Adv. Oper. Theory 3 (2018), no. 2, 388–399
https://doi.org/10.15352/aot.1708-1212
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

ON THE TRUNCATED TWO-DIMENSIONAL MOMENT PROBLEM

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Communicated by V. Bolotnikov

ABSTRACT. We study the truncated two-dimensional moment problem (with rectangular data) to find a non-negative measure $\mu(\delta)$, $\delta \in \mathfrak{B}(\mathbb{R}^2)$, such that $\int_{\mathbb{R}^2} x_1^m x_2^n d\mu = s_{m,n}$, $0 \leq m \leq M, \quad 0 \leq n \leq N$, where $\{s_{m,n}\}_{0\leq m\leq M,\ 0\leq n\leq N}$ is a prescribed sequence of real numbers; $M, N \in \mathbb{Z}_+$. For the cases $M = N = 1$ and $M = 1, N = 2$ explicit numerical necessary and sufficient conditions for the solvability of the moment problem are given. In the cases $M = N = 2$; $M = 2, N = 3$; $M = 3, N = 2$; $M = 3, N = 3$ some explicit numerical sufficient conditions for the solvability are obtained. In all the cases some solutions (not necessarily atomic) of the moment problem can be constructed.

1. Introduction and preliminaries

In this paper we consider the truncated two-dimensional moment problem. A general approach for this moment problem was presented by Curto and Fialkow in their books [2] and [3]. These books entailed a series of papers by a group of mathematicians, see recent papers [4], [6], [8] and references therein. This approach includes an extension of the matrix of prescribed moments, which has the same rank. While positive extensions are easy to build, the Hankel-type structure is hard to inherit. This aim needed an involved analysis. Effective optimization algorithms for the multidimensional moment problems were given in the book of Lasserre [5]. Another approaches for truncated moment problems were presented by Vasilescu in [7] and by Cichoń, Stochel and Szafraniec in [1].

Date: Received: Aug. 4, 2017; Accepted: Oct. 22, 2017.
2010 *Mathematics Subject Classification.* Primary 47A57; Secondary 44A60.
Key words and phrases. Hankel matrix, moment problem, non-linear inequalities.

For arbitrary $k,l \in \mathbb{Z}$ we denote $\mathbb{Z}_{k,l} := \{j \in \mathbb{Z} : \ k \leq j \leq l\}$. Consider the following problem: find a non-negative measure $\mu(\delta)$, $\delta \in \mathfrak{B}(\mathbb{R}^2)$, such that

$$\int_{\mathbb{R}^2} x_1^m x_2^n d\mu = s_{m,n}, \qquad m \in \mathbb{Z}_{0,M}, \quad n \in \mathbb{Z}_{0,N}, \tag{1.1}$$

where $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,M},\ n\in\mathbb{Z}_{0,N}}$ is a prescribed sequence of real numbers; $M,N \in \mathbb{Z}_+$. This problem is said to be *the truncated two-dimensional moment problem (with rectangular data)*.

In the case of an arbitrary size of truncations, the approach of Curto and Fialkow for the truncated two-dimensional moment problem (with triangular data) gives some special conditions for the solvability of the moment problem, see [3, p. 51]. A more comprehensive analysis can be performed for small sizes of truncations ([3, p. 49–51]). A similar situation appears for the moment problem (1.1).

Let K be a subset of $\mathbb{R}^2$. The problem of finding a solution μ of the truncated two-dimensional moment problem (1.1) such that

$$\mathrm{supp}\mu \subseteq K,$$

is said to be *the truncated (two-dimensional) K-moment problem (with rectangular data)*. Since no other types of truncations will appear in the sequel, we shall omit the words about rectangular data.

As a tool for the study of the truncated two-dimensional moment problem we shall use the truncated K-moment problem on parallel lines (see Theorem 2.1). For the case of arbitrary M,N, Theorem 2.1 allows to perform some numerical tests for the existence of solutions of the moment problem (1.1) (see Remark 2.2). Similar to [9], this also allows us to consider a set of Hamburger moment problems and then to analyze the corresponding systems of non-linear inequalities. For the cases $M = N = 1$ and $M = 1, N = 2$ this approach leads to the necessary and sufficient conditions of the solvability of the truncated two-dimensional moment problem. In the cases $M = N = 2$; $M = 2, N = 3$; $M = 3, N = 2$; $M = 3, N = 3$ some explicit numerical sufficient conditions for the solvability are obtained. In all these cases a set of solutions (not necessarily atomic) can be constructed.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\max\{a,b\}$ we denote the maximal number of a and b. For arbitrary $k,l \in \mathbb{Z}$ we set

$$\mathbb{Z}_{k,l} := \{j \in \mathbb{Z} : \ k \leq j \leq l\}.$$

By $\mathfrak{B}(M)$ we denote the set of all Borel subsets of M, where $M \subseteq \mathbb{R}$ or $M \subseteq \mathbb{R}^2$.

