Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 2, 60–68
http://doi.org/10.15352/afa/06-2-6
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

NOTES ABOUT SUBSPACE-SUPERCYCLIC OPERATORS

LIANG ZHANG[1] AND ZE-HUA ZHOU[*,1,2]

Communicated by M. de Jeu

Abstract. A bounded linear operator T on a Banach space X is called subspace-hypercyclic for a nonzero subspace M if $orb\,(T,x)\cap M$ is dense in M for a vector $x \in X$, where $orb(T,x) = \{T^n x : n = 0, 1, 2, \cdots\}$. Similarly, the bounded linear operator T on a Banach space X is called subspace-supercyclic for a nonzero subspace M if there exists a vector whose projective orbit intersects the subspace M in a relatively dense set. In this paper we provide a Subspace-Supercyclicity Criterion and offer two equivalent conditions of this criterion. At the same time, we also characterize other properties of subspace-supercyclic operators.

1. Introduction

Let X be a separable infinite dimensional Banach space over the scalar field $\mathbb{C}$ and let B (X) denote the set of all bounded linear operators on X and we will usually refer to elements of B (X) as just operators. A bounded linear operator $T : X \to X$ is called hypercyclic (respectively, supercyclic) if there is some vector $x \in X$ such that $orb(T,x) = \{T^n x : n = 0, 1, \cdots\}$ (respectively, the projective orbit $\{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, 2, \cdots\}$) is dense in X. Such a vector x is said to hypercyclic (respectively, supercyclic) for T. The study of hypercyclicity goes back a long way, and has been investigated in more general settings, for example in topological vector spaces. This phenomenon appears in separable spaces, and it is connected with the invariant subspace problem, dynamical systems, and approximation theory. One can refer to [3, 5] for more information about hypercyclicity.

Date: Received: Dec. 1, 2013; Revised: Apr. 24, 2014; Accepted: Jun. 10, 2014.
[*] Corresponding author.
2010 *Mathematics Subject Classification.* Primary: 47A16; Secondary: 46E15.
Key words and phrases. Subspace-supercyclic, Subspace-Supercyclicity Criterion, Banach space.

The definition and the properties of supercyclic operators were introduced by Hilden and Wallen [6]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic. The study of supercyclic operators has experienced a great of development in recent years. Salas gave a characterization of supercyclic bilateral backward weighted shifts via the Supercyclicity Criterion in [12]. Montes and Salas [10] refined the Supercyclicity Criterion and proved that it is equivalent to the former one given by Salas. Besides, T. Bermúdez, A. Bonilla and A. Peris [2] showed that the equivalence of two supercyclicity criteria given by N. Feldman, V. Miller and L. Miller in [4] to the Supercyclicity Criterion.

Recently, B. F. Madore and R. A. Martínez-Avendaño in [9] introduced the concept of subspace-hypercyclicity and proved several results analogous to the hypercyclic case.

For example, there is a Kitai-like Criterion: if T is a subspace-hypercyclic, then its spectrum must intersect the unit circle; subspace-hypercyclicity is a strictly infinite dimensional phenomenon; compact or hyponormal operators are not subspace-hypercyclic and provided a Subspace-Hypercyclicity Criterion and they constructed examples to show that subspace-hypercyclicity is interesting, including a nontrivial subspace-hypercyclic operator that is not hypercyclic. Besides, C.M. Le [8] gave an improvement of the Kitai-like Criterion and showed that if an operator T satisfies a strong condition than the Hypercyclicity Criterion, it is subspace-hypercyclic for any finite co-dimensional subspace. Other sources of examples and some properties of notions relating subspace-hypercyclicity and subspace-supercyclicity are [7, 11].

Similarly, for subspace-supercyclicity, Zhao et al. in [13] provided a Subspace-Supercyclicity Criterion and offered two necessary and sufficient conditions for a path of bounded linear operators to have a dense G_δ set of common subspace-hypercyclic vectors and common subspace-supercyclic vectors and they also constructed examples to show that subspace-supercyclic is not a strictly infinite dimensional phenomenon and that some subspace-supercyclic operators are not supercyclic.

Note that if the operator T is invertible, T is supercyclic if and only if T^{-1} is supercyclic (see, [3, Theorem 1.12]). And if $T \in B(X)$ is supercyclic, then there exists $R \geq 0$ such that the circle $\{|z| = R\}$ intersects each component of the spectrum of T. The circle $\{|z| = R\}$ is called a supercyclic circle of T (see, [3, Theorem 1.24]).

