \\ &= A^{1/2}h(A^{-1/2}A^{1/2}g_1(A^{-1/2}BA^{-1/2})A^{1/2}A^{-1/2})A^{1/2} \\ &= A^{1/2}h(A^{-1/2}(A\sigma_{g_1}B)A^{-1/2})A^{1/2} = A\sigma_h(A\sigma_{g_1}B). \end{aligned}$$

On the other hand, it follows from the first inequality of (2.5) that

$$g_1(A^{-1/2}BA^{-1/2}) \le f_1(A^{-1/2}BA^{-1/2})$$

and next

$$A\sigma_{g_1}B \le A\sigma_{f_1}B. \tag{2.12}$$

It is seen from (2.5) that

$$\min_{t\in J} g_1(t) \le \min_{t\in J} f_1(t),$$

which together with (2.6) gives

$$f_1(J) \subset g_1(J). \tag{2.13}$$

Denote

$$Z_0 = A^{-1/2}(A\sigma_{g_1}B)A^{-1/2} = g_1(A^{-1/2}BA^{-1/2})$$

and

$$W_0 = A^{-1/2}(A\sigma_{f_1}B)A^{-1/2} = f_1(A^{-1/2}BA^{-1/2}).$$

Then $\mathrm{Sp}\,(Z_0) \subset g_1(J)$ and $\mathrm{Sp}\,(W_0) \subset f_1(J)$, because $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2}) \subset J$.

Since $h = g_2 \circ g_1^{-1}$ is operator monotone on $J' = g_1(J)$, from (2.12) and (2.13) we obtain

$$h(A^{-1/2}(A\sigma_{g_1}B)A^{-1/2}) \le h(A^{-1/2}(A\sigma_{f_1}B)A^{-1/2})$$

and next

$$A\sigma_h(A\sigma_{g_1}B) \le A\sigma_h(A\sigma_{f_1}B). \tag{2.14}$$

Therefore, by (2.11) and (2.14), we deduce that

$$\Phi(A\sigma_{g_2}B) \le \Phi(A\sigma_{g_2g_1^{-1}}(A\sigma_{f_1}B)). \tag{2.15}$$

Now, by combining (2.10) and (2.15), we conclude that (2.7) holds true. □

Remark 2.4. In Theorem 2.3, if in addition f_1 and g_1 are nondecreasing on $[m, M]$, then condition (2.6) simplifies to

$$g_1(M) = f_1(M).$$

Likewise, if f_1 and g_1 are nonincreasing on $[m, M]$, then (2.6) means

$$g_1(m) = f_1(m).$$

Corollary 2.5. *Under the assumptions of Theorem 2.3.*

(i): *If $g_2g_1^{-1}$ is an affine function, i.e., $g_2g_1^{-1}(s) = as + b$ for $s \in g_1(J)$, $a > 0$, then (2.7) reduces to*

$$c_{g_2}\,\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_{g_2}B) \le a\,\Phi(A\sigma_{f_1}B) + b\,\Phi(A). \tag{2.16}$$

(ii): *If $g_2g_1^{-1}$ is a power function, i.e., $g_2g_1^{-1}(s) = s^\alpha$ for $s \in g_1(J)$, $\alpha \in [0,1]$, then (2.7) reduces to*

$$c_{g_2}\,\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_{g_2}B) \le \Phi(A\sigma_{\sharp_\alpha}(A\sigma_{f_1}B)). \tag{2.17}$$

(iii): *If $g_2g_1^{-1}$ is an inverse function of the form $g_2g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1}$ for $s \in g_1(J)$, $\alpha \in [0,1]$, then (2.7) reduces to*

$$c_{g_2}\,\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_{g_2}B) \le \Phi([(1-\alpha)A^{-1} + \alpha(A\sigma_{f_1}B)^{-1}]^{-1}). \tag{2.18}$$

Proof. **(i)**. To show (2.16), observe that $a > 0$ implies the operator monotonicity of $g_2g_1^{-1}(s) = as + b$ (see [4, p. 113]).

