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SINGULAR RIESZ MEASURES ON SYMMETRIC CONES

ABDELHAMID HASSAIRI[1*] and SALLOUHA LAJMI[2]

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Abstract. A fondamental theorem due to Gindikin [Russian Math. Surveys, 29 (1964), 1-89] says that the generalized power $\Delta_s(-\theta^{-1})$ defined on a symmetric cone is the Laplace transform of a positive measure R_s if and only if s is in a given subset Ξ of $\mathbb{R}^r$, where r is the rank of the cone. When s is in a well defined part of Ξ, the measure R_s is absolutely continuous with respect to Lebesgue measure and has a known expression. For the other elements s of Ξ, the measure R_s is concentrated on the boundary of the cone and it has never been explicitly determined. The aim of the present paper is to give an explicit description of the measure R_s for all s in Ξ. The work is motivated by the importance of these measures in probability theory and in statistics since they represent a generalization of the class of measures generating the famous Wishart probability distributions.

1. Introduction

Many interesting results of analysis on Jordan algebras and their symmetric cones have been not only used as powerful mathematical tools in the development of other fields but also sources of inspiration. This seems to be due to the importance, in certain areas of these fields, of the special case of the algebra of symmetric matrices and of its symmetric cone of positive definite matrices. In probability theory and statistics, besides the developments realized in random matrix theory (see [2]), many results have been established in the general framework of a Jordan algebra. For instance in 2001, Hassairi and Lajmi [4]

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*Corresponding author.
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have introduced a class of natural exponential families of probability distributions generated by measures related to the so-called Riesz integrals in analysis on symmetric cones (see [1], p.137). These measures have been called by these authors Riesz measures and the generated probability distributions called Riesz probability distributions. The Riesz measures are in fact defined in a famous Theorem due to Gindikin [3] by their Laplace transforms. More precisely, for s in a given subset Ξ of $\mathbb{R}^r$, where r is the rank of the cone, the Riesz measure R_s is the measure of which the Laplace transform estimated in θ is equal to the so called generalized power $\Delta_s(-\theta^{-1})$. The generalized power $\Delta_s(-\theta^{-1})$ reduces to $(det(-\theta^{-1}))^p$ when all the components of s are equal to p. In this particular case, the generated probability distributions are the well known Wishart distributions. According to the position of s in the set Ξ, the Riesz measure R_s is either absolutely continuous with respect to the Lebesgue measure on the symmetric cone or it is singular and concentrated on the boundary of the cone. While the absolutely continuous Riesz measures are known, the singular ones are of complicated nature and their structure has never been explicitly determined. The aim of the present paper is to give an explicit description of the Riesz measure R_s for all s in Ξ. The question is very interesting from a mathematical point of view, in fact, besides the use of many important known facts from the analysis on symmetric cones, we have been led to develop many other useful results. The main motivation stems from the fact the knowledge of the way in which a singular Riesz measure is built allows the extension to a general singular Riesz probability distribution of some interesting results established for the singular Wishart distributions. We mention here that the structure of the singular Wishart distributions have been described in [8], and it is shown in [9] that these distributions are very useful; they arise for example in Bayesian analysis of some interesting problems.

2. Preliminaries

In this section, we first recall some facts concerning Jordan algebras and their symmetric cones. For more details, we refer the reader to the book by Faraut and Korányi [1] which is a complete and self-contained exposition on the subject. We then establish some new results on symmetric cones which we need in the description of the Riesz measures.

Recall that a Euclidean Jordan algebra is a Euclidean space V with scalar product $\langle x, y\rangle$ and a bilinear map

$$V \times V \to V, \qquad (x, y) \longmapsto x.y$$

called Jordan product such that, for all x, y, z in V,

(1) $x.y = y.x$,
(2) $\langle x, y.z\rangle = \langle x.y, z\rangle$,
(3) there exists e in V such that $e.x = x$,
(4) $x.(x^2.y) = x^2.(x.y)$, where we used the abbreviation $x^2 = x.x$.