2. The truncated two-dimensional moment problems for the cases $M = N = 1$ and $M = 1, N = 2$.

Choose an arbitrary $N \in \mathbb{Z}_+$ and arbitrary real numbers a_j, $j \in \mathbb{Z}_{0,N}$: $a_0 < a_1 < a_2 < ... < a_N$. Set

$$K_N = K_N(a_0,...,a_N) = \bigcup_{j=0}^{N} L_j, \qquad L_j := \{(x_1,x_2) \in \mathbb{R}^2 : \ x_2 = a_j\}.$$

Thus, K_N is a union on $N+1$ parallel lines in the plane. In this case the K-moment problem is reduced to a set of Hamburger moment problems (cf. [9, Theorems 2 and 4]).

Theorem 2.1. *Let $M,N \in \mathbb{Z}_+$ and a_j, $j \in \mathbb{Z}_{0,N}$: $a_0 < a_1 < a_2 < ... < a_N$, be arbitrary. Consider the truncated K-moment problem (1.1) with $K = K_N(a_0,...,a_N)$. Let*

$$W = W(a_0,a_1,...,a_N) = \begin{vmatrix} 1 & 1 & ... & 1 \\ a_0 & a_1 & ... & a_N \\ a_0^2 & a_1^2 & ... & a_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^N & a_1^N & ... & a_N^N \end{vmatrix},$$

and $\Delta_{j;m}$ be the determinant obtained from W by replacing j-th column with

$$\begin{pmatrix} s_{m,0} \\ s_{m,1} \\ \vdots \\ s_{m,N} \end{pmatrix}, \qquad j \in \mathbb{Z}_{0,N},\ m \in \mathbb{Z}_{0,M}.$$

Set

$$s_m(j) := \frac{\Delta_{j;m}}{W}, \qquad j \in \mathbb{Z}_{0,N},\ m \in \mathbb{Z}_{0,M}. \tag{2.1}$$

The truncated $K_N(a_0,a_1,...,a_N)$-moment problem has a solution if and only if for each $j \in \mathbb{Z}_{0,N}$, the truncated Hamburger moment problem with moments $s_m(j)$:

$$\int_{\mathbb{R}} x^m d\sigma_j = s_m(j), \qquad m = 0,1,...,M, \tag{2.2}$$

is solvable. Here σ_j is a non-negative measure on $\mathfrak{B}(\mathbb{R})$.

Moreover, if σ_j is a solution of the Hamburger moment problem (2.2), $j \in \mathbb{Z}_{0,N}$, then we may define a measure $\widetilde{\sigma}_j$ by

$$\widetilde{\sigma}_j(\delta) = \sigma_j(\delta \cap \mathbb{R}), \qquad \delta \in \mathfrak{B}(\mathbb{R}^2). \tag{2.3}$$

Here $\mathbb{R}$ means the set $\{(x_1,x_2) \in \mathbb{R}^2 :\ x_2 = 0\}$. We define

$$\widetilde{\sigma}_j'(\delta) = \widetilde{\sigma}_j(\theta_j^{-1}(\delta)), \qquad \delta \in \mathfrak{B}(\mathbb{R}^2), \tag{2.4}$$

where

$$\theta_j((x_1,x_2)) = (x_1, x_2 + a_j) :\ \mathbb{R}^2 \to \mathbb{R}^2. \tag{2.5}$$

Then we can define μ in the following way:

$$\mu(\delta) = \sum_{j=0}^{N} \widetilde{\sigma}_j'(\delta), \qquad \delta \in \mathfrak{B}(\mathbb{R}^2), \tag{2.6}$$

to get a solution μ of the truncated $K_N(a_0,a_1,...,a_N)$-moment problem.

Proof. Suppose that the truncated $K_N(a_0,a_1,...,a_N)$-moment problem has a solution μ. For an arbitrary $j \in \mathbb{Z}_{0,N}$ we denote:

$$\pi_j((x_1,x_2)) = (x_1, x_2 - a_j) :\ \mathbb{R}^2 \to \mathbb{R}^2,$$

and

$$\mu_j'(\delta) = \mu(\pi_j^{-1}(\delta)), \qquad \delta \in \mathfrak{B}(\mathbb{R}^2).$$

Using the measure $\mu_j'(\delta)$ on $\mathfrak{B}(\mathbb{R}^2)$, we define the measure σ_j as a restriction of $\mu_j'(\delta)$ to $\mathfrak{B}(\mathbb{R})$. Here by $\mathbb{R}$ we mean the set $\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = 0\}$. With these notations, using the change of variables for measures and the definition of the integral, for arbitrary $m \in \mathbb{Z}_{0,M}$, $n \in \mathbb{Z}_{0,N}$, we may write:

$$s_{m,n} = \int_{\mathbb{R}^2} x_1^m x_2^n d\mu = \sum_{j=0}^N a_j^n \int_{L_j} x_1^m d\mu = \sum_{j=0}^N a_j^n \int_{\mathbb{R}} x_1^m d\mu_j' = \sum_{j=0}^N a_j^n \int_{\mathbb{R}} x^m d\sigma_j.$$

