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M^k-TYPE SHARP MAXIMAL FUNCTION INEQUALITIES AND BOUNDEDNESS FOR TOEPLITZ TYPE OPERATOR ASSOCIATED TO PSEUDO-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we establish the M^k-type sharp maximal function inequalities for the Toeplitz type operator associated to the pseudo-differential operator. As an application, we obtain the boundedness of the operator on Lebesgue and Morrey spaces.

1. Introduction and preliminaries

As the development of singular integral operators(see [6, 17]), their commutators have been well studied. In [2, 15], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. In [7, 8, 10], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators are obtained. In this paper, we will study the Toeplitz type operator generated by the pseudo-differential operator and BMO functions.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

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where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [6, 17])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. It has been known that (see [17])

$$||f - f_{2^kQ}||_{BMO} \le Ck||f||_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$ and $M_\eta^{\#}(f) = M^{\#}(|f|^\eta)^{1/\eta}$.

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f) = M(f)$ and

$$M^k(f) = M(M^{k-1}(f)) \text{ when } \ k \ge 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q}||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1+logt)$ and $\tilde{\Phi}(t) = exp(t)$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL),Q}$, $M_{L(logL)}$ and $||\cdot||_{expL,Q}$, M_{expL}. Following [15], we know the generalized Hölder's inequality and the following inequalities hold:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q},$$

$$||f||_{L(logL),Q} \le M_{L(logL)}(f) \le CM^2(f),$$

$$||f - f_Q||_{expL,Q} \le C||f||_{BMO}$$

and

$$||f - f_Q||_{expL,2^kQ} \le Ck||f||_{BMO}.$$

The A_p weight is defined by (see [6]), for $1 < p < \infty$,

$$A_p = \left\{w \in L^1_{loc}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\}$$

and

$$A_1 = \{w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e.\}.$$

Given a weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

Definition 1.1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for} \quad t \ge 0.$$

Let w be a weight function and f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x \in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\eta$, $\eta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\eta}(R^n, w)$, which is the classical weighted Morrey spaces (see [13, 14]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [6]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3, 4, 9, 12]).

In this paper, we will study certain pseudo-differential operator as following (see [1]).

We say a symbol $\sigma(x,\xi)$ is in the class $S^m_{\rho,\delta}$ or $\sigma \in S^m_{\rho,\delta}$, if for $x, \xi \in R^n$,

$$\left| \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial \xi^\nu} \sigma(x,\xi) \right| \le C_{\mu,\nu}(1 + |\xi|)^{m - \rho|\nu| + \delta|\mu|},$$

where μ, ν are multi-indices and $|\mu| = |\mu_1| + \cdots + |\mu_n|$. A pseudo-differential operator with symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$ is defined by

$$T(f)(x) = \int_{R^n} e^{2\pi i x\cdot\xi} \sigma(x,\xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. We know there exists a kernel $K(x,y)$ such that (see [1])

$$T(f)(x) = \int_{R^n} K(x, x-y) f(y) dy,$$

where, formally,

$$K(x,y) = \int_{R^n} e^{2\pi i (x-y)\cdot\xi} \sigma(x,\xi) d\xi.$$

In [5], the boundedness of the pseudo-differential operators with symbol $\sigma \in S^{-\beta}_{1-\theta,\delta}(\beta < n\theta/2, 0 \le \delta < 1-\theta)$ are obtained. In [11], the boundedness of the pseudo-differential operators with symbol of order 0 and $-\infty$ are obtained. In [1], the sharp function estimate of the pseudo-differential operators with symbol

$\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ are obtained. In [5, 16, 18], the boundedness of the pseudo-differential operators and their commutators with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ are obtained. Our study are motivated by these papers.

Suppose T is a pseudo-differential operator with symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$. Let b be a locally integrable function on R^n. The Toeplitz type operator associated to T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are the pseudo-differential operator T with symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$ or $\pm I$(the identity operator), $T^{k,2}$ are the bounded linear operators on $L^p(w)$ for $1 < p < \infty$ and $w \in A_1$, $k = 1, \cdots, m$, $M_b(f) = bf$.

