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EXTENSIONS OF THEORY OF REGULAR AND WEAK REGULAR SPLITTINGS TO SINGULAR MATRICES

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Abstract. Matrix splittings are useful in finding a solution of linear systems of equations, iteratively. In this note, we present some more convergence and comparison results for recently introduced matrix splittings called index-proper regular and index-proper weak regular splittings. We then apply to theory of double index-proper splittings.

1. Introduction and preliminaries

The need to solve linear systems of algebraic equations arises in many mathematical models. The most common methods used to solve such systems are iterative methods. A large class of iterative methods for solving $Ax = b$, where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, can be formulated by using matrix splittings. A matrix splitting is an expression of a given matrix as a sum or difference of matrices. Historically, the idea of matrix splittings has its origin in the regular splitting theory. This technique was devised by Varga, [15]. Thereafter, the theory of splittings have been extended and studied by many authors for nonsingular, singular and rectangular matrices (see [1, 4, 6, 7, 9, 12, 15, 16] and the references cited therein). The goal of this paper is to study convergence and comparison results for recently introduced matrix splittings called *index-proper regular* and *index-proper weak regular splittings* for real square singular matrices using the theory of Drazin inverse.

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Throughout, all our matrices are real square unless otherwise stated. The *index* of A is the least nonnegative integer k such that $\text{rank}(A^{k+1})=\text{rank}(A^k)$, and we denote it by $\text{ind}(A)$. We declare that here onwards the symbol k always represents the index of the respective matrix. The *Drazin inverse* of a matrix A is the unique solution A^D satisfying the equations: $A^k = A^kA^DA$, $A^D = A^DAA^D$ and $AA^D = A^DA$, where k is the index of A. We next recall the definition of an index-proper splitting. A splitting $A = U - V$ is called an *index-proper splitting* of A if $R(U^k) = R(A^k)$ and $N(U^k) = N(A^k)$, where $R(A)$ and $N(A)$ denote the range and the kernel of A, respectively, and k is the index of A; see [6]. It reduces to an *index splitting* if $\text{ind}(U) = 1$; see [16]. When $k = 1$, then an index-proper splitting becomes a *proper splitting*; cf. [4]. The asymptotic behaviors of the iterative sequences (obtained by using the index-proper splitting $A = U - V$):

$$x^{i+1} = U^DVx^i + U^Db,\ i = 0, 1, 2, \dots$$

and

$$Y^{j+1} = U^DVY^j + U^D,\ j = 0, 1, 2, \dots$$

are governed by $\rho(U^DV)$, where $\rho(A)$ is the spectral radius of the matrix A. For an index-proper splitting, $\rho(U^DV) < 1$ if and only if the above schemes converge to A^Db and A^D, respectively. More on index-proper splitting can be found in the recent articles [6, 7]. In this note, we add a few more new results to theory of index-proper regular and weak regular splittings which are introduced in [1].

The organization of this paper is as follows. In Section 2, we present convergence and comparison theorems for index-proper regular and weak regular splittings. In Section 3, we introduce the notion of double index-proper regular and weak regular splittings for real $n \times n$ singular matrices. Convergence and comparison results for double index-proper regular and weak regular splitting are also discussed in the same section.

Before proceeding further, let us recall a few results which are to be used in further discussions. The first six results deal with non-negative matrices.

Theorem 1.1. ([15, Theorem 2.20]) Let $A \geq 0$. Then
(i) A has a nonnegative real eigenvalue equal to its spectral radius.
(ii) There exists a nonnegative eigenvector for its spectral radius.

Theorem 1.2. ([15, Theorem 2.21])
Let $A \geq B \geq 0$. Then $\rho(A) \geq \rho(B)$.

Theorem 1.3. ([15, Theorem 3.16])
Let $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0$.

Theorem 1.4. ([11, Corollary 3.2])
If $B \geq 0$ and $x \geq 0$ is such that $Bx - \alpha x \geq 0$, then $\alpha \leq \rho(B)$.

Theorem 1.5. ([14, Lemma 2.2])
Let $X = \begin{pmatrix} B & C \\ I & O \end{pmatrix} \geq 0$ and $\rho(B+C) < 1$. Then $\rho(X) < 1$.

Let us denote the nonnegative orthants of $\mathbb{R}^n$ by $\mathbb{R}^n_+$, and the set of all interior points of $\mathbb{R}^n_+$ by $\operatorname{int}(\mathbb{R}^n_+)$. Next result uses these notation.