A Euclidean Jordan algebra is said to be simple if it does not contain a non-trivial ideal. Actually to each Euclidean simple Jordan algebra, one attaches the set of Jordan squares

$$\overline{\Omega} = \left\{x^2; x \in V\right\}.$$

Its interior Ω is a symmetric cone i.e. a cone which is

(1) self dual, i.e., $\Omega = \left\{x \in V; \quad \langle x, y\rangle > 0 \quad \forall y \in \overline{\Omega} \setminus \{0\}\right\}$
(2) homogeneous, i.e. the subgroup $G(\Omega)$ of the linear group $GL(V)$ of linear automorphisms which preserve Ω acts transitively on Ω.
(3) salient, i.e., Ω does not contain a line. Furthermore, it is irreducible in the sense that it is not the product of two cones.

Let now x be in V. If $L(x)$ is the endomorphism of V ; $y \longmapsto x.y$ and $P(x) = 2L(x)^2 - L(x^2)$, then $L(x)$ and $P(x)$ are symmetric for the Euclidean structure of V, the map $x \longmapsto P(x)$ is called the quadratic representation of V.

An element c of V is said to be idempotent if $c^2 = c$, it is a primitive idempotent if furthermore $c \neq 0$ and is not the sum $t + u$ of two non null idempotents t and u such that $t.u = 0$.

A Jordan frame is a set $\{c_1, c_2, \ldots, c_r\}$ such that $\displaystyle\sum_{i=1}^{r} c_i = e$ and $c_i.c_j = \delta_{i,j}c_i$, for $1 \leq i, j \leq r$. It is an important result that the size r of such a frame is a constant called the rank of V. For any element x of a Euclidean simple Jordan algebra, there exists a Jordan frame $(c_i)_{1\leq i\leq r}$ and $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ such that $x = \displaystyle\sum_{i=1}^{r} \lambda_i c_i$. The real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ depend only on x, they are called the eigenvalues of x and this decomposition is called its spectral decomposition. The trace and the determinant of x are then respectively defined by $\mathrm{tr}(x) = \displaystyle\sum_{i=1}^{r} \lambda_i$ and $\det x = \displaystyle\prod_{i=1}^{r} \lambda_i$. If c is an idempotent of V, the only possible eigenvalues of $L(c)$ are 0, $\frac{1}{2}$ and 1. The corresponding spaces are respectively denoted by $V(c, 0)$, $V(c, \frac{1}{2})$ and $V(c, 1)$ and the decomposition

$$V = V(c, 1) \oplus V(c, \frac{1}{2}) \oplus V(c, 0)$$

is called the Peirce decomposition of V with respect to c. An element x of V can then be written in a unique way as

$$x = x_1 + x_{12} + x_0$$

with x_1 in $V(c, 1)$, x_{12} in $V(c, \frac{1}{2})$ and x_0 in $V(c, 0)$, which is also called the Peirce decomposition of x with respect to the idempotent c. We will denote by Ω_c the symmetric cone associated to the sub-algebra $V(c, 1)$, and when x is in $V(c, 1)$, $\det(x)$ denotes the determinant of x in the sub-algebra $V(c, 1)$.
Suppose now that $(c_i)_{1\leq i\leq r}$ is a Jordan frame in V and, for $1\leq i, j \leq r$, we set

$$V_{ij} = \begin{cases} V(c_i, 1) = \mathbb{R}c_i & \text{if } i = j \\ V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}) & \text{if } i \neq j \end{cases}$$

Then (See [1, Theorem IV.2.1]) we have the Peirce decomposition $V = \bigoplus_{i\leq j} V_{ij}$ with respect to the Jordan frame $(c_i)_{1\leq i\leq r}$. The dimension of V_{ij} is, for $i \neq j$, a constant d called the Jordan constant or multiplicity, it is related to the dimension n and the rank r of V by the relation $n = r + \frac{d}{2}r(r-1)$.
For $1 \leq k \leq r$, we have

$$V(c_1 + \ldots + c_k, 1) = \bigoplus_{i\leq j\leq k} V_{ij}\ ,\ V(c_1 + \ldots + c_k, \frac{1}{2}) = \bigoplus_{1\leq i\leq k<j} V_{ij}$$

In the following proposition, we establish some useful intermediary results.