For the subspace-supercyclic operator T, the similar questions were posed from [13] as follows:

Question 1.1. If $T \in B(X)$ is subspace-supercyclic for some infinite dimensional subspace M, then does T have a subspace-supercyclic circle?

Question 1.2. Let T be invertible and T be subspace-supercyclic for some subspace M. Is T^{-1} subspace-supercyclic for some space?

Question 1.3. If T is subspace-supercyclic for some subspace M, is T^* subspace-supercyclic for some space?

In this present paper, we will partially answer these questions and we provide a Subspace-Supercyclicity Criterion and offer two equivalent conditions of this criterion. At the same time, we also obtain a slight improvement of the Subspace-Supercyclicity Criterion.

2. Subspace-supercyclicity

In this section, we will discuss the subspace-supercyclicity.

Definition 2.1. Let $T \in B(X)$ and let M be a nonzero subspace of X. We say that T is M-supercyclic for M if there exists $x \in X$ such that $\mathbb{C}orb\,(T, x) \cap M$ is dense in M. We call x a M-supercyclic vector.

Remark 2.2. The definition above reduces to the classical definition of supercyclicity if $M = X$ and we may assume that the subspace-supercyclic vector $x \in M$, if needed.

First, there are some simple examples of subspace-supercyclic operators.

Example 1. Consider the operator $T := \mathbb{B}$, where $\mathbb{B}$ is the backwad shift on $l^2(\mathbb{N})$ and I is the identity operator on $l^2(\mathbb{N})$. Then the operator $T \oplus I$ is subspace-supercyclic for the subspace $M := l^2 \oplus \{0\}$ of $l^2 \oplus l^2$; whereas it cannot be supercyclic on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, since the identity operator is not supercyclic on $l^2(\mathbb{N})$. But, $T \oplus I$ is not subspace-hypercyclic for the subspace $M := l^2 \oplus \{0\}$ of $l^2 \oplus l^2$, since the operator $\mathbb{B}$ is not hypercyclic on $l^2(\mathbb{N})$.

Example 2. It can easily be shown that the operator $\left(\frac{1}{2}\mathbb{B}\right) \oplus (4I) : l^2(\mathbb{N}) \oplus l^2(\mathbb{N}) \to l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ is subspace-supercyclic for $l^2(\mathbb{N}) \oplus \{0\}$ with subspace-supercyclic vector $f \oplus 0$, where f is a supercyclic vector for $\frac{1}{2}\mathbb{B}$. Observe that the spectrum $\sigma\left(\left(\frac{1}{2}\mathbb{B}\right) \oplus (4I)\right)$ is the closed disk of radius $\frac{1}{2}$ union the singleton $\{4\}$. Thus, it is not true that every component of the spectrum must intersect some circle for this subspace-supercyclic operator, which partially solves Question 1.1.

As far as we know, if T is subspace-hypercyclic for M, then M is not finite-dimensional (see, [9, Theorem 4.10]); but if T is subspace-supercyclic for some space M, then M may be finite-dimensional (see, [13, Proposition 1.3]). Besides, the adjoint of a supercyclic operator T cannot have more than one eigenvalue (see, [3, Proposition 1.26]). Let $\mathcal{H}$ denote a separable Hilbert space over $\mathbb{C}$, the field of complex numbers. Similarly, we have the following result:

Theorem 2.3. *Let $T \in B(\mathcal{H})$, M be a nonzero reducing subspace of X. If T is a subspace-supercyclic for M, then* $\ker(T^* - \lambda) \subseteq M^\perp$ *for all $\lambda \in \mathbb{C}$ or* $\ker(T^* - \lambda) \subseteq M$ *for some $\lambda \in \mathbb{C} \backslash \{0\}$. In particular,* $\ker(T^* - \lambda) = \ker(T^* - \lambda)^p$, *for all $\lambda \in \mathbb{C}$ and all $p \in \mathbb{N}$.*

The proof relies on the following elementary lemma.

Lemma 2.4. [3, Lemma 1.27] *Let $a, b, c, \lambda, \mu \in \mathbb{C}$. Then the sets $\mathbb{C} \cdot \{(a\lambda^n, b\mu^n); n \in \mathbb{N}\}$ and $\mathbb{C} \cdot \{(a, cn + b); n \in \mathbb{N}\}$ are not dense in $\mathbb{C}^2$.*

Proof. of Theorem 2.3.