Theorem 1.6. ([8, Theorem 25.4])
Suppose that $C \leq B$, B^{-1} exists, and $B^{-1} \geq 0$. Then C^{-1} exists and $C^{-1} \geq 0$ if and only if $C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^n_+) \neq \emptyset$.

Next four results show a few properties of an index-proper splitting.

Theorem 1.7. ([6, Theorem 3.2])
Let $A = U - V$ be an index-proper splitting. Then
(a) $AA^D = UU^D = U^D U = A^D A$;
(b) $I - U^D V$ is invertible;
(c) $A^D = (I - U^D V)^{-1} U^D$.

Since $A = U - V$ is an index-proper splitting, so is $U = A + V$. Hence, we have the following results.

Theorem 1.8. ([7, Theorem 1.6])
Let $A = U - V$ be an index-proper splitting. Then
(a) $I + A^D V$ and $I + VA^D$ are invertible;
(b) $A^D = (I + A^D V)U^D = U^D(I + VA^D)$;
(c) $U^D = (I + A^D V)^{-1} A^D = A^D (I + VA^D)^{-1}$;
(d) $U^D V A^D = A^D V U^D$;
(e) $U^D V A^D V = A^D V U^D V$;
(f) $VU^D V A^D = VA^D V U^D$.

Remark 1.9. Let $A = U - V$ be an index-proper splitting. Then the matrices $U^D V$ and $A^D V$ (or VU^D and VA^D) have the same eigenvectors.

Lemma 1.10. ([7, Lemma 1.8])
Let $A = U - V$ be an index-proper splitting. Let μ_i, $1 \leq i \leq s$ and λ_j, $1 \leq j \leq s$ be the eigenvalues of the matrices $U^D V$ (VU^D) and $A^D V$ (VA^D), respectively. Then for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.

2. Index-proper regular and weak regular splittings

We begin with the definition of an index-proper regular and weak regular splitting.

Definition 2.1. ([6, Definition 4.13], [1, Definition 3.1])
A splitting $A = U - V$ of $A \in \mathbb{R}^{n \times n}$ is called an *index-proper regular splitting* (or also called a *D-regular splitting*) if it is an index-proper splitting such that $U^D \geq 0$ and $V \geq 0$.

Definition 2.2. ([1, Definition 3.5])
A splitting of the form $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is called an *index-proper weak regular splitting* if it is an index-proper splitting such that $U^D \geq 0$ and $U^D V \geq 0$.

We next present one example of a matrix splitting which is an index-proper weak regular but not an index-proper regular.

Example 2.3. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Then $\mathrm{ind}(A) = 2$. Taking $U = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, we have $\mathrm{ind}(U) = 2$ and $R(U^2) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = R(A^2)$. Hence A is an index-proper splitting. Here $U^D = (1/27)U^2 = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{pmatrix} \geq 0$ and $U^D V = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{pmatrix} \geq 0$. Hence, the splitting $A = U - V$ is index-proper weak regular splitting and not index-proper regular splitting as $V \ngeq 0$.

We now recall a convergence theorem for an index-proper regular splitting.

Theorem 2.4. ([1, Theorem 3.2])
Let $A = U - V$ be an index-proper regular splitting. If $A^D \geq 0$, then $\rho(U^D V) < 1$.

The converse of the above result is also true, and was stated in [1, Theorem 3.4]. For an index-proper weak regular splitting, we have the following convergence theorem.

Theorem 2.5. ([1, Theorem 3.8])
Let $A = U - V$ be an index-proper weak regular splitting. If $\rho(U^D V) < 1$, then $A^D \geq 0$.

We next present a new proof to [1, Theorem 3.7].

Theorem 2.6. ([1, Theorem 3.7])
Let $A = U - V$ be an index-proper weak regular splitting with $N(A^k) \subseteq N(V)$. If $A^D \geq 0$, then $\rho(U^D V) < 1$.