It is not hard to verify that

$$A\sigma_{g_2g_1^{-1}}(A\sigma_{f_1}B) = a\,A\sigma_{f_1}B + b\,A.$$

Hence

$$\Phi(A\sigma_{g_2g_1^{-1}}(A\sigma_{f_1}B)) = a\,\Phi(A\sigma_{f_1}B) + b\,\Phi(A).$$

Now, it is sufficient to apply (2.7).

(ii). To see (2.17), it is enough to use (2.7) together with the operator monotonicity of $g_2g_1^{-1}(s) = s^\alpha$ with $\alpha \in [0,1]$ (see [4, p. 115]).

(iii). Finally, (2.18) is an easy consequence (2.7) for the operator monotone function $g_2g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1}$ with $\alpha \in [0,1]$ (see [4, p. 114]). □

The next result develops some ideas in [12, 14].

Corollary 2.6. *Let f_1, f_2, g be continuous real functions defined on an interval $J = [m, M]$ with invertible operator monotone concave $g > 0$ on J. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and

$$f_2(t) \le g(t) \le f_1(t) \quad \text{for } t \in J,$$
$$\max_{t\in J} g(t) = \max_{t\in J} f_1(t),$$

then

$$c_g\,\Phi(A)\sigma_{f_2}\Phi(B) \le \Phi(A\sigma_g B) \le \Phi(A\sigma_{f_1}B), \tag{2.19}$$

where c_g is defined by (2.8) for $g_2 = g$.

In particular, if g is an affine function, i.e., $g(t) = at + b$ for $t \in J$, $a > 0$, then (2.19) reduces to

$$\Phi(A)\sigma_{f_2}\Phi(B) \le b\Phi(A) + a\Phi(B) \le \Phi(A\sigma_{f_1}B). \tag{2.20}$$

Proof. It is enough to apply Theorem 2.3 with $g_1 = g_2 = g$. Then the superposition $g_2 \circ g_1^{-1}$ is the identity function $s \to s$, $s \in g(J)$. So, (2.16) reads as (2.19).

To see (2.20), use (2.19) with $c_g = 1$ (see Remark 2.2). □

Remark 2.7. The right-hand inequality in (2.20) can be used to obtain Diaz-Metcalf type inequalities [8, 14].

Remark 2.8. A specialization of Corollary 2.6 leads to [8, Theorem 2.1] (see Theorem **D** in Section 1).

Namely, it is easy to verify that the spectrum $\mathrm{Sp}(Z) \subset J$, where $Z = A^{-1/2}BA^{-1/2}$ and $J = [m, M]$ with $m = \frac{b_2}{a_1}$ and $M = \frac{a_2}{b_1}$.

By weighted arithmetic-geometric inequality (see [8])

$$t^\alpha \omega^{1-\alpha} \le \alpha t + (1-\alpha)\omega \quad \text{for } \alpha \in [0,1] \text{ and } t > 0,\ \omega > 0. \tag{2.21}$$

Since $\sigma = \sigma_f$ with operator monotone function f on $[0, \infty)$, f must be strictly increasing and concave. Hence

$$\mu t + \nu \le f(t) \quad \text{for } t \in J.$$

As a consequence,

$$\alpha t + (1-\alpha)\omega \le \frac{\alpha}{\mu} f(t) \quad \text{for } t \in J. \tag{2.22}$$

By setting

$$f_1(t) = \frac{\alpha}{\mu} f(t), \quad f_2(t) = t^\alpha \omega^{1-\alpha}, \quad g(t) = (1-\alpha)\omega + \alpha t, \quad t \in J,$$

we see that conditions (2.5)-(2.6) are satisfied (cf. (2.21)-(2.22) and Remark 2.4). Moreover,

$$\sigma_{f_2} = \sharp_\alpha \quad \text{and} \quad \sigma_g = \nabla_\alpha.$$

Now, it is not hard to check that inequalities (2.20) in Corollary 2.6 applied to the matrices ωA and B yield (1.5), as required.

The special case of Theorem 2.3 for $f_1 = f_2 = f$ gives the following result.