Let x be a M-supercyclic vector for T and $\mathbb{C}orb\,(T,x)\cap M$ be dense in M. The proof follows ideas of the proof of [3, Proposition 1.26]. Let $x^*, y^* \in \mathcal{H}^*$. Note that if $x^*|_M,\ y^*|_M$ are linearly independent and $x^* \notin M^\perp$ and $y^* \notin M^\perp$, then the set

$$A_{x^*,y^*} := \mathbb{C}\cdot\{(\langle x^*, T^n x\rangle, \langle y^*, T^n x\rangle);\ T^n x \in M,\ n\in\mathbb{N}\}$$

is dense in $\mathbb{C}^2$. Indeed, the linear map $\Phi_{x^*,y^*} := M \to \mathbb{C}^2$ defined by $\Phi_{x^*,y^*}(z) := (\langle x^*, z\rangle, \langle y^*, z\rangle)$ is continuous and onto, so it maps the dense set $\mathbb{C}\cdot orb(x,T)\cap M$ in M onto a dense subset of $\mathbb{C}^2$.

If x^* and y^* are any two eigenvectors of the adjoint of T and $x^* \notin M^\perp$ and $y^* \notin M^\perp$, there exist $\lambda, \mu \in \mathbb{C}$ such that $T^*(x^*) = \lambda x^*$ and $T^*(y^*) = \mu y^*$. By Lemma 2.4, $A_{x^*,y^*} = \mathbb{C}\cdot\{(\langle x^*, x\rangle\lambda^n, \langle y^*, x\rangle\mu^n);\ T^n x\in M,\ n\in\mathbb{N}\}$ is not dense in $\mathbb{C}^2$. So $x^*|_M,\ y^*|_M$ are linearly dependent. It is easy to know that M-supercyclic operator has dense range in M. Note that $T(M)\subseteq M$ and $T\left(M^\perp\right)\subseteq M^\perp$, so $\lambda = \mu \neq 0$. Therefore, $\ker(T^* - \lambda)\subseteq M^\perp$ for all $\lambda\in\mathbb{C}$ or $\ker(T^*-\lambda)\subseteq M$ for some $\lambda\in\mathbb{C}\backslash\{0\}$.

If $x_0^* \in \ker(T^* - \lambda I)$, $T^* x_0^* = \lambda x_0^*$. For $a\in\mathbb{C}\backslash\{0\}$, since T is a subspace-supercyclic for M if and only if aT is also subspace-supercyclic for M, we may in fact assume that $\lambda = 1$, that is, $T^* x_0^* = x_0^*$. Let $y^*\in\ker(T^*-I)^2$ be arbitrary. Then $(T^*-I)y^*\in\ker(T^*-I) = \mathbb{C}x_0^*$, so one can write $T^*(y^*) = y^* + \gamma x_0^*$, where $\gamma\in\mathbb{C}$. By induction, $T^{*n}(y^*) = y^* + n\gamma x_0^*$ for all $n\in\mathbb{N}$. By Lemma 2.4,

$$A_{x_0^*,y^*} = \mathbb{C}\cdot\{(\langle x_0^*, x\rangle, \gamma\langle x_0^*, x\rangle n + \langle y^*, x\rangle);\ T^n x\in M,\ n\in\mathbb{N}\}$$

is not dense in $\mathbb{C}^2$, so that $y^*\in\mathbb{C}x_0^*$. Thus $\ker(T^*-I) = \ker(T^*-I)^2$. Then we get $\ker(T^*-I)^n = \ker(T^*-I)$ for all $n\geq 1$ by straightforward induction. The proof is complete. □

Definition 2.5. Let $T\in B(X)$, M be a nonzero subspace of X. For every $x\in M$, the set
$J_S(T,M,x) := \{y\in M :$ *for every relatively open neighborhoods* U, V *of* x , y *in* M *respectively, and every positive integer* N, *there exist* $n > N$ *and* $\lambda\in\mathbb{C}\backslash\{0\}$ *such that* $\lambda T^n(U)\cap V$ *is nonempty and* $T^n(M)\subseteq M.\}$
denotes the M-generalized limit set of x under T.