Proof. Suppose that $A^D \geq 0$. Let $C = U^D V$. Then $C \geq 0$. Also, $CU^D U = U^D V U^D U = U^D V = C$ as $VU^D U = V$, which follows from the condition $N(A^k) \subseteq N(V)$. Set $B_m = (I + C + C^2 + C^3 + \cdots + C^m)U^D$ for any positive integer m. Then $B_m \geq 0$ and $B_m \leq B_{m+1}$, since $C \geq 0$. Then by (c) of Theorem 1.7, we have $U^D = (I - C)A^D$. Using $U^D = (I - C)A^D$ in $B_m = (I + C + C^2 + C^3 + \cdots + C^m)U^D$, we have $B_m = (I - C^{m+1})A^D$. Then it follows from $B_m = (I - C^{m+1})A^D$ that $B_m \leq A^D$ since $C \geq 0$, $A^D \geq 0$. Hence, the sequence $\{B_m\}$ is a monotonically increasing sequence, which is bounded above. Hence, the sequence $\{B_m\}$ is convergent with respect to any matrix norm $||\cdot||$. Also, $B_{m+1}U - B_m U = C^{m+1}U^D U = C^m C U^D U = C^m C = C^{m+1}$ since

$N(A^k) \subseteq N(V)$ gives $VU^DU = V$. Hence, $||B_{m+1}U - B_mU|| = ||C^{m+1}|| \leq ||B_{m+1} - B_m||||U||$. We conclude that C^{m+1} converges to the zero matrix. It then follows that $\rho(U^DV) < 1$. □

The following Lemma will be used to prove Theorem 2.9.

Lemma 2.7. ([6, Lemma 4.1]) The system $Ax = b$ has a solution if $AA^Db = b$. In that case, the general solution is given by $x = A^Db + z$ for some $z \in N(A)$.

Before presenting the next theorems, we would like to recall the notion of Drazin monotonicity: an extension of inverse positive matrices to the singular case. $A \in \mathbb{R}^{n\times n}$ is said to be *Drazin monotone* if $A^D \geq 0$. For invertible matrices A, Drazin monotonicity reduces to monotonicity. A real square matrix A is called *monotone* if A is inverse positive. The book by Collatz [5] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. The problem of characterizing monotone matrices has been extensively dealt with in the literature.
Pye [13] showed the following equivalence for the matrices having nonnegative Drazin inverse (i.e. $A^D \geq 0$).

Theorem 2.8. ([13, Theorem 1]) $A^D \geq 0$ if and only if $Ax \in \mathbb{R}^n_+ + N(A^k)$ and $x \in R(A^k)$ imply $x \geq 0$.

A new characterization of Drazin monotone matrices is shown next.

Theorem 2.9. Consider the following statements.
(a) $A^D \geq 0$.
(b) $Ax \in \mathbb{R}^n_+ + N(A^k)$ and $x \in R(A^k) \Rightarrow x \geq 0$.
(c) $\mathbb{R}^n_+ \subseteq A\mathbb{R}^n_+ + N(A^k)$.
(d) There exist $x^0 \in \mathbb{R}^n_+$ and $z^0 \in N(A^k)$ such that $Ax^0 + z^0 \in \text{int}(\mathbb{R}^n_+)$. Then we have (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).
Suppose that A has an index-proper regular splitting such that $R(V) \subseteq R(A^k)$. Then each of the above is equivalent to the following:
(e) $\rho(U^DV) = \rho(VU^D) < 1$.

Proof. *(a)* $\Leftrightarrow$ *(b)*: By Theorem 2.8.
(b) $\Rightarrow$ *(c)*: Let $p \in \mathbb{R}^n_+$ and $q = A^Dp$. Then $q \in R(A^k)$ and by Lemma 2.7, $p = Aq + r$, $r \in N(A) \subseteq N(A^k)$ so that $Aq = p - r \in \mathbb{R}^n_+ + N(A^k)$. Therefore $q \in \mathbb{R}^n_+$ by (b) . Hence $p \in A\mathbb{R}^n_+ + N(A^k)$.
(c) $\Rightarrow$ *(d)*: Let $u^0 \in \text{int}(\mathbb{R}^n_+)$. Then there exist $x^0 \in \mathbb{R}^n_+$ and $z^0 \in N(A^k)$ such that $u^0 = Ax^0 + z^0$. Thus $Ax^0 + z^0 \in \text{int}(\mathbb{R}^n_+)$.
(d) $\Rightarrow$ (e): The fact $R(V) \subseteq R(A^k)$ implies $A = U - V = U - U^DUV = U(I - U^DV)$. Since A has an index-proper regular splitting, i.e., $U^D \geq 0$, $V \geq 0$, $R(U^k) = R(A^k)$ and $N(U^k) = N(A^k)$, we have $U^DV \geq 0$. Then $V^TU^{D^T} \geq 0$. Therefore $I - V^TU^{D^T} \leq I$. Set $C = I - V^TU^{D^T}$ and $B = I$. Then $C \leq B$, B^{-1} exists and $B^{-1} \geq 0$. We show that there exists a vector $w^0 \in \mathbb{R}^n_+$ such that $Cw^0 \in \text{int}(\mathbb{R}^n_+)$ is nonzero. It would then follow from Theorem 1.6 that C^{-1} exists and $C^{-1} \geq 0$. The fact $A^D \geq 0$ implies $A^{T^D} \geq 0$. The implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are also true for $A^{T^D} \geq 0$. Hence, the corresponding