Denote $\mathbf{s}_m(j) = \int_{\mathbb{R}} x^m d\sigma_j$, $j \in \mathbb{Z}_{0,N}$, $m \in \mathbb{Z}_{0,M}$. Then

$$\left\{ \begin{array}{c} \mathbf{s}_m(0) + \mathbf{s}_m(1) + \mathbf{s}_m(2) + ... + \mathbf{s}_m(N) = s_{m,0}, \\ a_0\mathbf{s}_m(0) + a_1\mathbf{s}_m(1) + a_2\mathbf{s}_m(2) + ... + a_N\mathbf{s}_m(N) = s_{m,1}, \\ a_0^2\mathbf{s}_m(0) + a_1^2\mathbf{s}_m(1) + a_2^2\mathbf{s}_m(2) + ... + a_N^2\mathbf{s}_m(N) = s_{m,2}, \\ \cdots \\ a_0^N\mathbf{s}_m(0) + a_1^N\mathbf{s}_m(1) + a_2^N\mathbf{s}_m(2) + ... + a_N^N\mathbf{s}_m(N) = s_{m,N}, \end{array} \right. \qquad (m \in \mathbb{Z}_{0,M}). \tag{2.7}$$

By Cramer's formulas numbers $\mathbf{s}_m(j)$ coincide with numbers $s_m(j)$ from (2.1). We conclude that the truncated Hamburger moment problems (2.2) are solvable.

On the other hand, suppose that the truncated Hamburger moment problems (2.2) have solutions σ_j. We define measures $\widetilde{\sigma}_j$, $\widetilde{\sigma}_j'$, μ by (2.3), (2.4) and (2.6), respectively. Observe that $\widetilde{\sigma}_j(\mathbb{R}^2 \backslash \mathbb{R}) = 0$. Then $\widetilde{\sigma}_j'(\mathbb{R}^2 \backslash L_j) = 0$, and $\mathrm{supp}\mu \subseteq \bigcup_{j=0}^N L_j$. Using the change of the variable (2.5) and the definition of μ we see that

$$s_m(j) = \int_{\mathbb{R}} x_1^m d\sigma_j = \int_{L_j} x_1^m d\mu, \qquad j \in \mathbb{Z}_{0,N},\ m \in \mathbb{Z}_{0,M}.$$

Observe that $s_m(j)$ are solutions of the linear system of equations (2.7). Then

$$s_{m,n} = \sum_{j=0}^N a_j^n \int_{L_j} x_1^m d\mu = \int_{\mathbb{R}^2} \sum_{j=0}^N a_j^n \chi_{L_j}(x_1, x_2) x_1^m d\mu =$$

$$= \int_{\mathbb{R}^2} x_1^m x_2^n d\mu, \qquad m \in \mathbb{Z}_{0,M},\ n \in \mathbb{Z}_{0,N}.$$

Here by χ_{L_j} we denote the characteristic function of the set L_j. Thus, μ is a solution of the truncated $K_N(a_0, a_1, ..., a_N)$-moment problem. □

Remark 2.2. (Numerical tests).

Consider the truncated two-dimensional moment problem (1.1) with some $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,M},\ n\in\mathbb{Z}_{0,N}}$ $(M, N \in \mathbb{Z}_+)$. How to use Theorem 2.1 in our search of its solutions?

Firstly, we can choose arbitrary real numbers a_j, $j \in \mathbb{Z}_{0,N}$: $a_0 < a_1 < a_2 < ... < a_N$, and consider the truncated K-moment problem (1.1) with $K = K_N(a_0, ..., a_N)$, and the same $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,M},\ n\in\mathbb{Z}_{0,N}}$ as above. Then we calculate the $s_m(j)$'s by formula (2.1). It remains to check that the corresponding truncated Hamburger moment problems (2.2) are solvable.

Of course, such a test is specific. It can be powered in the following way. Choose an arbitrary real interval $[-T, T]$ and its partition:

$$-T = y_0 < y_1 < ... < y_g = T,$$

with a uniform step h. Then we can choose real numbers a_j, $j \in \mathbb{Z}_{0,N}$: $a_0 < a_1 < a_2 < ... < a_N$, taking a_j's from the latter partition. For each choice of a_j's we perform the above test.

If these tests do not help, we can increase T and/or decrease h. Finally, we can consider more that $N+1$ lines by increasing the given N and by introducing some additional moments.

Observe that the positive result of tests in Remark 2.2 is not guaranteed. However, for small M and N there are some conditions which guarantee the existence of a solution of the moment problem (1.1). At first we consider the case $M = 1, N = 1$ of the truncated two-dimensional moment problem.

Theorem 2.3. *Let the truncated two-dimensional moment problem* (1.1) *with* $M = 1, N = 1$ *and some* $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,1}}$ *be given. This moment problem has a solution if and only if one of the following conditions holds:*

(*i*) $s_{0,0} = s_{0,1} = s_{1,0} = s_{1,1} = 0$;

(*ii*) $s_{0,0} > 0$.