Proposition 2.6. *An equivalent definition of* $J_S(T,M,x)$ *is the following.*
$J_S(T,M,x) := \{y\in M :$ *there exists a strictly increasing sequence of positive integers* $(k_n)_n$, *a sequence* $(x_n)_n\subseteq M$ *and* $(\lambda_{k_n})_n\subseteq\mathbb{C}\backslash\{0\}$ *such that* $x_n\to x$ *and* $\lambda_{k_n}T^{k_n}x_n\to y$ *and for every* $n, T^{k_n}(M)\subseteq M\}$.

Proof. Let $\forall y\in J_S(T,M,x)$ and consider the open balls

$$U_n = B\left(x,\frac{1}{n}\right)\cap M, V_n = B\left(y,\frac{1}{n}\right)\cap M,\ \text{for } n = 1,2,\cdots$$

and $N = k_{n-1},\ k_0 = 1$. Then there exist $k_n > N = k_{n-1}$ and $(\lambda_{k_n})_n\subseteq\mathbb{C}\backslash\{0\}$ such that

$$\lambda_{k_n}T^{k_n}(U_n)\cap V_n\neq\emptyset\ \text{ and }\ T^{k_n}(M)\subseteq M.$$

Hence there exists $x_n \in U_n$ such that $\lambda_{k_n} T^{k_n} x_n \in V_n$ and $T^{k_n}(M) \subseteq M$. Therefore, there exist a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(x_n)_n \subseteq M$ and $(\lambda_{k_n})_n \subseteq \mathbb{N}$ such that $x_n \to x$ and $\lambda_{k_n} T^{k_n} x_n \to y$ and for every n, $T^{k_n}(M) \subseteq M$. The converse is obvious. The proof is complete. $\square$

Theorem 2.7. *Let $T \in B(X)$ and M be a nonzero subspace of X. For every $x \in M$, $J_S(T, M, x) = M$. Then T is subspace-supercyclic for M.*

Proof. For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, consider $x_0 \in V$, $y_0 \in U$. Since $J_S(T, M, x_0) = M$, there exist $n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda T^n(V) \cap U \neq \emptyset$ and $T^n(M) \subseteq M$. By Theorem 1.8 in [13], T is subspace-supercyclic for M. $\square$

Next, we partially solve the Question 1.2.

Theorem 2.8. *Let $T \in B(X)$, T be an invertible operator and M be a nonzero subspace of X. If for every $x \in M$, $J_S(T, M, x) = M$, then T^{-1} is also subspace-supercyclic for M.*

Proof. First, by Theorem 2.7, T is subspace-supercyclic for M. For any $x, y \in M$, by assumption, $J_S(T, M, x) = M$. For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open such that contain x, y respectively, then there exist $n > 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda T^n(U) \cap V \neq \emptyset$ and $T^n(M) \subseteq M$. By the invertibility of T, $\lambda^{-1}(T^{-1})^n(V) \cap U \neq \emptyset$ and $T^{-n}(M) \subseteq M$. Hence for every $y \in M$, $J_S(T^{-1}, M, y) = M$. By Theorem 2.7, T^{-1} is also subspace-supercyclic for M. $\square$

We know from [5] that there exists an operator T on $l^2(\mathbb{Z})$ such that T and its adjoint T^* are weakly mixing and hence hypercyclic. The following example shows that there exists the operator T such that T and its adjoint T^* are subspace-supercyclic and partially answer Question 1.3.

Example 3. As usual, we identify the dual of $l^2(\mathbb{Z})$ with itself. Let $T = B_w$ be a bilateral shift and $B_w e_n = w_n e_{n-1}\,(n \in \mathbb{Z})$. If the w_n, $n \in \mathbb{Z}$, are bounded, then $T = B_w$ defines an operator on $l^2(\mathbb{Z})$. It is easy to know that the adjoint $T^* = B_w^*\ of\ B_w$ is the forward shift $F_{(w_{n+1})}$. Let

$$v_n = \left(\prod_{i=1}^{n} w_i\right)^{-1} (n \geq 1), v_n = \left(\prod_{i=n+1}^{0} w_i\right) (n \leq -1), v_0 = 1.$$

We choose the symmetric sequence $(v_n)_{n \in z}$ with

$$(v_n)_{n \geq 0} = \left(1, 1, 2, 1, \frac{1}{2}, 1, 2, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \cdots\right).$$

By Proposition 4.16 in [5], T and its adjoint T^* are hypercyclic. Then the operator $T_1 = T \oplus I$ is subspace-supercyclic for the subspace $M := l^2(\mathbb{Z}) \oplus \{0\}$ of $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ and $T_1^* = T^* \oplus I$ is also subspace-supercyclic for the subspace $M := l^2(\mathbb{Z}) \oplus \{0\}$ of $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$.