(d^T) implies that there exist $x^0 \in \mathbb{R}^n_+$ and $z^0 \in N(A^{k^T})$ such that $A^T x^0 + z^0 \in \mathrm{int}(\mathbb{R}^n_+)$. Set $w^0 = U^T x^0 + z^0$. Then $w^0 = (A^T + V^T)x^0 + z^0 = A^T x^0 + z^0 + V^T x^0$. Since $V^T \geq 0$ and $x^0 \in \mathbb{R}^n_+$, we have $V^T x^0 \in \mathbb{R}^n_+$. Thus $w^0 \in \mathbb{R}^n_+$. Further, $Cw^0 = (I - V^T U^{D^T})w^0 = (I - V^T U^{D^T})(U^T x^0 + z^0) = (I - V^T U^{D^T})U^T x^0 + (I - V^T U^{D^T})z^0 = A^T x^0 + z^0 \in \mathrm{int}(\mathbb{R}^n_+)$. Here we have used the fact that $z^0 \in N(A^{k^T}) = N(U^{k^T}) = N(U^{D^T})$. Hence $(I - V^T U^{D^T})^{-1} = C^{-1} \geq 0$. By Theorem 1.3, it now follows that $\rho(U^D V) = \rho(V^T U^{D^T}) < 1$.

(e) $\Rightarrow$ (a): The proof is same as the converse part of the proof of [1, Theorem 3.4]. □

We next present a convergence result for an index-proper regular splitting under a sufficient condition $A^D U \geq 0$. The sketch of the proof is similar to [12, Lemma 3.4] which is for proper nonnegative splittings of rectangular matrices.

Theorem 2.10. Let $A = U - V$ be an index-proper regular splitting and $A^D U \geq 0$. Then $\rho(U^D V) = \frac{\rho(A^D U) - 1}{\rho(A^D U)} < 1$.

Proof. Since $U^D \geq 0$, $V \geq 0$ imply $U^D V \geq 0$, by (ii) of Theorem 1.1, there exists a nonnegative vector x ($x \neq 0$) such that $U^D V x = \rho(U^D V)x$. Then $x \in R(U^D) = R(U^k) = R(A^k)$. Therefore $U^D U x = x$. However $A^D = (I - U^D V)^{-1} U^D$, by Theorem 1.7 (c). Hence $A^D U = (I - U^D V)^{-1} U^D U$. Then $A^D U x = (I - U^D V)^{-1} U^D U x = (I - U^D V)^{-1} x = \frac{1}{1 - \rho(U^D V)} x$ which implies $\frac{1}{1 - \rho(U^D V)} \geq 0$ and is an eigenvalue of $A^D U$. Hence $0 \leq \frac{1}{1 - \rho(U^D V)} \leq \rho(A^D U)$, i.e., $\rho(U^D V) \leq \frac{\rho(A^D U) - 1}{\rho(A^D U)}$. Similarly, $A^D U \geq 0$ guaranties the existence of a nonnegative vector y ($y \neq 0$) such that $A^D U y = \rho(A^D U) y$. Then $y \in R(A^k) = R(U^k)$ implies $y = U^D U y$. Hence $(I - U^D V)^{-1} y = (I - U^D V)^{-1} U^D U y = A^D U y = \rho(A^D U) y$. Thus $\frac{1}{\rho(A^D U)} y = y - U^D V y$, i.e., $U^D V y = \frac{\rho(A^D U) - 1}{\rho(A^D U)} y$ which yields $\rho(U^D V) \geq \frac{\rho(A^D U) - 1}{\rho(A^D U)}$. Therefore $\rho(U^D V) = \frac{\rho(A^D U) - 1}{\rho(A^D U)} < 1$. □

We remark that Theorems 2.9 and 2.10 are also true for index-proper weak regular splittings.