Proposition 2.1. *Let c be an idempotent of V. Then*

(1) $\Omega_c = P(c)(\Omega)$,
(2) *for all x in $V(c,1)$, $2L(x)_{|V(c,\frac{1}{2})}$ is an endomorphism of $V(c,\frac{1}{2})$ with determinant equal to $\det(x)^{d(r-k)}$, where k is the rank of c,*
(3) *if x in $V(c,1)$ is invertible, then $2L(x)|_{V(c,\frac{1}{2})}$ is an automorphism of $V(c,\frac{1}{2})$ with inverse equal to $2L(x^{-1})|_{V(c,\frac{1}{2})}$.*
(4) *for all x in $V(c,1)$, $L(x^2)|_{V(c,\frac{1}{2})} = \frac{1}{2}L(x^2)|_{V(c,\frac{1}{2})}$*

Proof. (1) From [1, Theorem III.2.1], we have that the symmetric cone of a Jordan algebra is the set of element x in V for which $L(x)$ is positive definite.
Let x be in Ω. For $y \in V(c,1),\ y \neq 0$, we have:

$$\begin{aligned} \langle L(P(c)x)(y), y\rangle &= \langle P(c)(x)y, y\rangle \\ &= \langle P(c)(x), y^2\rangle \\ &= \langle x, P(c)y^2\rangle \\ &= \langle y, xy\rangle > 0 \end{aligned}$$

Thus $P(c)\Omega \subseteq \Omega_c$.
Now, let $w \in \Omega_c$, then $w + (e-c)$ is an element of Ω. Since $P(c)(w + (e-c)) = P(c)(w) = w$, we obtain that $\Omega_c \subseteq P(c)\Omega$.

(2) Let $x \in V(c,1)$. It is known (see [1], Prop IV.1.1) that $V(c,1).V(c,\frac{1}{2}) \subseteq V(c,\frac{1}{2})$, hence $2L(x)|_{V(c,\frac{1}{2})}$ is an endomorphism of $V(c,\frac{1}{2})$.
As c is an idempotent of rank k, there exit $c_1, c_2, \ldots, c_k$ orthogonal idempotents and $(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$ such that $c = \sum_{i=1}^{k} c_i$ and $x = \sum_{i=1}^{k} \lambda_i c_i$, so that $\det(x) = \prod_{i=1}^{k} \lambda_i$. Similarly, since $e-c$ is an idempotent with rank $r-k$, there exit $c_{k+1}, c_{k+2}, \ldots, c_r$ orthogonal idempotents such that $e - c = \sum_{i=1}^{r-k} c_{k+i}$. The system $(c_i)_{1\leq i\leq r}$ is a Jordan frame of V. If for $1 \leq i \leq k$, we set $V_{i,k+1} = \bigoplus_{j=k+1}^{r} V_{ij}$, then $V(c,\frac{1}{2}) = \bigoplus_{i=1}^{k} V_{i,k+1}$. We can

easily show that $2L(x)|_{V_{i,k+1}} = \lambda_i Id_{i,k+1}$, where $Id_{i,k+1}$ is the identity on the space $V_{i,k+1}$. As the dimension of $V_{i,k+1}$ is equal to $(r-k)d$, we have that the determinant of $2L(x)|_{V(c,\frac{1}{2})}$ is equal to $\prod_{i=1}^{k} \lambda_i^{(r-k)d} = \det(x)^{(r-k)d}$.