Next we get the following theorem, which is the Subspace-Supercyclicity Criterion, which is similar to the Supercyclictiy Criterion that was stated in [2].

Theorem 2.9. *(Subspace-Supercyclicity Criterion) Let $T \in B(X)$ and M be a nonzero subspace of X. Assume that there exist M_0 and M_1, dense subsets of M, an increasing sequence of positive integers $(n_k)_k$, a sequence $(\lambda_{n_k})_{k\in\mathbb{N}} \subseteq \mathbb{C}\backslash\{0\}$ and a sequence of mappings $S_{n_k} : M_1 \to M$ such that*

(i) $\lambda_{n_k} T^{n_k} x \to 0$ for every $x \in M_0$,

(ii) $\frac{1}{\lambda_{n_k}} S_{n_k} y \to 0$ for every $y \in M_1$,

(iii) $(T^{n_k} \circ S_{n_k})y \to y$, for every $y \in M_1$.

(iv) M is an invariant subspace for T^{n_k} for all $k \in \mathbb{N}$.

Then T is subspace-supercyclic for M.

Proof. Let U and V be non-empty open subsets of M. Since M_0 and M_1 are dense in M, there exist $x \in M_0 \cap V$, $y \in M_1 \cap U$. And since U and V are nonempty open subsets, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq V$ and $B(y,\varepsilon) \subseteq U$. By assumption, there exist n_k and $\lambda_{n_k} \in \mathbb{C}\backslash\{0\}$ such that

$$\|\lambda_{n_k} T^{n_k} x\| < \frac{\varepsilon}{2}, \ \left\|\lambda_{n_k}^{-1} S_{n_k} y\right\| < \frac{\varepsilon}{2} \text{ and } \ \|T^{n_k} S_{n_k} y - y\| < \frac{\varepsilon}{2}.$$

Define $u = x + \lambda_{n_k}^{-1} S_{n_k} y$. We know that $u \in M$ and $u \in V$, since $\|u - x\| = \left\|\lambda_{n_k}^{-1} S_{n_k} y\right\| < \frac{\varepsilon}{2}$. Observe that $\lambda_{n_k} T^{n_k} u = \lambda_{n_k} T^{n_k} x + T^{n_k} S_{n_k} y$, so $\lambda_{n_k} T^{n_k} u \in M$. Since

$$\|\lambda_{n_k} T^{n_k} u - y\| = \|\lambda_{n_k} T^{n_k} x\| + \|T^{n_k} S_{n_k} y - y\| < \varepsilon,$$

we have that $\lambda_{n_k} T^{n_k} u \in U$. Then $(\lambda_{n_k} T^{n_k})^{-1} U \cap V \neq \emptyset$ and T is subspace-supercyclic for M. $\square$

Lemma 2.10. [13, Theorem 1.7] *Let $T \in B(X)$, M be a nonzero subspace of X, then $SC(T,M) \bigcap M = \left(\cap_j \cup_{\lambda,n} (\lambda T^n)^{-1}(V_j)\right) \cap M$, where $\{V_j\}_{j\in N^+}$ is a countable basis of open sets for M.*

T. Bermúdez, A. Bonilla and A. Peris in [2] showed that the equivalence of two supercyclicity criteria given by N. Feldman, V. Miller and L. Miller in [4] to the Supercyclicity Criterion. Now, we will show that they hold for subspace M.

Theorem 2.11. *Let $T \in B(X)$ and M be a nonzero subspace of X. Then the following (a), (b) and (c) are equivalent:*

(a) T satisfies Subspace-Supercyclicity Criterion.

(b) (Outer Subspace-Supercyclicity Criterion) There exist an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, a dense linear subspace $Y_0 \subseteq M$ and, for each $y \in Y_0$, a dense linear subspace $X_y \subseteq M$ such that:

(i). There exists a sequence of maps $S_{n_k} : Y_0 \to M$ such that $(T^{n_k} \circ S_{n_k})y \to y$, for every $y \in Y_0$ and

(ii). $\|T^{n_k} x\| \|S_{n_k} y\| \to 0$ for every $y \in Y_0$ and $x \in X_y$.