Comparison theorems between the spectral radii of matrices are useful for choosing a better splitting between two given splittings. An accepted rule for preferring one iteration scheme to another is to choose the scheme having the smaller spectral radius of the respective iteration matrix. Many authors such as Jena and Mishra [6], Jena and Pani [7], Jena et al. [9] and Mishra [12], etc. have introduced various comparison results for different matrix splittings. Here we take index-proper regular and weak regular splittings for our analysis.

First we present two comparison results for index-proper regular splittings. The proofs are analogous to proofs of [9, Theorems 3.2 & 3.3]. Hence, we omit the proofs.

Theorem 2.11. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper regular splittings. If $A^D \geq 0$ and $V_1 \leq V_2$, then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$.

d^T means the corresponding result Theorem 2.9(d) when A is replaced by A^T.

The two conditions $A^D \geq 0$ and $V_1 \leq V_2$ in the above theorem can be merged, and is shown next.

Remark 2.12. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper regular splittings. If $A^D V_1 \leq A^D V_2$, then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$.

Theorem 2.13. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper regular splittings. If $A^D \geq 0$ and $U_1^D \geq U_2^D$, then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$.

Next two comparison results are obtained for two index-proper regular splittings.

Theorem 2.14. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper regular splittings. If $0 \leq A^D U_1 \leq A^D U_2$, then

$$\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1.$$

Proof. By Theorem 2.10, we have $\rho(U_i^D V_i) = \frac{\rho(A^D U_i) - 1}{\rho(A^D U_i)} < 1$ for $i = 1, 2$. Again, the condition $A^D U_1 \leq A^D U_2$ and Theorem 1.2 together yield $\rho(A^D U_1) \leq \rho(A^D U_2)$. Let λ_i be the eigenvalues of $A^D U_i$ for $i = 1, 2$. Since $\frac{\lambda_i - 1}{\lambda_i}$ is a strictly increasing function for $\lambda_i > 0$, we have $\frac{\rho(A^D U_2) - 1}{\rho(A^D U_2)} \geq \frac{\rho(A^D U_1) - 1}{\rho(A^D U_1)}$. Therefore $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$. $\square$

Theorem 2.15. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper regular splittings of A with $A^D \geq 0$. If $V_2 U_2^D \geq U_1^D V_1$, then

$$\rho(V_1 U_1^D) \leq \rho(V_2 U_2^D) < 1.$$

Proof. We have $\rho(U_i^D V_i) = \rho(V_i U_i^D) < 1$ for $i = 1, 2$ by Theorems 2.4. Also, we have $(I + A^D V_1)^{-1} A^D = U_1^D$ and $U_2^D = A^D (I + V_2 A^D)^{-1}$ by Theorem 1.8. Now $V_2 U_2^D \geq U_1^D V_1$ implies $V_2 A^D (I + V_2 A^D)^{-1} \geq (I + A^D V_1)^{-1} A^D V_1$. Then premultiplying both the sides by $I + A^D V_1$ and post-multiplying them by $I + V_2 A^D$, as $I + A^D V_1 \geq 0$ and $I + V_2 A^D \geq 0$, we obtain $V_2 A^D \geq A^D V_1$. Then, by Theorem 1.2, we get $\rho(V_2 A^D) = \rho(A^D V_2) \geq \rho(A^D V_1)$. Since $\frac{\lambda}{\lambda+1}$ is a strictly increasing function for $\lambda \geq 0$, and $\rho(V_2 A^D) = \rho(A^D V_2) \geq \rho(A^D V_1)$, then we have $\frac{\rho(A^D V_2)}{1+\rho(A^D V_2)} \geq \frac{\rho(A^D V_1)}{1+\rho(A^D V_1)}$. Therefore $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$. $\square$