In the case (*i*) *the unique solution is* $\mu \equiv 0$. *In the case* (*ii*) *a solution* μ *can be constructed as a solution of the truncated* $K_1(a_0, a_1)$*-moment problem with the same* $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,1}}$, *and arbitrary* $a_0 < \frac{s_{0,1}}{s_{0,0}}$; $a_1 > \frac{s_{0,1}}{s_{0,0}}$.

Proof. Suppose that the truncated two-dimensional moment problem with $M = N = 1$ has a solution μ. Of course, $s_{0,0} = \int d\mu \geq 0$. If $s_{0,0} = 0$ then $\mu \equiv 0$ and condition (*i*) holds. If $s_{0,0} > 0$ then condition (*ii*) is true.

On the other hand, if condition (*i*) holds then $\mu \equiv 0$ is a solution of the moment problem. Of course, it is the unique solution (one can repeat the arguments at the beginning of this Proof). If condition (*ii*) holds, choose arbitrary real a_0, a_1 such that $a_0 < \frac{s_{0,1}}{s_{0,0}}$ and $a_1 > \frac{s_{0,1}}{s_{0,0}}$. Consider the truncated $K_1(a_0, a_1)$-moment problem with $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,1}}$. Let us check by Theorem 2.1 that this problem is solvable. We have: $W = a_1 - a_0$,

$$s_0(0) = \frac{a_1 s_{0,0} - s_{0,1}}{a_1 - a_0} > 0, \quad s_0(1) = \frac{s_{0,1} - a_0 s_{0,0}}{a_1 - a_0} > 0,$$

$$s_1(0) = \frac{a_1 s_{1,0} - s_{1,1}}{a_1 - a_0}, \quad s_1(1) = \frac{s_{1,1} - a_0 s_{1,0}}{a_1 - a_0}.$$

The Hamburger moment problems (2.2) are solvable [10, Theorem 8]. Their solutions can be used to construct a solution μ of the truncated two-dimensional moment problem. □

We now turn to the case $M = 1, N = 2$ of the truncated two-dimensional moment problem.

Theorem 2.4. *Let the truncated two-dimensional moment problem* (1.1) *with* $M = 1, N = 2$ *and some* $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,1},\ n\in\mathbb{Z}_{0,2}}$ *be given. This moment problem has a solution if and only if one of the following conditions holds:*

(*a*) $s_{0,0} = s_{0,1} = s_{0,2} = s_{1,0} = s_{1,1} = s_{1,2} = 0$;
(*b*) $s_{0,0} > 0$, *and*

$$s_{m,n} = \alpha^n s_{m,0}, \qquad m = 0,1;\ n = 1,2, \tag{2.8}$$

for some $\alpha \in \mathbb{R}$.
(*c*) $s_{0,0} > 0$, $s_{0,0}s_{0,2} - s_{0,1}^2 > 0$.

In the case (*a*) *the unique solution is* $\mu \equiv 0$.

In the case (*b*) *a solution* μ *can be constructed as a solution of the truncated* $K_0(\alpha)$*-moment problem with moments* $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,1},\ n=0}$.

In the case (*c*) *a solution* μ *can be constructed as a solution of the truncated* $K_2(a_0,a_1,a_2)$*-moment problem with the same* $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,1},\ n\in\mathbb{Z}_{0,2}}$, *arbitrary* $a_2 > \sqrt{\frac{s_{0,2}}{s_{0,0}}}$ *and* $a_1 = \frac{s_{0,1}}{s_{0,0}}$, $a_0 = -a_2$.

Proof. Suppose that the truncated two-dimensional moment problem with $M = 1, N = 2$ has a solution μ. Choose $p(x_2) = b_0 + b_1x_2$, where b_0, b_1 are arbitrary real numbers. Since

$$0 \le \int p^2 d\mu = s_{0,0}b_0^2 + 2s_{0,1}b_0b_1 + s_{0,2}b_1^2,$$

then $\Gamma_1 := \begin{pmatrix} s_{0,0} & s_{0,1} \\ s_{0,1} & s_{0,2} \end{pmatrix} \ge 0$. If $s_{0,0} = 0$ then $\mu \equiv 0$ and condition (*a*) is true. If $s_{0,0} > 0$ and $s_{0,0}s_{0,2} - s_{0,1}^2 = 0$, then 0 is an eigenvalue of the matrix Γ_1 with an eigenvector $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$, $c_0, c_1 \in \mathbb{R}$. Observe that $c_1 \ne 0$. Denote $\alpha = -\frac{c_0}{c_1}$. From the equation $\Gamma_1 \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 0$, it follows that relation (2.8) holds for $m = 0$. Observe that $\int_{\mathbb{R}^2} (\alpha - x_2)^2 d\mu = 0$. Then $\mu(\{(x_1,x_2) \in \mathbb{R}^2 :\ x_2 \ne \alpha\}) = 0$. For $n = 1,2$, we get $s_{1,n} = \int_{\mathbb{R}^2} x_1x_2^n d\mu = \alpha^n s_{1,0}$. Thus, condition (*b*) is true. Finally, it remains the case (*c*).