(iii). M is an invariant subspace for T^{n_k} for all $k \in \mathbb{N}$.

(c) (Inner Subspace-Supercyclicity Criterion) There exist an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, a dense linear subspace $Y_0 \subseteq M$ and, for each $y \in Y_0$, a dense linear subspace $X_y \subseteq M$ such that:

(i). There exists a sequence of maps $S_{y,n_k} : X_y \to M$ such that $T^{n_k} \circ S_{y,n_k} x \to x$ for all $x \in X_y$, and

(ii). $\|T^{n_k} y\| \|S_{y,n_k} x\| \to 0$ for every $y \in Y_0$ and $x \in X_y$.

(iii). M is an invariant subspace for T^{n_k} for all $k \in \mathbb{N}$.

Proof. It is obvious that any operator satisfying the Subspace-Supercyclicity Criterion also satisfies the criteria of (b) and (c). It suffices to show that (b) implies (a), since the other case is analogous. Let $U_i,\ V_i \subseteq M$ non-empty open sets with $i = 1, 2$. The same argument as in the proof of Theorem 3.2 in [2] can be used to show that there exist $n_k \in \mathbb{N}$ and $\lambda_{n_k} \in \mathbb{C}\backslash\{0\}$ such that

$$(\lambda_{n_k} T^{n_k})^{-1} (U_i) \cap V_i \neq \emptyset, \text{ for } i = 1, 2.$$

Then we can know that $T \bigoplus T$ is subspace-supercyclic for $M \bigoplus M$ and (x, y) is subspace-supercyclic vector for $T \bigoplus T$. In particular, x be subspace-supercyclic vector for T and $SC(T, M) \bigcap M$ is a dense G_δ subset of M. Now fix $(U_k)_{k\geq 1}$ a decreasing 0-basis in M. Proceeding by induction we find $u_k \in U_k$, for all $k \in \mathbb{N}$, an increasing subsequence $(m_k)_{k\geq 1}$ of $(n_k)_{k\geq 1}$ and $(\lambda_{m_k})_k \subseteq \mathbb{C}\backslash\{0\}$ satisfying

$$\lambda_{m_k} T^{m_k} x \in U_k \text{ and } \lambda_{m_k} T^{m_k} u_k \in x + U_k, \tag{2.1}$$

for all $k \in \mathbb{N}$. Let $M_0 = M_1 = \mathbb{C}orb(T, x) \cap M$, which is dense in M. From (2.1), we have that $\lambda_{m_k} T^{m_k} x \to 0$ and then $\lambda_{m_k} T^{m_k} y \to 0$ for every $y \in M_0$. Define $S_{m_k}(\lambda T^n x) = \lambda_{m_k}.\lambda T^n u_k$, for all $n, k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then $\lambda_{m_k}^{-1} S_{m_k} v \to 0$, for every $v \in M_1$. Finally, given $n \in \mathbb{N}, \lambda \in \mathbb{C}$, by (2.1), we have that

$$T^{m_k} S_{m_k}(\lambda T^n x) = \lambda_{m_k} T^{m_k} \left(\lambda_{m_k}^{-1} S_{m_k}(\lambda T^n x)\right) = \lambda_{m_k} T^{m_k} (\lambda T^n u_k) \to \lambda T^n x.$$

Hence (a) holds. We complete the proof. □

Next, we provide a slight improvement of the Subspace-Supercyclicity Criterion.

Theorem 2.12. *Let $T \in B(X)$ and M be a nonzero closed subspace of X. Assume that there exist subsets X_0 and Y_0 of M, where Y_0 is dense in M, an increasing sequence $(n_k)_k$ of positive integers and $(\lambda_{n_k})_k \subseteq \mathbb{C}\backslash\{0\}$ such that the following hold:*

(i) $\lambda_{n_k} T^{n_k} x \to 0$ for all $x \in X_0$,

(ii) for each $y \in Y_0$, there exists a sequence $(x_k)_{k=1}^\infty$ in X_0 such that

$$\frac{1}{\lambda_{n_k}} x_k \to 0 \text{ and } T^{n_k} x_k \to y,$$

(iii) $X_0 \subseteq \cap_{k=1}^\infty T^{-n_k} \left(\frac{1}{\lambda_{n_k}} M\right)$

Then T is subspace-supercyclic for M.