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [15]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the L^p-norm inequality and Morrey spaces boundedness for the Toeplitz type operator T_b.

2. Some lemmas

We begin with the following lemmas.

Lemma 2.1. *(see [6], p. 485) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 2.2. *(see [15]) We have*

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \leq ||f||_{expL,Q}||g||_{L(logL),Q}.$$

Lemma 2.3. *(see [1]) Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$. Then T is bounded on $L^p(w)$ for $1 < p < \infty$, $w \in A_1$ and weak (L^1, L^1) bounded.*

Lemma 2.4. *(see [1]) Let $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ and K be the kernel of the pseudo-differential operator T with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$. Then, for*

$|x_0 - x| \le d < 1$ *and* $j \ge 1$,

$$\left(\int_{(2^j d)^{1-\theta} \le |y-x_0| < (2^{j+1} d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy\right)^{1/2}$$
$$\le \quad C \frac{|x_0 - x|^{(1-\theta)(m-n/2)}}{(2^j d)^{m(1-\theta)}},$$

provided m *is an integer such that* $n/2 < m < n/2 + 1/(1-\theta)$.

Lemma 2.5. *(see [1]) Let* $\sigma \in S^0_{\rho,\delta}(0 < \rho < 1)$ *and*

$$K(x, w) = \int_{R^n} e^{2\pi i w \cdot \xi} \sigma(x, \xi) d\xi.$$

Then, for $|w| \ge 1/4$ *and any integer* $N \ge 1$,

$$|K(x, w)| \le C_N |w|^{-2N}.$$

Lemma 2.6. *(see [6]) Let* $0 < p < \infty$, $0 < \eta < \infty$ *and* $w \in \cup_{1 \le r < \infty} A_r$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p w(x) dx \le C \int_{R^n} M^\#_\eta(f)(x)^p w(x) dx.$$

Lemma 2.7. *(see [3, 4]) Let* $1 < p < \infty$, $w \in A_1$ *and* $0 < D < 2^n$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}(w)} \le C ||f||_{L^{p,\varphi}(w)}.$$

Lemma 2.8. *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_1$ *and* $0 < D < 2^n$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \le C ||M^\#_\eta(f)||_{L^{p,\varphi}(w)}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [6]. By Lemma 2.6, we have, for $f \in L^{p,\varphi}(R^n, w)$,

$$\begin{aligned}
&\int_Q |M_\eta(f)(y)|^p w(y)dy \\
=\ & \int_{R^n} |M_\eta(f)(y)|^p w(y)\chi_Q(y)dy \\
\le\ & \int_{R^n} |M_\eta(f)(y)|^p M(w\chi_Q)(y)dy \\
\le\ & C\int_{R^n} |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy \\
=\ & C\Big(\int_Q |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy \\
& \quad + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M_\eta^\#(f)(y)|^p M(w\chi_Q)(y)dy\Big) \\
\le\ & C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M_\eta^\#(f)(y)|^p \frac{w(Q)}{|2^{k+1}Q|}dy\right) \\
\le\ & C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M_\eta^\#(f)(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right) \\
\le\ & C\left(\int_Q |M_\eta^\#(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M_\eta^\#(f)(y)|^p \frac{w(y)}{2^{nk}}dy\right) \\
\le\ & C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d) \\
\le\ & C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d) \\
\le\ & C||M_\eta^\#(f)||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \le C||M_\eta^\#(f)||_{L^{p,\varphi}(w)}.$$

This finishes the proof. $\square$

Lemma 2.9. *Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \le \delta < 1-\theta)$, $1 < p < \infty$, $w \in A_1$ and $0 < D < 2^n$. Then*

$$||T(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

The proof of the Lemma is similar to that of Lemma 2.8 by Lemma 2.3, we omit the details.