Conversely, if condition (*a*) holds then $\mu \equiv 0$ is a solution of the moment problem. Since $s_{0,0} = 0$, then any solution is equal to $\mu \equiv 0$.

Suppose that condition (*b*) holds. Consider the truncated $K_0(\alpha)$-moment problem with moments $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,1},\ n=0}$. Let us check by Theorem 2.1 that this problem is solvable. In fact, $W = 1$, $\Delta_{0;m} = s_m(0) = s_{m,0}$, $m = 0,1$. Since $s_0(0) = s_{0,0} > 0$, then the truncated Hamburger moment problem (2.2) has a solution. Then we may construct μ as it was described in the statement of the theorem. Remaining moment equalities then follow from relations (2.8) and the fact that $\mathrm{supp}\mu \subseteq \{(x_1,x_2) \in \mathbb{R}^2 :\ x_2 = \alpha\}$.

Suppose that condition (*c*) holds. Consider the truncated $K_2(a_0,a_1,a_2)$-moment problem with the same $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,1},\ n\in\mathbb{Z}_{0,2}}$, arbitrary $a_2 > \sqrt{\frac{s_{0,2}}{s_{0,0}}}$ and $a_1 = \frac{s_{0,1}}{s_{0,0}}$, $a_0 = -a_2$. We shall check by Theorem 2.1 that this problem is solvable. Observe that $W(a_0,a_1,a_2) = 2a_2(a_2^2 - a_1^2) > 0$, and

$$s_0(0) = \frac{a_2 - a_1}{W}(a_1a_2s_{0,0} - (a_1+a_2)s_{0,1} + s_{0,2}),$$

$$s_0(1) = \frac{a_2 - a_0}{W}(-a_0 a_2 s_{0,0} + (a_2 + a_0)s_{0,1} + s_{0,2}),$$
$$s_0(2) = \frac{a_1 - a_0}{W}(a_0 a_1 s_{0,0} - (a_0 + a_1)s_{0,1} + s_{0,2}).$$

For the solvability of the corresponding three truncated Hamburger moment problems it is sufficient the validity of the following inequalities: $s_0(j) > 0$, $j = 0, 1, 2$, which are equivalent to

$$a_1 a_2 s_{0,0} - (a_1 + a_2)s_{0,1} + s_{0,2} > 0,$$
$$a_2^2 s_{0,0} - s_{0,2} > 0,$$
$$-a_1 a_2 s_{0,0} - (a_1 - a_2)s_{0,1} + s_{0,2} > 0.$$

All these inequalities are true. Then the solution of the truncated $K_2(a_0, a_1, a_2)$-moment problem exists and provides us with a solution of the truncated two-dimensional moment problem. □

3. The truncated two-dimensional moment problems for the cases $M = N = 2$; $M = 2, N = 3$; $M = 3, N = 2$; $M = N = 3$.

Consider arbitrary real numbers $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$, such that

$$s_{0,0} > 0, \quad s_{0,0}s_{0,2} - s_{0,1}^2 > 0, \quad s_{0,0}s_{2,0} - s_{1,0}^2 > 0. \tag{3.1}$$

Let us study the truncated two-dimensional $K_3(a_0, a_1, a_2, a_3)$-moment problem with the moments $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$ and with some $a_0 < a_1 < a_2 < a_3$:

$$a_2 \in \left(\frac{|s_{0,1}|}{s_{0,0}}, \sqrt{\frac{s_{0,2}}{s_{0,0}}} \right); \tag{3.2}$$

$$a_3 > \max\left\{ \frac{|s_{0,3} - a_2^2 s_{0,1}|}{-a_2^2 s_{0,0} + s_{0,2}}, \sqrt{\frac{a_2 s_{0,2} + |s_{0,3}|}{a_2 s_{0,0} - |s_{0,1}|}} \right\}; \tag{3.3}$$

$$a_0 = -a_3, \quad a_1 = -a_2.$$

Observe that condition (3.1) ensures the correctness of all expressions in (3.2), (3.3). Let us study by Theorem 2.1, when this moment problem has a solution. We have: $W = \prod_{1\le j<i\le 4}(a_{i-1} - a_{j-1}) > 0$, and for $m \in \mathbb{Z}_{0,3}$,

$$s_m(0) = \frac{2a_2(a_3 - a_2)(a_3 + a_2)}{W}\{-a_2^2 a_3 s_{m,0} + a_2^2 s_{m,1} + a_3 s_{m,2} - s_{m,3}\},$$
$$s_m(1) = -\frac{(a_2 + a_3)(a_3 - a_2)2a_3}{W}\{-a_3^2 a_2 s_{m,0} + a_3^2 s_{m,1} + a_2 s_{m,2} - s_{m,3}\},$$
$$s_m(2) = \frac{(-a_2 + a_3)(a_3 + a_2)2a_3}{W}\{a_2 a_3^2 s_{m,0} + a_3^2 s_{m,1} - a_2 s_{m,2} - s_{m,3}\},$$
$$s_m(3) = -\frac{(-a_2 + a_3)2a_2(a_2 + a_3)}{W}\{a_3 a_2^2 s_{m,0} + a_2^2 s_{m,1} - a_3 s_{m,2} - s_{m,3}\}.$$