(3) If x is invertible in $V(c,1)$, then $\lambda_1, \dots, \lambda_k$ are different from zero and $x^{-1} = \sum_{p=1}^{k} \lambda_p^{-1} c_p$. Therefore, $2L(x)|_{V_{i,k+1}}$ is an automorphism of $V_{i,k+1}$ with inverse $\lambda_i^{-1} Id_{i,k+1}$ and it follows that $2L(x)|_{V(c,\frac{1}{2})}$ is an automorphism of $V(c,\frac{1}{2})$ with inverse $2L(x^{-1})|_{V(c,\frac{1}{2})}$.

(4) We have that $x^2 = \sum_{p=1}^{k} \lambda_p^2 c_p$ and for all $1 \leq i \leq k$, $2L(x)|_{V_{i,k+1}} = \lambda_i Id_{i,k+1}$. Then

$$L(x)^2|_{V_{i,k+1}} = \frac{1}{4}\lambda_i^2 Id_{i,k+1} = \frac{1}{2}\frac{\lambda_i^2}{2} Id_{i,k+1} = \frac{1}{2} L(x^2)|_{V_{i,k+1}}.$$

Thus, we conclude that $L(x^2)|_{V(c,\frac{1}{2})} = \frac{1}{2} L(x^2)|_{V(c,\frac{1}{2})}$.

□

Besides, the results shown above, we will use the facts stated in the following proposition due to Massam and Neher [7].

Proposition 2.2. *Let c be an idempotent of V, u_1 in $V(c,1)$, v_{12} in $V(c,\frac{1}{2})$, and u_0, z_0 in $V(c,0)$. Then*

(1) $\langle u_1, P(v_{12}) z_0 \rangle = 2\langle v_{12}, L(z_0) L(u_1) v_{12} \rangle$
(2) $L(z_0) L(u_1) = L(u_1) L(z_0)$
(3) *If $u_1 \in \Omega_c$ and $z_0 \in \Omega_{e-c}$, then $L(u_1) L(z_0)|_{V(c,\frac{1}{2})}$ is a positive definite endomorphism.*

Throughout, we suppose that the Jordan frame $(c_i)_{1\leq i\leq r}$ is fixed in V. For $1\leq k \leq r$, let P_k denote the orthogonal projection on the Jordan subalgebra

$$V^{(k)} = V(c_1 + c_2 + \dots + c_k, 1),$$

$\det^{(k)}$ the determinant in the subalgebra $V^{(k)}$ and, for x in V, $\Delta_k(x) = \det^{(k)}(P_k(x))$. The real number $\Delta_k(x)$ is called the principal minor of order k of x with respect to the frame $(c_i)_{1\leq i\leq r}$.

The generalized power with respect to the Jordan frame $(c_i)_{1\leq i\leq r}$ is the polynomial function defined in x of V by

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \dots \Delta_r(x)^{s_r}.$$

Note that $\Delta_s(x) = (\det(x))^p$ if $s = (p, p, \dots, p)$ with $p \in \mathbb{R}$, and if $x = \sum_{i=i}^{r} \lambda_i c_i$, then $\Delta_s(x) = \lambda_1^{s_1} \lambda_2^{s_2} \dots \lambda_r^{s_r}$. It is also easy to see that $\Delta_{s+s'}(x) = \Delta_s(x).\Delta_{s'}(x)$. In particular, if $m \in \mathbb{R}$ and $s + m = (s_1 + m, s_2 + m, \dots, s_r + m)$, we have

$\Delta_{s+m}(x) = \Delta_s(x)\det(x)^m$.
Now for the fixed Jordan frame $(c_i)_{1\leq i\leq r}$, and for $1 \leq l \leq r$ we define

$$\sigma_l = \sum_{i=1}^{l} c_i, \tag{2.1}$$

and we suppose that $V(\sigma_l, 1)$ and $V(e-\sigma_l, 1)$ are respectively equipped with the Jordan frames $(c_i)_{1\leq i\leq l}$ and $(c_i)_{l+1\leq i\leq r}$. Then we have the following result which allows the calculation of the general power of some projections. For the proof we refer the reader to Hassairi and Lajmi [5].