3. Double index-proper regular and weak regular splittings

Motivated by the idea of Jena et al. [9] we now introduce the double index-proper splitting $A = P - R - S$ of A to $Ax = b$ which leads to the following iterative scheme spanned by three iterates:

$$x^{i+1} = P^D R x^i + P^D S x^{i-1} + P^D b, \ i = 1, 2, \ldots. \tag{1}$$

Then

$$\begin{pmatrix} x^{i+1} \\ x^i \end{pmatrix} = \begin{pmatrix} P^D R & P^D S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^i \\ x^{i-1} \end{pmatrix} + \begin{pmatrix} P^D b \\ 0 \end{pmatrix},$$

Let $y^{i+1} = \begin{pmatrix} x^{i+1} \\ x^i \end{pmatrix}$, $y^i = \begin{pmatrix} x^i \\ x^{i-1} \end{pmatrix}$, $W = \begin{pmatrix} P^D R & P^D S \\ I & 0 \end{pmatrix}$ and $d = \begin{pmatrix} P^D b \\ 0 \end{pmatrix}$. Then we have

$$y^{i+1} = W y^i + d, \ i = 1, 2, \ldots. \tag{2}$$

The iteration scheme (2) is *convergent* if $\rho(W) < 1$, and then $A = P - R - S$ is called a *convergent double splitting*.

We next introduce definitions of double index-proper regular and weak regular splittings.

Definition 3.1. A double splitting $A = P - R - S$ of A is called *double index-proper regular splitting* if $R(A^k) = R(P^k)$, $N(A^k) = N(P^k)$, $P^D \geq 0$, $R \geq 0$ and $S \geq 0$.

Definition 3.2. A double splitting $A = P - R - S$ of A is called *double index-proper weak regular splitting* if $R(A^k) = R(P^k)$, $N(A^k) = N(P^k)$, $P^D \geq 0$, $P^D R \geq 0$ and $P^D S \geq 0$.

The above definitions reduce to index-proper regular and weak regular splittings by setting $P = U$ and $R + S = V$. A convergence result which relates convergence of single and double splitting is shown next.

Theorem 3.3. Let $A^D \geq 0$ and $A = P - R - S$ is a double index-proper regular splitting (or double index-proper weak regular splitting), then $\rho(W) < 1$.

Proof. Since $A = P - R - S$ is a double proper regular splitting (or double proper weak regular splitting) of A, so for both the cases $W = \begin{pmatrix} P^D R & P^D S \\ I & 0 \end{pmatrix} \geq 0$. Setting $U = P$ and $V = R + S$, $A = U - V$ is an index-proper regular splitting (or an index-proper weak regular splitting) of A. By Theorem 2.4, we then have $\rho(P^D(R+S)) = \rho(U^D V) < 1$. By Theorem 1.5, it now follows that $\rho(W) < 1$. $\square$

Comparison of the spectral radii of the iteration matrices are necessary to study the convergence rate of two different systems of linear equations by iterative methods. The scheme with the smaller spectral radius will converge faster. We next discuss the above issue and the Drazin inverse analog of [9, Theorem 3.7].

Theorem 3.4. Let A_1 and A_2 be two square matrices with $N(A_1^k) = N(A_2^k)$, where $k = ind(A_1) = ind(A_2)$. Suppose that $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are double index-proper weak regular splittings such that $A_1^D \geq 0$ and $A_2^D \geq 0$. If $P_1^D A_1 \geq P_2^D A_2$ and $P_1^D R_1 \geq P_2^D R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$, where $W_1 = \begin{pmatrix} P_1^D R_1 & P_1^D S_1 \\ I & 0 \end{pmatrix}$ and $W_2 = \begin{pmatrix} P_2^D R_2 & P_2^D S_2 \\ I & 0 \end{pmatrix}$.

Proof. By Theorem 3.3, we have $\rho(W_i) < 1$ for $i = 1, 2$. If $\rho(W_1) = 0$, then our claim holds trivially. Suppose that $\rho(W_1) \neq 0$. Since A_1 and A_2 possess double index-proper weak regular splitting, we have $W_1 \geq 0$ and $W_2 \geq 0$. Now, applying Theorem 1.1 (ii) to W_1, we have $W_1 x = \rho(W_1)x$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0$, i.e.,

$$\begin{aligned} P_1^D R_1 x_1 + P_1^D S_1 x_2 &= \rho(W_1) x_1, \\ x_1 &= \rho(W_1) x_2. \end{aligned}$$