Proof. Since X is separable and Y_0 is dense in M, we assume that $Y_0 = (y_j)_{j=1}^\infty$. Let $(\varepsilon_j)_{j=1}^\infty$ be a sequence of positive numbers such that

$$\lim_{j\to\infty} \left(j\varepsilon_j + \sum_{i=j+1}^\infty \varepsilon_i \right) = 0.$$

By assumption, choose $x_{k_1} \in X_0,\ n_{k_1} \in (n_k)_k$ and $\lambda_{n_{k_1}} \in (\lambda_{n_k})_k$ such that

$$\left\| \frac{1}{\lambda_{n_{k_1}}} x_{k_1} \right\| + \| T^{n_{k_1}} x_{k_1} - y_1 \| < \varepsilon_1.$$

For each j, choose $x_{k_j} \in X_0$, $n_{k_j} \in (n_k)_k$ and $\lambda_{n_{k_j}} \in (\lambda_{n_k})_k$ such that

$$\left\|\frac{1}{\lambda_{n_{k_j}}} x_{k_j}\right\| + \left\|\lambda_{n_{k_j}} T^{n_{k_j}} \frac{x_{k_i}}{\lambda_{n_{k_i}}}\right\| + \left\|\lambda_{n_{k_i}} T^{n_{k_i}} \frac{x_{k_j}}{\lambda_{n_{k_j}}}\right\| + \left\|T^{n_{k_j}} x_{k_j} - y_j\right\| < \varepsilon_j, \text{ for all } i < j.$$

By induction, we construct a sequence $\left(x_{k_j}\right)_{j=1}^{\infty} \subseteq X_0$, a subsequence $\left(n_{k_j}\right)_{j=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$ and a subsequence $\left(\lambda_{n_{k_j}}\right)_{j=1}^{\infty}$ of $(\lambda_{n_k})_k \subseteq \mathbb{C}\backslash\{0\}$.

Let $x = \sum_{i=1}^{\infty} \frac{1}{\lambda_{n_{k_i}}} x_{k_i}$, where it is well defined. Obviously, $\lambda_{n_{k_j}} T^{n_{k_j}} x \in M$, for every j. Then

$$\begin{aligned}
\left\|\lambda_{n_{k_j}} T^{n_{k_j}} x - y_j\right\| &= \left\|T^{n_{k_j}} x_{k_j} - y_j + \sum_{i=1}^{j-1} \lambda_{n_{k_j}} T^{n_{k_j}} \frac{x_{k_i}}{\lambda_{n_{k_i}}} + \sum_{i=1+j}^{\infty} \lambda_{n_{k_j}} T^{n_{k_j}} \frac{x_{k_i}}{\lambda_{n_{k_i}}}\right\| \\
&\leq \varepsilon_j + \left\|\sum_{i=1}^{j-1} \lambda_{n_{k_j}} T^{n_{k_j}} \frac{x_{k_i}}{\lambda_{n_{k_i}}}\right\| + \left\|\sum_{i=1+j}^{\infty} \lambda_{n_{k_j}} T^{n_{k_j}} \frac{x_{k_i}}{\lambda_{n_{k_i}}}\right\| \\
&< j\varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon_i.
\end{aligned}$$

Therefore, T is subspace-supercyclic for M. □

Remark 2.13. Notice that in Theorem 2.12 the set X_0 is not assumed to be dense in M.

We end this section by mentioning related problems.

Question 2.14. If T is supercyclic, must there be a proper subspace M such that T is subspace-supercyclic for M?

We should mention that the authors in [9] showed that compact or hyponormal operators are not subspace-hypercyclic. But, there exists compact subspace-supercyclic operator T, as showed by [13]. Besides, Hilden and Wallen showed in [6] that unitary operators are not supercyclic; and hyponormal operators are not supercyclic, as showed by [1]. Similarly, the question for subspace-supercyclicity can be given:

Question 2.15. Can hyponormal operators be subspace-supercyclic for some nonzero subspace M of the Hilbert space H? and can unitary operators be subspace-supercyclic for some nonzero subspace M of the Hilbert space H?

Acknowledgments. We would like to thank the editor and referee for useful comments and suggestions which improved the presentation of this paper. The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11201331).

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[1]Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China.
E-mail address: 168zhangliang2011@163.com

[2]Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China.
Center for Applied Mathematics, Tianjin University, Tianjin 300072, P.R. China.
E-mail address: zehuazhoumath@aliyun.com;zhzhou@tju.edu.cn