3. Theorems and proofs

Theorem 3.1. *Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$, $0 < r < 1$, $2 < s < \infty$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M_r^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{BMO} \sum_{k=1}^{m} (M^2(T^{k,2}(f))(\tilde{x}) + M_s(T^{k,2}(f))(\tilde{x})).$$

Proof. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^r \, dx\right)^{1/r} \leq C||b||_{BMO} \sum_{k=1}^{m} (M^2(T^{k,2}(f))(\tilde{x}) + M_s(T^{k,2}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \cdots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We consider the following two cases:

Case 1. $d \leq 1$. In this case, let $\tilde{Q}$ be the cube concentric with Q of side length $d^{1-\theta}$. We write, by $T_1(g) = 0$,

$$T_b(f)(x) = T_{b-b_{2\tilde{Q}}}(f)(x) = T_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}(f)(x) + T_{(b-b_{2\tilde{Q}})\chi_{(2\tilde{Q})^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^r \, dx\right)^{1/r} \leq \left(\frac{C}{|Q|}\int_Q |f_1(x)|^r dx\right)^{1/r}$$
$$+ \left(\frac{C}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^r dx\right)^{1/r} = I_1 + I_2.$$

For I_1, recalling $s > 2$, for $1 < t_1 < \infty$ with $1/s + 1/t_1 = 1/2$, let $\sigma(x,\xi) = \sigma(x,\xi)|\xi|^{n\theta/2}|\xi|^{-n\theta/2} = q(x,\xi)|\xi|^{-n\theta/2}$, we have $q(x,\xi) \in S^0_{1-\theta,\delta}$, set S be the pseudo-differential operator with symbol $q(x,\xi)$, by the Hardy-Littlewood-Soboleve fractional integration theorem and the L^2-boundedness of S (see [1]), we obtain, for $1/s = 1/2 - \theta/2$,

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}T^{k,2}(f)(x)|dx \\
\leq & \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq & |Q|^{-1/s}\left(\int_{R^n} |SM_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}T^{k,2}(f)(x)|^2 dx\right)^{1/2} \\
\leq & C|Q|^{-1/s}\left(\int_{R^n} |M_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}T^{k,2}(f)(x)|^2 dx\right)^{1/2} \\
\leq & C|Q|^{-1/s}\left(\int_{2\tilde{Q}} (|b(x)-b_{2\tilde{Q}}||T^{k,2}(f)(x)|)^2 dx\right)^{1/2} \\
\leq & C\frac{|\tilde{Q}|^{1/2}}{|Q|^{1/s}}\left(\frac{1}{|2\tilde{Q}|}\int_{2\tilde{Q}} |b(x)-b_{2\tilde{Q}}|^{t_1} dx\right)^{1/t_1}\left(\frac{1}{|2\tilde{Q}|}\int_{2\tilde{Q}} |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq & C||b||_{BMO}\sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\leq C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{2\tilde{Q}}}T^{k,2}(f)(x)|^r dx\right)^{1/r} \\
&\leq C||b||_{BMO}\sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_2, recalling $n/2 < m$, by Lemma 2.4, we get, for $1 < t_2 < \infty$ with $1/s + 1/t_2 = 1/2$ and $x \in Q$,