Sufficient conditions for the solvability of the corresponding Hamburger moment problems (2.2) are the following ([10, Theorem 8]):

$$s_0(j) > 0, \quad s_0(j)s_2(j) - (s_1(j))^2 > 0, \qquad j = 0, 1, 2, 3. \tag{3.4}$$

The first inequality in (3.4) for $j = 0, 1, 2, 3$ is equivalent to the following system:

$$\begin{cases} -a_2^2 a_3 s_{0,0} + a_2^2 s_{0,1} + a_3 s_{0,2} - s_{0,3} > 0 \\ a_3^2 a_2 s_{0,0} - a_3^2 s_{0,1} - a_2 s_{0,2} + s_{0,3} > 0 \\ a_2 a_3^2 s_{0,0} + a_3^2 s_{0,1} - a_2 s_{0,2} - s_{0,3} > 0 \\ -a_3 a_2^2 s_{0,0} - a_2^2 s_{0,1} + a_3 s_{0,2} + s_{0,3} > 0 \end{cases}. \tag{3.5}$$

The second inequality in (3.4) for $j = 0, 1, 2, 3$ is equivalent to the following inequalities:

$$(-a_2^2 a_3 s_{0,0} + a_2^2 s_{0,1} + a_3 s_{0,2} - s_{0,3})(-a_2^2 a_3 s_{2,0} + a_2^2 s_{2,1} + a_3 s_{2,2} - s_{2,3}) >$$
$$> (-a_2^2 a_3 s_{1,0} + a_2^2 s_{1,1} + a_3 s_{1,2} - s_{1,3})^2,$$
$$(a_3^2 a_2 s_{0,0} - a_3^2 s_{0,1} - a_2 s_{0,2} + s_{0,3})(a_3^2 a_2 s_{2,0} - a_3^2 s_{2,1} - a_2 s_{2,2} + s_{2,3}) >$$
$$> (a_3^2 a_2 s_{1,0} - a_3^2 s_{1,1} - a_2 s_{1,2} + s_{1,3})^2,$$
$$(a_3^2 a_2 s_{0,0} + a_3^2 s_{0,1} - a_2 s_{0,2} - s_{0,3})(a_3^2 a_2 s_{2,0} + a_3^2 s_{2,1} - a_2 s_{2,2} - s_{2,3}) >$$
$$> (a_3^2 a_2 s_{1,0} + a_3^2 s_{1,1} - a_2 s_{1,2} - s_{1,3})^2,$$
$$(-a_2^2 a_3 s_{0,0} - a_2^2 s_{0,1} + a_3 s_{0,2} + s_{0,3})(-a_2^2 a_3 s_{2,0} - a_2^2 s_{2,1} + a_3 s_{2,2} + s_{2,3}) >$$
$$> (-a_2^2 a_3 s_{1,0} - a_2^2 s_{1,1} + a_3 s_{1,2} + s_{1,3})^2.$$

Dividing by a_3 or a_3^2 we obtain that the latter inequalities are equivalent to the following inequalities:

$$\left(-a_2^2 s_{0,0} + s_{0,2} + \frac{a_2^2 s_{0,1} - s_{0,3}}{a_3}\right)\left(-a_2^2 s_{2,0} + s_{2,2} + \frac{a_2^2 s_{2,1} - s_{2,3}}{a_3}\right) >$$
$$> \left(-a_2^2 s_{1,0} + s_{1,2} + \frac{a_2^2 s_{1,1} - s_{1,3}}{a_3}\right)^2,$$
$$\left(a_2 s_{0,0} - s_{0,1} + \frac{-a_2 s_{0,2} + s_{0,3}}{a_3^2}\right)\left(a_2 s_{2,0} - s_{2,1} + \frac{-a_2 s_{2,2} + s_{2,3}}{a_3^2}\right) >$$
$$> \left(a_2 s_{1,0} - s_{1,1} + \frac{-a_2 s_{1,2} + s_{1,3}}{a_3^2}\right)^2,$$
$$\left(a_2 s_{0,0} + s_{0,1} - \frac{a_2 s_{0,2} + s_{0,3}}{a_3^2}\right)\left(a_2 s_{2,0} + s_{2,1} - \frac{a_2 s_{2,2} + s_{2,3}}{a_3^2}\right) >$$
$$> \left(a_2 s_{1,0} + s_{1,1} - \frac{a_2 s_{1,2} + s_{1,3}}{a_3^2}\right)^2,$$
$$\left(-a_2^2 s_{0,0} + s_{0,2} + \frac{s_{0,3} - a_2^2 s_{0,1}}{a_3}\right)\left(-a_2^2 s_{2,0} + s_{2,2} + \frac{s_{2,3} - a_2^2 s_{2,1}}{a_3}\right) >$$
$$> \left(-a_2^2 s_{1,0} + s_{1,2} + \frac{s_{1,3} - a_2^2 s_{1,1}}{a_3}\right)^2. \tag{3.6}$$