Theorem 2.3. *Let $1 \leq l \leq r-1$, and denote θ_0 the orthogonal projection of an element θ of the cone Ω on $V(e-\sigma_l, 1)$. Then*

(1) $\Delta_l(\theta^{-1}) = \det(\theta^{-1})\det(\theta_0)$,

(2) *for* $l+1 \leq k \leq r-1$, $\frac{\Delta_{k+1}(\theta^{-1})}{\Delta_k(\theta^{-1})} = \frac{\Delta^{e-\sigma_l}_{k+1-l}(\theta_0^{-1})}{\Delta^{e-\sigma_l}_{k-l}(\theta_0^{-1})}$, *and* $\frac{\Delta_{l+1}(\theta^{-1})}{\Delta_l(\theta^{-1})} = \Delta_1^{e-\sigma_l}(\theta_0^{-1})$.

We now introduce the set Ξ of elements $s = (s_1, \ldots, s_r)$ in $\mathbb{R}^r$ defined in the following way:
For a given real number $u \geq 0$, we set

$$\varepsilon(u) = 0 \quad \text{if} \quad u = 0$$
$$\varepsilon(u) = 1 \quad \text{if} \quad u > 0$$

Given $u = (u_1, \ldots, u_r) \in \mathbb{R}_+^r$, we define

$$s_1 = u_1 \quad \text{and} \quad s_i = u_i + \frac{d}{2}(\varepsilon(u_1) + \ldots + \varepsilon(u_{i-1})) \quad \text{for} \quad 2 \leq i \leq r. \tag{2.2}$$

Note that the set Ξ contains $\prod_{i=1}^{r}](i-1)\frac{d}{2}, +\infty[$, and that

$$\Lambda = \{p \in \mathbb{R} \ \text{ such that } \ (p, \ldots, p) \in \Xi\} = \left\{\frac{d}{2}, \ldots, \frac{d}{2}(r-1)\right\} \cup](r-1)\frac{d}{2}, +\infty[$$

The set Λ is the so called Wallach set.
The definition of the Riesz measure is based on the following theorem due to Gindikin [3] where the Laplace transform of a positive measure μ on V is defined by

$$L_\mu(\theta) = \int_V \exp(\langle\theta, x\rangle)\mu(dx).$$

Theorem 2.4. *There exists a positive measure R_s on V with Laplace transform defined on -Ω by $L_{R_s}(\theta) = \Delta_s(-\theta^{-1})$ if and only if s is in the set Ξ.*

Hassairi and Lajmi [4] have called the measure R_s Riesz measure and they have used it to introduce a class of probability distributions which is an important extension of the celebrated Wishart ones.
When $s = (s_1, s_2, \ldots, s_r)$ is in $\prod_{i=1}^{r}](i-1)\frac{d}{2}, +\infty[$, the measure R_s has an explicit density. In fact, if for s such that for all i, $s_i > (i-1)\frac{d}{2}$, we consider the measure

$$R_s = \frac{1}{\Gamma_\Omega(s)} \Delta_{s-\frac{n}{r}}(x)\mathbf{1}_\Omega(x)dx,$$

where $\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}}\prod_{j=1}^{r}\Gamma(s_j - (j-1)\frac{d}{2})$, then it is proved in [1, Theorem VII.1.2], that the Laplace transform of R_s is equal to $\Delta_s(-\theta^{-1})$ for $\theta \in -\Omega$, that is for all $\theta \in -\Omega$,

$$\frac{1}{\Gamma_\Omega(s)}\int \exp(\langle\theta, x\rangle)\Delta_{s-\frac{n}{r}}(x)\mathbf{1}_\Omega(x)(dx) = \Delta_s(-\theta^{-1}).$$