Now $N(A_1^k) = N(A_2^k)$ implies $R({A_1^k}^T) = R({A_2^k}^T)$ which yields $R({P_1^k}^T) = R({P_2^k}^T)$ gives $R(P_1^k) = R(P_2^k)$ by taking transpose on both the sides. Then $P_1^D P_1 =$

$P_{R(P_1^k),N(P_1^k)} = P_{R(P_2^k),N(P_2^k)} = P_2^D P_2$. The conditions $P_1^D R_1 \geq P_2^D R_2$ and $0 < \rho(W_1) < 1$ imply $(P_2^D R_2 - P_1^D R_1)x_1 \geq \frac{1}{\rho(W_1)}(P_2^D R_2 - P_1^D R_1)x_1$. Therefore

$$\begin{aligned}
W_2 x - \rho(W_1)x &= \begin{pmatrix} P_2^D R_2 x_1 + P_2^D S_2 x_2 - \rho(W_1)x_1 \\ x_1 - \rho(W_1)x_2 \end{pmatrix} \\
&= \begin{pmatrix} P_2^D R_2 x_1 + P_2^D S_2 x_2 - P_1^D R_1 x_1 - P_1^D S_1 x_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} (P_2^D R_2 - P_1^D R_1)x_1 - \frac{1}{\rho(W_1)}(P_1^D S_1 - P_2^D S_2)x_1 \\ 0 \end{pmatrix} \\
&\geq \begin{pmatrix} \frac{1}{\rho(W_1)}(P_2^D R_2 - P_1^D R_1)x_1 - \frac{1}{\rho(W_1)}(P_1^D S_1 - P_2^D S_2)x_1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\rho(W_1)}[P_2^D(R_2 + S_2) - P_1^D(R_1 + S_1)]x_1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\rho(W_1)}[P_2^D(P_2 - A_2) - P_1^D(P_1 - A_1)]x_1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\rho(W_1)}(P_1^D A_1 - P_2^D A_2)x_1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Condition $P_1^D A_1 \geq P_2^D A_2$ now yields that $W_2 x \geq \rho(W_1)x$. Thus, by Theorem 1.4, we have $\rho(W_1) \leq \rho(W_2) < 1$. □

The next two examples show that the converse of Theorem 3.4 is not necessarily true.

Example 3.5. Let $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Here $k = 2$, $N(A_1^2) = N(A_2^2)$, $A_1^D \geq 0$ and $A_2^D \geq 0$.
Set $P_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Again, $P_2 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are two double index-proper weak regular splittings with $0.5000 = \rho(W_1) \leq \rho(W_2) = 0.5774 < 1$. Here $P_1^D A_1 \ngeq P_2^D A_2$, but $P_1^D R_1 \geq P_2^D R_2$, where $P_1^D = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$, $P_1^D R_1 = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$,

$P_1^D A_1 = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$, $P_2^D = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$, $P_2^D A_2 = \begin{pmatrix} 0 & 2/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & 2/3 & 0 \end{pmatrix}$, $P_2^D R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Example 3.6. Let $A_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Here $k = 2$, $N(A_1^2) = N(A_2^2)$, $A_1^D \geq 0$ and $A_2^D \geq 0$. Set $P_1 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 6 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $P_1^D = \begin{pmatrix} 0 & 2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 2/3 & 0 \end{pmatrix}$, $P_1^D R_1 = \begin{pmatrix} 0 & 2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 2/3 & 0 \end{pmatrix}$, $P_1^D A_1 = \begin{pmatrix} 0 & 4/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & 4/3 & 0 \end{pmatrix}$. Again for $P_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ we get $P_2^D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,$P_2^D A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $P_2^D R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hence $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are two double index-proper weak regular splittings with $0.3333 = \rho(W_1) \leq \rho(W_2) = 0.5000 < 1$ and here $P_1^D A_1 \geq P_2^D A_2$, but $P_1^D R_1 \ngeq P_2^D R_2$.

Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double index-proper weak regular splitting of A. Then, we have $W_1 = \begin{pmatrix} P_1^D R_1 & P_1^D S_1 \\ I & 0 \end{pmatrix} \geq 0$ and $W_2 = \begin{pmatrix} P_2^D R_2 & P_2^D S_2 \\ I & 0 \end{pmatrix} \geq 0$. A comparison theorem for a single system of equations whose coefficient matrix A has two different double index-proper weak regular splitting is presented next.