We additionally assume that

$$(-a_2^2 s_{0,0} + s_{0,2})(-a_2^2 s_{2,0} + s_{2,2}) > (-a_2^2 s_{1,0} + s_{1,2})^2, \tag{3.7}$$

$$(a_2 s_{0,0} - s_{0,1})(a_2 s_{2,0} - s_{2,1}) > (a_2 s_{1,0} - s_{1,1})^2, \tag{3.8}$$

$$(a_2 s_{0,0} + s_{0,1})(a_2 s_{2,0} + s_{2,1}) > (a_2 s_{1,0} + s_{1,1})^2. \tag{3.9}$$

In this case inequalities (3.6) will be valid, if a_3 is sufficiently large. In fact, inequalities (3.6) have the following obvious structure:

$$(r_j + \psi_j(a_3))(l_j + \xi_j(a_3)) > (t_j + \eta_j(a_3))^2, \qquad j \in \mathbb{Z}_{0,3},$$

while inequalities (3.7), (3.8), (3.9) mean that

$$r_j l_j > t_j^2, \qquad j \in \mathbb{Z}_{0,3}.$$

Since $\psi_j(a_3)$, $\xi_j(a_3)$ and $\eta_j(a_3)$ tend to zero as $a_3 \to \infty$, then there exists $A = A(a_2) \in \mathbb{R}$ such that inequalities (3.6) hold, if $a_3 > A$.

System (3.5) can be written in the following form:

$$\begin{cases} \pm(a_2^2 s_{0,1} - s_{0,3}) < a_3(-a_2^2 s_{0,0} + s_{0,2}) \\ \pm(a_3^2 s_{0,1} - s_{0,3}) < a_2(a_3^2 s_{0,0} - s_{0,2}) \end{cases}. \tag{3.10}$$

System (3.10) is equivalent to the following system:

$$\begin{cases} |a_2^2 s_{0,1} - s_{0,3}| < a_3(-a_2^2 s_{0,0} + s_{0,2}) \\ |a_3^2 s_{0,1} - s_{0,3}| < a_2(a_3^2 s_{0,0} - s_{0,2}) \end{cases}. \tag{3.11}$$

If

$$a_3 > \frac{|a_2^2 s_{0,1} - s_{0,3}|}{-a_2^2 s_{0,0} + s_{0,2}},$$

and

$$a_3 > \sqrt{\frac{|s_{0,3}| + a_2 s_{0,2}}{a_2 s_{0,0} - |s_{0,1}|}}, \tag{3.12}$$

then inequalities (3.11) are true. Observe that relation (3.12) ensures that

$$a_3^2 |s_{0,1}| + |s_{0,3}| < a_2(a_3^2 s_{0,0} - s_{0,2}).$$

Quadratic (with respect to a_3 or a_3^2) inequalities (3.7)-(3.9) can be verified by elementary means, using their discriminants. Let us apply our considerations to the truncated two-dimensional moment problem.

Theorem 3.1. *Let the truncated two-dimensional moment problem (1.1) with $M = N = 3$ and some $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$ be given and conditions (3.1) hold. Denote by I_1, I_2 and I_3 the sets of positive real numbers a_2 satisfying inequalities (3.7), (3.8) and (3.9), respectively. If*

$$\left(\frac{|s_{0,1}|}{s_{0,0}}, \sqrt{\frac{s_{0,2}}{s_{0,0}}} \right) \cap I_1 \cap I_2 \cap I_3 \neq \emptyset, \tag{3.13}$$

then this moment problem has a solution.

A solution μ of the moment problem can be constructed as a solution of the truncated $K_3(-a_3, -a_2, a_2, a_3)$-moment problem with the same $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$, with arbitrary a_2 from the interval $\left(\frac{|s_{0,1}|}{s_{0,0}}, \sqrt{\frac{s_{0,2}}{s_{0,0}}} \right) \cap I_1 \cap I_2 \cap I_3$, and some positive large a_3.

Proof. The proof follows from the preceding considerations. □

Let the truncated two-dimensional moment problem (1.1) with $M, N \in \mathbb{Z}_{2,3}$ and some $\{s_{m,n}\}_{m\in\mathbb{Z}_{0,M},\ n\in\mathbb{Z}_{0,N}}$ be given, and conditions (3.1) hold. Notice that conditions (3.1), (3.7), (3.8), (3.9) and the first interval in (3.13) do not depend on $s_{m,n}$ with indices $m = 3$ or $n = 3$. Thus, we can check conditions of Theorem 3.1 for this moment problem (keeping undefined moments as parameters).