3. Description of the Riesz measures

In this section, we give a complete description of the Riesz measure R_s including the ones corresponding to s in $\Xi\backslash\prod_{i=1}^{r}](i-1)\frac{d}{2}, +\infty[$ which are concentrated on the boundary $\partial\Omega$ of the symmetric cone Ω. In order to do so, we need to recall some facts concerning the boundary structure of the cone Ω. More precisely, we have the following useful decomposition of the closed cone $\overline{\Omega}$ into orbits under the action of the group G, connected component of the identity in $G(\Omega)$, which appears in [6]. Recall that for the fixed Jordan frame $(c_i)_{1\leq i\leq r}$ and $1 \leq l \leq r$, $\sigma_l = \displaystyle\sum_{i=1}^{l} c_i$.

Proposition 3.1. (1) *An element x of $\overline{\Omega}$ is of rank l if and only if $x \in G\sigma_l$*

(2) *We have that $\overline{\Omega} = \displaystyle\bigcup_{l=1}^{r} G\sigma_l$.*

More precisely, $\Omega = G\sigma_r = Ge$ and $\partial\Omega = \displaystyle\bigcup_{l=1}^{r-1} G\sigma_l$

(3) *Denote for $1 \leq l \leq r-1$*

$$J_l = \{x \in G\sigma_l;\ \Delta_l(x) \neq 0\}.$$

then J_l is an open subset dense in $G\sigma_l$.

(4) *Suppose that $x = x_1 + x_{12} + x_0$ is the Peirce decomposition of x with respect to σ_l, then the map*

$$\Omega_{\sigma_l} \times V(\sigma_l, \tfrac{1}{2}) \to J_l \quad ; \quad (x_1, x_{12}) \longmapsto x_1 + x_{12} + 2(e - \sigma_l)[x_{12}(x_1^{-1}x_{12})]$$

is a bijection.

As a corollary of the last point, we have that an element x of J_l can be written in a unique way as $x = x_1 + x_{12} + (e - \sigma_l)v^2$, where $v = \frac{1}{2}\sqrt{x_1^{-1}}x_{12}$.

We now give the description of the Riesz measures R_s when s has a particular form, we then give the general case.

Theorem 3.2. *Let l be in $\{1, \ldots, r\}$, $\sigma_l = \displaystyle\sum_{i=1}^{l} c_i$, and $u = (u_1, \ldots, u_l)$ in $\mathbb{R}^l$ such that $u_i > (i-1)\frac{d}{2}$, for $1 \leq i \leq l$. Consider the measure on $\Omega_{\sigma_l} \times V(\sigma_l, \frac{1}{2})$ given by*

$$\gamma_l(dx_1, dv) = \frac{\Delta_u^{\sigma_l}(x_1)(\det(x_1))^{-1-(l-1)\frac{d}{2}}}{(2\pi)^{l(r-l)\frac{d}{2}}\Gamma_{\Omega_{\sigma_l}}(u)}\mathbf{1}_{\Omega_{\sigma_l}\times V(\sigma_l,\frac{1}{2})}(x_1, v)dx_1dv$$

and the map

$$\alpha_l : \Omega_{\sigma_l} \times V(\sigma_l, \tfrac{1}{2}) \to V \quad ; \quad (x_1, v) \longmapsto x_1 + 2v\sqrt{x_1} + (e - \sigma_l)v^2.$$

Then the Laplace transform of the image $\mu_l = \alpha_l(\gamma_l)$ of γ_l by α_l is defined on $-\Omega$ and is given by

$$L_{\mu_l}(\theta) = \Delta_s(-\theta^{-1}),$$

where $s = (u_1, \ldots, u_l, \frac{dl}{2}, \ldots, \frac{dl}{2}) \in \mathbb{R}^r$.