$$\begin{aligned}
& |T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x_0)| \\
\leq & \int_{(2\tilde{Q})^c} |b(y)-b_{2\tilde{Q}}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy \\
\leq & \sum_{j=1}^\infty \int_{(2^jd)^{1-\theta}\leq|y-x_0|<(2^{j+1}d)^{1-\theta}} |b(y)-b_{2\tilde{Q}}||K(x,x-y)-K(x_0,x_0-y)| \\
& \times|T^{k,2}(f)(y)|dy \\
\leq & C\sum_{j=1}^\infty \left(\int_{2^{j+1}\tilde{Q}} |b(y)-b_{2\tilde{Q}}|^{t_1} dy\right)^{1/t_1}\left(\int_{2^{j+1}\tilde{Q}} |T^{k,2}(f)(y)|^s dy\right)^{1/s} \\
& \times\left(\int_{(2^jd)^{1-\theta}\leq|y-x_0|<(2^{j+1}d)^{1-\theta}} |K(x,x-y)-K(x_0,x_0-y)|^2 dy\right)^{1/2} dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}||b||_{BMO}j(2^jd)^{n(1-\theta)/r_1}(2^jd)^{n(1-\theta)/s}\frac{d^{(1-\theta)(m-n/2)}}{(2^jd)^{m(1-\theta)}}\\
&\quad\times\left(\frac{1}{|2^{j+1}\tilde{Q}|}\int_{2^{j+1}\tilde{Q}}|T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\le C||b||_{BMO}\sum_{j=1}^{\infty}j2^{j(1-\theta)(n/2-m)}\left(\frac{1}{|2^{j+1}\tilde{Q}|}\int_{2^{j+1}\tilde{Q}}|T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\le C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_2 &\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_{2\tilde{Q}})\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x_0)|dx\\
&\le C||b||_{BMO}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

Case 2. $d>1$. Similar to the proof of **Case 1**, we have

$$
T_b(f)(x)=T_{b-b_{2Q}}(f)(x)=T_{(b-b_{2Q})\chi_{2Q}}(f)(x)+T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x)=g_1(x)+g_2(x)
$$

and

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q|T_b(f)(x)|^r\,dx\right)^{1/r} &\le \left(\frac{C}{|Q|}\int_Q|g_1(x)|^r dx\right)^{1/r}\\
+\left(\frac{C}{|Q|}\int_Q|g_2(x)|^r dx\right)^{1/r} &= I_3+I_4.
\end{aligned}
$$

For I_3, by Lemma 2.1, 2.2 and 2.3, we obtain

$$
\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{2Q}}(f)(x)|\\
\le&\left(\frac{1}{|Q|}\int_Q|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r}\\
\le&|Q|^{-1}\frac{||T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)\chi_Q||_{L^r}}{|Q|^{1/r-1}}\\
\le&C|Q|^{-1}||T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)||_{WL^1}\\
\le&C|Q|^{-1}||M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)||_{L^1}\\
\le&C|Q|^{-1}\int_{2Q}|b(x)-b_{2Q}||T^{k,2}(f)(x)|dx\\
\le&C||b-b_{2Q}||_{expL,2Q}||T^{k,2}(f)||_{L(logL),2Q}\\
\le&C||b||_{BMO}M^2(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$\begin{aligned} I_3 &\leq \sum_{k=1}^{m}\left(\frac{C}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r} \\ &\leq C||b||_{BMO}\sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

For I_4, by Lemma 2.5, we obtain

$$\begin{aligned} I_4 &\leq C\sum_{k=1}^{m}\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)||T^{k,2}(f)(y)|dy \\ &\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ}|b(y)-b_{2Q}||x-y|^{-2n}|T^{k,2}(f)(y)|dy \\ &\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}(2^jd)^{-2n}(2^jd)^n\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'} \\ &\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s dy\right)^{1/s} \\ &\leq C||b||_{BMO}d^{-n}\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-jn}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s dy\right)^{1/s} \\ &\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$, $2 < p < \infty$, $w \in A_1$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then T_b is bounded on $L^p(w)$.*

Proof. Choose $1 < s < p$ in Theorem 3.1. We have, by Lemmas 2.3 and 2.6,

$$\begin{aligned} &||T_b(f)||_{L^p(w)} \leq \|M_\eta(T_b(f))\|_{L^p(w)} \leq C\|M^\#_\eta(T_b(f))\|_{L^p(w)} \\ &\leq C||b||_{BMO}\sum_{k=1}^{m}(\|M^2(T^{k,2}(f))\|_{L^p(w)} + \|M_s(T^{k,2}(f))\|_{L^p(w)}) \\ &\leq C||b||_{BMO}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p(w)} \\ &\leq C||b||_{BMO}\|f\|_{L^p(w)}. \end{aligned}$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$, $0 < D < 2^n$, $2 < p < \infty$, $w \in A_1$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then T_b is bounded on $L^{p,\varphi}(R^n, w)$.*