Theorem 2.9. *Let f, g_1, g_2 be continuous real functions defined on an interval $J = [m, M]$. Assume $g_2 > 0$ and $g_2 g_1^{-1}$ are operator monotone on J and $J' = g_1(J)$, respectively, with invertible g_1 and concave g_2. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and

$$g_1(t) \le f(t) \le g_2(t) \quad \text{for } t \in J,$$
$$\max_{t\in J} g_1(t) = \max_{t \in J} f(t),$$

then

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A\sigma_{g_2} B) \le \Phi(A\sigma_{g_2 g_1^{-1}}(A\sigma_f B)), \tag{2.23}$$

where $c_{g_2} > 0$ is given by (2.8).

Proof. Apply Theorem 2.3 for $f_1 = f_2 = f$. □

Corollary 2.10. *Under the assumptions of Theorem 2.9.*

(i): *If $g_2 g_1^{-1}$ is an affine function, i.e., $g_2 g_1^{-1}(s) = as + b$ for $s \in g_1(J)$, $a > 0$, then (2.23) reduces to*

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A\sigma_{g_2} B) \le a\, \Phi(A\sigma_f B) + b\, \Phi(A). \tag{2.24}$$

(ii): *If $g_2 g_1^{-1}$ is a power function, i.e., $g_2 g_1^{-1}(s) = s^\alpha$ for $s \in g_1(J)$, $\alpha \in [0,1]$, then (2.23) reduces to*

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A\sigma_{g_2} B) \le \Phi(A\sigma_{\sharp_\alpha}(A\sigma_f B)). \tag{2.25}$$

(iii): *If $g_2 g_1^{-1}$ is an inverse function of the form $g_2 g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1}$ for $s \in g_1(J)$, $\alpha \in [0,1]$, then (2.23) reduces to*

$$c_{g_2}\,\Phi(A)\sigma_f\Phi(B) \le \Phi(A\sigma_{g_2}B) \le \Phi([(1-\alpha)A^{-1} + \alpha(A\sigma_{f_1}B)^{-1}]^{-1}). \tag{2.26}$$

Proof. Apply Theorem 2.9. □

Remark 2.11. **(i):** It is worth emphasing that the above inequality (2.24) can be viewed as a reverse inequality of Aujla and Vasudeva [3]:

$$\Phi(A\sigma_f B) \le \Phi(A)\sigma_f\Phi(B)$$

for an operator monotone function $f : (0,\infty) \to (0,\infty)$.

(ii): In the case $f(t) = t^{1/2}$ inequality (2.24) is similar to that in [11, Corollary 3.7].

By employing the second part of Theorem 2.9 for some special functions g_1 and g_2 we obtain the following.

Corollary 2.12. *Let $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous real functions with interval $J = [m, M]$ and invertible operator monotone concave g on J. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and

$$a_1 g(t) + b_1 \le f(t) \le a_2 g(t) + b_2 \quad \text{for } t \in J,\ a_1 > 0,\ a_2 > 0,$$

$$\max_{t\in J}(a_1 g(t) + b_1) = \max_{t\in J} f(t),$$

then

$$c_{g_2}\,\Phi(A)\sigma_f\Phi(B) \le a_2\Phi(A\sigma_g B) + b_2\Phi(A) \le \frac{a_2}{a_1}\,\Phi(A\sigma_f B) + \left(b_2 - \frac{a_2}{a_1}b_1\right)\,\Phi(A), \tag{2.27}$$

where $c_{g_2} > 0$ is given by (2.8) with $g_2 = a_2 g + b_2 > 0$.

If in addition $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ *then (2.27) becomes*

$$c_{g_2}\,\Phi(A)\sigma_f\Phi(B) \le a_2\Phi(A\sigma_g B) + b_2\Phi(A) \le \frac{a_2}{a_1}\,\Phi(A\sigma_f B). \tag{2.28}$$

Proof. By putting $g_1(t) = a_1 g(t) + b_1$ and $g_2(t) = a_2 g(t) + b_2$ for $t \in J$, we find that $g_2 g_1^{-1} : g_1(J) \to \mathbb{R}$ is an affine function, i.e.,

$$g_2 g_1^{-1}(s) = \frac{a_2}{a_1}s + b_2 - \frac{a_2}{a_1}b_1 \quad \text{for } s \in g_1(J). \tag{2.29}$$

Making use of (2.29) and Theorem 2.9, eq. (2.24), with $a = \frac{a_2}{a_1}$ and $b = b_2 - \frac{a_2}{a_1}b_1$ yields (2.27).