Example 3.2. Consider the truncated two-dimensional moment problem (1.1) with $M = N = 2$, and

$$s_{0,0} = 4ab,\ s_{0,1} = 0,\ s_{0,2} = \frac{4}{3}ab^3,\ s_{1,0} = s_{1,1} = s_{1,2} = 0,$$
$$s_{2,0} = \frac{4}{3}a^3b,\ s_{2,1} = 0,\ s_{2,2} = \frac{4}{9}a^3b^3,$$

where a, b are arbitrary positive numbers. In this case, condition (3.1) holds. Moreover, we have:

$$I_1 = (0, +\infty)\backslash\left\{\frac{1}{\sqrt{3}}b\right\}, \quad I_2 = I_3 = (0, +\infty);$$
$$\left(\frac{|s_{0,1}|}{s_{0,0}}, \sqrt{\frac{s_{0,2}}{s_{0,0}}}\right) = \left(0, \frac{1}{\sqrt{3}}b\right).$$

By Theorem 3.1 we conclude that this moment problem has a solution.

Let us construct a solution of the moment problem. For simplicity we set $a = 1, b = 3$. Thus, we have the following moments:

$$s_{0,0} = 12,\ s_{0,1} = 0,\ s_{0,2} = 36,\ s_{1,0} = s_{1,1} = s_{1,2} = 0,$$
$$s_{2,0} = 4,\ s_{2,1} = 0,\ s_{2,2} = 12.$$

We consider the truncated two-dimensional moment problem (1.1) with $M = N = 3$ and with $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$, where new moments (with indices $m = 3$ or $n = 3$) are zeros. According to Theorem 3.1 we choose $a_2 = 1$, and consider the truncated $K_3(-a_3, -1, 1, a_3)$-moment problem with $\{s_{m,n}\}_{m,n\in\mathbb{Z}_{0,3}}$. The value of a_3 (> 1) will be specified later.

We next calculate W, $\Delta_{j;m}$ and $s_m(j)$ from Theorem 2.1. A direct calculation of the determinants gives the following formulas for $s_m(j)$:

$$s_m(0) = \frac{1}{2a_3(a_3^2 - 1)}\left(-a_3 s_{m,0} + s_{m,1} + a_3 s_{m,2}\right),$$
$$s_m(1) = \frac{-1}{2(a_3^2 - 1)}\left(-a_3^2 s_{m,0} + a_3^2 s_{m,1} + s_{m,2}\right),$$
$$s_m(2) = \frac{1}{2(a_3^2 - 1)}\left(a_3^2 s_{m,0} + a_3^2 s_{m,1} - s_{m,2}\right),$$
$$s_m(3) = \frac{-1}{2a_3(a_3^2 - 1)}\left(a_3 s_{m,0} + s_{m,1} - a_3 s_{m,2}\right).$$

Then

$$s_0(0) = \frac{12}{a_3^2 - 1},\ s_1(0) = 0,\ s_2(0) = \frac{4}{a_3^2 - 1},\ s_3(0) = 0;$$
$$s_0(1) = \frac{6a_3^2 - 18}{a_3^2 - 1},\ s_1(1) = 0,\ s_2(1) = \frac{2a_3^2 - 6}{a_3^2 - 1},\ s_3(1) = 0;$$

$$s_0(2) = \frac{6a_3^2 - 18}{a_3^2 - 1},\ s_1(2) = 0,\ s_2(2) = \frac{2a_3^2 - 6}{a_3^2 - 1},\ s_3(2) = 0;$$

$$s_0(3) = \frac{12}{a_3^2 - 1},\ s_1(3) = 0,\ s_2(3) = \frac{4}{a_3^2 - 1},\ s_3(3) = 0.$$

We set $a_3 = 2$ to get

$$s_0(0) = 4,\ s_1(0) = 0,\ s_2(0) = \frac{4}{3},\ s_3(0) = 0;$$

$$s_0(1) = 2,\ s_1(1) = 0,\ s_2(1) = \frac{2}{3},\ s_3(1) = 0;$$

$$s_0(2) = 2,\ s_1(2) = 0,\ s_2(2) = \frac{2}{3},\ s_3(2) = 0;$$

$$s_0(3) = 4,\ s_1(3) = 0,\ s_2(3) = \frac{4}{3},\ s_3(3) = 0.$$

Of course, the latter truncated Hamburger moment problems are solvable. As $\sigma_0 = \sigma_3$ (see Theorem 2.1) we can take the two-atomic measure with atoms at points $\pm\frac{1}{\sqrt{3}}$ and masses equal to 2. As $\sigma_1 = \sigma_2$ we take the two-atomic measure with atoms at points $\pm\frac{1}{\sqrt{3}}$ and masses equal to 1. By the construction in the formulation of Theorem 2.1 we get a solution μ of the moment problem. The measure μ is 8-atomic with atoms at points $\left(\pm\frac{1}{\sqrt{3}}, \pm 2\right)$, $\left(\pm\frac{1}{\sqrt{3}}, \pm 1\right)$. The masses at points $\left(\pm\frac{1}{\sqrt{3}}, \pm 2\right)$ are equal to 2, while the masses at points $\left(\pm\frac{1}{\sqrt{3}}, \pm 1\right)$ are equal to 1.

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