Theorem 3.7. Let $A^D \geq 0$ and $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double index-proper regular splittings. Suppose that $P_1 \leq P_2$. Then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $A^D \geq 0$ and the double splittings are index-proper regular, by Theorem 3.3, $\rho(W_i) < 1$ for $i = 1, 2$. Using the given condition $P_1 \leq P_2$, we have $A + R_1 + S_1 \leq A + R_2 + S_2$, i.e., $R_2 + S_2 \geq R_1 + S_1 \geq 0$. Since the double splittings become single splittings by taking $U_i = P_i$, $V_i = R_i + S_i$ for $i = 1, 2$. Hence, by using Theorem 2.11, we have $\rho(P_1^D(R_1 + S_1)) = \rho(U_1^D V_1) \leq \rho(P_2^D(R_2 + S_2)) = \rho(U_2^D V_2) < 1$. Then by Theorem 1.5, it follows that $\rho(W_1) \leq \rho(W_2)$. □

A supportive example of above Theorem 3.7 is discussed below.

Example 3.8. Let $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Setting $P_1 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We get $R(A^2) = R(P_1^2) = R(P_2^2) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Hence, the splitting $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ are two double index-proper weak regular splittings of index 2. Thus, we have $P_1 \leq P_2$ with $\rho(W_1) \leq \rho(W_2) < 1$.

A corollary of the above theorem with $0 \leq S_1 \leq S_2$ is as follows.

Corollary 3.9. Let $A^D \geq 0$ and $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double index-proper regular splittings. Suppose that $R_1 \leq R_2$ and $S_1 \leq S_2$. Then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. The conditions $R_1 \leq R_2$ and $S_1 \leq S_2$ together gives $0 \leq R_1 + S_1 \leq R_2 + S_2$, which was obtained by component wise addition of $0 \leq R_1 \leq R_2$ and $0 \leq S_1 \leq S_2$. Then by Theorems 1.5 and 2.11, we have $\rho(W_1) \leq \rho(W_2) < 1$. $\square$

Another comparison result is obtained below.

Theorem 3.10. Let $A^D \geq 0$ and $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double index-proper regular splittings. Suppose that $P_1^D R_2 - P_1^D R_1 \geq P_1^D S_1 - P_1^D S_2$. Then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.

Proof. The double splitting $A = P_1 - R_1 - S_1$ is convergent, i.e., $\rho(W_1) < 1$, by Theorem 3.3 as $A^D \geq 0$. Since it is given that $P_1^D R_2 - P_1^D R_1 \geq P_1^D S_1 - P_1^D S_2$ from which we obtain $P_1^D (R_2 + S_2) \geq P_1^D (R_1 + S_1)$. Again, as we know that the double splittings become single splittings by taking $U_i = P_i$ and $V_i = R_i + S_i$ for $i = 1, 2$, we have $U_1^D V_2 \geq U_1^D V_1$. As the given splitting is index-proper, then by applying the property $U^D = (I + A^D V)^{-1} A^D$ to $U_1^D V_2 \geq U_1^D V_1$, we have $(I + A^D V_1)^{-1} A^D V_2 \geq (I + A^D V_1)^{-1} A^D V_1$. This implies $A^D V_2 \geq A^D V_1$. By Remark 2.12, we have $\rho(U_1^D V_1) \leq \rho(U_2^D V_2)$. Letting $V_i = R_i + S_i$ and $U_i = P_i$ for $i = 1, 2$, we get $\rho(U_1^D V_1) = \rho(P_1^D (R_1 + S_1)) \leq \rho(P_2^D (R_2 + S_2)) = \rho(U_2^D V_2) < 1$. Thus, by Theorem 1.5, we have $\rho(W_1) \leq \rho(W_2) < 1$. $\square$

The above result also holds for double index-proper weak regular splittings with $R_i + S_i \geq 0$ for $i = 1, 2$. Next example supports the above result.

Example 3.11. Taking $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ with P_1, P_2, R_1, R_2 and S_1, S_2 as in Example 3.8. Then $P_1^D R_2 - P_1^D R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P_1^D S_1 - P_1^D S_2 =$

$\begin{pmatrix} 0 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & -1/3 & 0 \end{pmatrix}$. So $P_1^D R_2 - P_1^D R_1 \geq P_1^D S_1 - P_1^D S_2$. Then $0.3333 = \rho(W_1) \leq \rho(W_2) = 0.6404 < 1$.

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