Inequality (2.28) is an easy consequence of (2.27). □

The special case of Corollary 2.12 for $g(t) = t$, $t \in J$, leads to some results of Kaur et al. [7, Theorems 1 and 2].

Corollary 2.13 (Cf. Kaur et al. [7, Theorems 1 and 2])**.** *Let $f : J \to \mathbb{R}$ be a continuous real function with interval $J = [m, M]$. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map, and

$$a_1 t + b_1 \le f(t) \le a_2 t + b_2 \quad \text{for } t \in J,\ a_1 > 0,\ a_2 > 0,$$

$$a_1 M + b_1 = \max_{t\in J} f(t),$$

then

$$\Phi(A)\sigma_f\Phi(B) \le a_2\Phi(B) + b_2\Phi(A) \le \frac{a_2}{a_1}\Phi(A\sigma_f B) + \left(b_2 - \frac{a_2}{a_1}b_1\right)\Phi(A). \quad (2.30)$$

If in addition $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ *then*

$$\Phi(A)\sigma_f\Phi(B) \le a_2\Phi(B) + b_2\Phi(A) \le \frac{a_2}{a_1}\Phi(A\sigma_f B). \quad (2.31)$$

Proof. Use Corollary 2.12, eq. (2.27) and (2.28) with $c_{g_2} = 1$ (see Remark 2.2). □

Remark 2.14. **(i):** With $a_1 = a_2$, inequality (2.31) can be restated as

$$\Phi(A)\sigma_f\Phi(B) \le a_2\Phi(B) + b_2\Phi(A) \le \Phi(A\sigma_f B).$$

This can be obtained for an operator monotone (concave) function f as in the Mond–Pečarić method [5, 11].

(ii): Inequality (2.30) with $a_1 = a_2$ and $f(t) = t^\alpha$, $\sigma_f = \sharp_\alpha$, $0 \le \alpha \le 1$, is of type as in Theorem **B** (see Section 1).

(iii): When $a_1 \ne a_2$ and $f(t) = t^\alpha$, $\sigma_f = \sharp_\alpha$, $0 \le \alpha \le 1$, then (2.31) leads to Theorem **C**.

(iv): With suitable choosen $a_1 \ne a_2$ and $\sigma_f = \sharp_{1/2}$, $f(t) = t^{1/2}$, inequality (2.31) can be used to derive Cassels, Kantorovich, Greub-Rheinbold type inequalities, etc. (cf. Theorem **A**, see also [12, 13, 14] and references therein).

We now consider consequences of Theorem 2.9 for case of geometric mean.

Corollary 2.15. *Let $f : J \to \mathbb{R}$ and $g : J \to (0, 1]$ be continuous real functions with interval $J = [m, M]$ and invertible operator monotone g. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and, $0 < \alpha \le \beta < 1$,

$$g^\beta(t) \le f(t) \le g^\alpha(t) \quad \text{for } t \in J,$$

$$\max_{t\in J} g^\beta(t) = \max_{t\in J} f(t),$$

then

$$c_{g_2}\Phi(A)\sigma_f\Phi(B) \le \Phi(A\sigma_{g^\alpha}B) \le \Phi(A\sharp_{\frac{\alpha}{\beta}}(A\sigma_f B)), \quad (2.32)$$

where $c_{g_2} > 0$ is given by (2.8) with concave $g_2 = g^\alpha$.