Proof. Let θ be in $-\Omega$ and let $\theta = \theta_1 + \theta_{12} + \theta_0$ be its Peirce decomposition with respect to σ_l. Then according to Proposition 2.1, (1), we have that $\theta_1 = P(\sigma_l)(\theta)$ is in $-\Omega_{\sigma_l}$ and $\theta_0 = P(e - \sigma_l)(\theta)$ is in $-\Omega_{e-\sigma_l}$. The Laplace transform of μ_l is given by.

$$\begin{aligned} L_{\mu_l}(\theta) &= \int_{\Omega_{\sigma_l} \times V(\sigma_l, \frac{1}{2})} \exp(\langle \theta, \alpha_l(x_1, v)\rangle \gamma_l(dx_1, dv) \\ &= \int_{\Omega_{\sigma_l} \times V(\sigma_l, \frac{1}{2})} \exp(\langle \theta_1, x_1\rangle + \langle \theta_{12}, 2v\sqrt{x_1}\rangle + \langle \theta_0, v^2\rangle) \\ &\quad \frac{\Delta_u^{\sigma_l}(x_1) \det(x_1)^{-1-(l-1)\frac{d}{2}}}{(2\pi)^{l(r-l)\frac{d}{2}} \Gamma_{\Omega_{\sigma_l}}(u)} dx_1 dv. \end{aligned}$$

This may be written as

$$L_{\mu_l}(\theta) = \int_{\Omega_{\sigma_l}} I(x_1) \exp(\langle \theta_1, x_1\rangle) \Delta_u^{\sigma_l}(x_1) \mathrm{det}(x_1)^{-1-(l-1)\frac{d}{2}} \frac{dx_1}{\Gamma_{\Omega_{\sigma_l}}(u)}, \tag{3.1}$$

where

$$I(x_1) = \frac{1}{(2\pi)^{l(r-l)\frac{d}{2}}} \int_{V(\sigma_l, \frac{1}{2})} \exp(\langle \theta_{12}, 2v\sqrt{x_1}\rangle + \langle \theta_0, v^2\rangle) dv.$$

According to Proposition 2.2, (3) and Proposition 2.1, (3), we have that $2L(-\theta_0)|_{V(\sigma_l, \frac{1}{2})} = L(4\sigma_l)L(-\theta_0)_{|V(\sigma_l, \frac{1}{2})}$ is an automorphism of $V(\sigma_l, \frac{1}{2})$ whose the inverse is equal to $2L(-\theta_0^{-1})|_{V(\sigma_l, \frac{1}{2})}$. Thus, one can write

$$I(x_1) = \frac{1}{(2\pi)^{l(r-l)\frac{d}{2}}} \int_{V(\sigma_l, \frac{1}{2})} \exp(\langle 2\sqrt{x_1}\theta_{12}, v\rangle - \frac{1}{2}\langle 2L(-\theta_0)v, v\rangle) dv.$$

Using [1, Lemma VII.2.5], then again Proposition 2.1, we get

$$\begin{aligned} I(x_1) &= \Big(\det 2L(-\theta_0^{-1})|_{V(\sigma_l, \frac{1}{2})}\Big)^{\frac{1}{2}} \exp(\frac{1}{2}\langle 2\sqrt{x_1}\theta_{12}, 2L(-\theta_0^{-1}) 2\sqrt{x_1}\theta_{12}\rangle) \\ &= \det(-\theta_0^{-1})^{l\frac{d}{2}} \exp(\frac{1}{2}\langle 2\sqrt{x_1}\theta_{12}, 4L(-\theta_0^{-1})L(\sqrt{x_1})\theta_{12}\rangle). \end{aligned}$$