Proof. Choose $1 < s < p$ in Theorem 3.1. We have, by Lemmas 2.7, 2.8 and 2.9,

$$\begin{aligned}
&||T_b(f)||_{L^{p,\varphi}(w)} \le \|M_\eta(T_b(f))\|_{L^{p,\varphi}(w)} \le C\|M_\eta^\#(T_b(f))\|_{L^{p,\varphi}(w)} \\
\le\ & C||b||_{BMO}\sum_{k=1}^m (\|M^2(T^{k,2}(f))\|_{L^{p,\varphi}(w)} + \|M_s(T^{k,2}(f))\|_{L^{p,\varphi}(w)}) \\
\le\ & C||b||_{BMO}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}(w)} \\
\le\ & C||b||_{BMO}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}$$

This completes the proof of Theorem 3.3. □

Corollary 3.4. *Let $[b,T](f) = bT(f) - T(bf)$ be the commutator generated by the pseudo-differential operator T with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \le \delta < 1-\theta)$ and b. Then Theorems 3.1-3.3 hold for $[b,T]$.*

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References

1. S. Chanillo and A. Torchinsky, *Sharp function and weighted L^p estimates for a class of pseudo-differential operators*, Ark. Math. **24** (1986), no. 1, 1–25.
2. R.R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), no. 5, 611–635.
3. G. Di FaZio and M.A. Ragusa, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. **5-A(7)** (1991), no. 2, 323–332.
4. G. Di Fazio and M.A. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (1993), no. 2, 241–256.
5. C. Fefferman, *L^p bounds for pseudo-differential operators*, Israel J. Math. **14** (1973), no. 3, 413–417.
6. J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math., **16**, Amsterdam, 1985.
7. S. Krantz and S. Li, *Boundedness and compactness of integral operators on spaces of homogeneous type and applications*, J. Math. Anal. Appl. **258** (2001), no. 5, 629–641.
8. Y. Lin and S.Z. Lu, *Toeplitz operators related to strongly singular Calderón-Zygmund operators*, Sci. China Ser. A **49** (2006), no. 8, 1048–1064.
9. L.Z. Liu, *Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators*, Acta Math. Scientia **25(B)** (2005), no. 1, 89–94.
10. S.Z. Lu and H.X. Mo, *Toeplitz type operators on Lebesgue spaces*, Acta Math. Sci. Ser. B Engl. Ed. **29** (2009), no. 1, 140–150.
11. N. Miller, *Weighted Sobolev spaces and pseudo-differential operators with smooth symbols*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 91–109.
12. T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, in "Harmonic Analysis", Proceedings of a conference held in Sendai, Japan, 1990, 183–189.
13. J. Peetre, *On convolution operators leaving $L^{p,\lambda}$-spaces invariant*, Ann. Mat. Pura. Appl. **72** (1966), no. 2, 295–304.
14. J. Peetre, *On the theory of $L^{p,\lambda}$-spaces*, J. Funct. Anal. **4** (1969), 71–87.

15. C. Pérez and R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. **65** (2002), no. 6, 672–692.
16. M. Saidani, A. Lahmar-Benbernou and S. Gala, *Pseudo-differential operators and commutators in multiplier spaces*, African Diaspora J. Math. **6** (2008), no. 1, 31–53.
17. E.M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.
18. M. Sugimoto and N. Tomita, *Boundedness properties of pseudo-differential and Calderón-Zygmund operators on modulation spaces*, J. Fourier Anal. Appl. **14** (2008), no. 1, 124–143.

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