Proof. By substituting $g_1(t) = g^\beta(t)$ and $g_2(t) = g^\alpha(t)$ for $t \in J$, we have

$$g_2 g_1^{-1} = (\cdot)^\alpha \circ g \circ g^{-1} \circ (\cdot)^{\frac{1}{\beta}} = (\cdot)^{\frac{\alpha}{\beta}},$$

where the symbol $\circ$ stands for superposition. Thus $g_2 g_1^{-1}(s) = s^{\frac{\alpha}{\beta}}$, $s \in g_1(J)$, is an operator monotone function. For this reason, Theorem 2.9, eq. (2.25), forces (2.32). □

Corollary 2.16. *Let $f : J \to \mathbb{R}$ be a continuous real function with interval $J = [m, M]$. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$, $0 < m < M \le 1$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and, $0 < \alpha \le \beta < 1$,

$$t^\beta \le f(t) \le t^\alpha \quad \textit{for } t \in J,$$

$$M^\beta = \max_{t \in J} f(t),$$

then

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A \sharp_\alpha B) \le \Phi(A \sharp_{\frac{\alpha}{\beta}} (A \sigma_f B)),$$

where $c_{g_2} > 0$ is given by (2.8) with $g_2(t) = t^\alpha$.

Proof. Employ Corollary 2.15 with $g(t) = t$. □

We now apply Theorem 2.9 in the context of harmonic mean (cf. [6, Lemma 3.3]).

Corollary 2.17. *Let $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}_+$ be continuous real functions with intervals $J = [m, M]$ and invertible operator monotone g on J. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and $0 < \alpha \le \beta < 1$,

$$(\beta(g(t))^{-1} + 1 - \beta)^{-1} \le f(t) \le (\alpha(g(t))^{-1} + 1 - \alpha)^{-1} \quad \textit{for } t \in J,$$

$$\max_{t \in J} (\beta(g(t))^{-1} + 1 - \beta)^{-1} = \max_{t \in J} f(t),$$

then

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A \sigma_{(\alpha(1/g)+1-\alpha)^{-1}} B) \le \Phi(A\,!_\gamma (A \sigma_f B)), \tag{2.33}$$

where $\gamma = \frac{\alpha}{\beta}$ and $c_{g_2} > 0$ is given by (2.8) with concave $g_2(t) = (\alpha(g(t))^{-1} + 1 - \alpha)^{-1}$.

Proof. By setting $g_1(t) = \left(\frac{\beta}{g(t)} + 1 - \beta\right)^{-1}$ and $g_2(t) = \left(\frac{\alpha}{g(t)} + 1 - \alpha\right)^{-1}$ for $t \in J$, we derive

$$g_2 g_1^{-1}(s) = \left[\frac{\alpha}{\beta} s^{-1} + \left((1-\alpha) - (1-\beta)\frac{\alpha}{\beta}\right)\right]^{-1} \quad \text{for } s \in g_1(J),$$

with $\frac{\alpha}{\beta} + (1 - \alpha - (1-\beta)\frac{\alpha}{\beta}) = 1$, $0 < \frac{\alpha}{\beta} \le 1$ and $0 \le 1 - \alpha - (1-\beta)\frac{\alpha}{\beta} < 1$. Therefore $g_2 g_1^{-1}(s) = (\gamma s^{-1} + 1 - \gamma)^{-1}$ is an operator monotone function. So, in accordance with Theorem 2.9, inequality (2.26) implies (2.33). □

Corollary 2.18. *Let $f : J \to \mathbb{R}$ be a continuous real function with interval $J = [m, M]$. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$, $0 < m < M$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map and, for $0 < \alpha \le \beta < 1$,

$$(\beta t^{-1} + 1 - \beta)^{-1} \le f(t) \le (\alpha t^{-1} + 1 - \alpha)^{-1} \quad \text{for } t \in J,$$

$$(\beta M^{-1} + 1 - \beta)^{-1} = \max_{t \in J} f(t),$$

then

$$c_{g_2}\, \Phi(A)\sigma_f \Phi(B) \le \Phi(A \,!_\alpha B) \le \Phi(A \,!_\gamma (A \sigma_f B)),$$

where $\gamma = \frac{\alpha}{\beta}$ and $c_{g_2} > 0$ is given by (2.8) with $g_2(t) = (\alpha t^{-1} + 1 - \alpha)^{-1}$.

Proof. Utilising Corollary 2.17 with $g(t) = t$ we get the desired result. □

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