As $L(\sqrt{x_1})$ is symmetric, we can write

$$I(x_1) = \det(-\theta_0^{-1})^{l\frac{d}{2}} \exp(\langle 4\theta_{12}, L(\sqrt{x_1})^2 L(-\theta_0^{-1})\theta_{12}\rangle).$$

Proposition 2.1 implies that

$$I(x_1) = \det(-\theta_0^{-1})^{l\frac{d}{2}} \exp(2\langle \theta_{12}, L(x_1)L(-\theta_0^{-1})\theta_{12}\rangle).$$

Finally, from Proposition 2.2 (1), we deduce that

$$I(x_1) = \det(-\theta_0^{-1})^{l\frac{d}{2}} \exp(\langle x_1, P(\theta_{12})(-\theta_0^{-1})\rangle).$$

Now inserting this in (3.1), we obtain

$$\begin{aligned} L_{\mu_l}(\theta) &= \frac{\det(-\theta_0)^{-l\frac{d}{2}}}{\Gamma_{\Omega_{\sigma_l}}(u)} \int_{\Omega_{\sigma_l}} \exp(\langle x_1, \theta_1 - P(\theta_{12})(\theta_0^{-1})\rangle)\Delta_u^{\sigma_l}(x_1) \\ &\quad \det(x_1)^{-1-(l-1)\frac{d}{2}} dx_1 \\ &= \det(-\theta_0)^{-l\frac{d}{2}} \Delta_u^{\sigma_l}\left(-(\theta_1 - P(\theta_{12})(\theta_0^{-1}))^{-1}\right). \end{aligned}$$

Since $(\theta^{-1})_1 = P(\sigma_l)(\theta^{-1}) = (\theta_1 - P(\theta_{12})(\theta_0^{-1}))^{-1}$ and according to Theorem 2.3, we can write

$$\begin{aligned} L_{\mu_l}(\theta) &= \det(-\theta_0)^{-l\frac{d}{2}} \Delta_u^{\sigma_l}(-(\theta^{-1})_1) \\ &= \left(\frac{\Delta_l(-\theta^{-1})}{\det(-\theta^{-1})}\right)^{-l\frac{d}{2}} \Delta_u^{\sigma_l}(-(\theta^{-1})_1), \end{aligned}$$

Therefore,

$$\begin{aligned} L_{\mu_l}(\theta) &= \Delta_1(-\theta^{-1})^{u_1-u_2} \dots \Delta_{l-1}(-\theta^{-1})^{u_{l-1}-u_l} \Delta_l(-\theta^{-1})^{u_l - l\frac{d}{2}} \det(-\theta^{-1})^{l\frac{d}{2}} \\ &= \Delta_s(-\theta^{-1}), \end{aligned}$$

where $s = (u_1, \dots, u_l, l\frac{d}{2}, \dots, l\frac{d}{2})$ in $\mathbb{R}^r$ □

Corollary 3.3. *For $1 \leq l \leq r-1$, the measure μ_l is concentrated on the boundary $\partial\Omega$ of the symmetric cone Ω.*

Proof. In fact, μ_l is concentrated on the set $J_l = \{x \in G\sigma_l;\ \Delta_l(x) \neq 0\}$, and from Proposition 3.1, this is dense in G. □

Theorem 3.4. *Let l be in $\{1, \dots, r-1\}$, and suppose that for $u = (u_1, \dots, u_{r-l}) \in \mathbb{R}_+^{r-l}$, there exists a measure μ_u on $V(e - \sigma_l, 1)$ such that the Laplace transform is defined on the set $-\Omega_{e-\sigma_l}$ and is equal to $\Delta_u^{e-\sigma_l}(-\theta_0^{-1})$. Then the Laplace transform of the measure μ image of μ_u by the injection of $V(e-\sigma_l, 1)$ into V is defined on $-\Omega$ and $L_\mu(\theta) = \Delta_s(-\theta^{-1})$, where $s = (0, \dots 0, u_1, \dots, u_{r-l}) \in \mathbb{R}_+^r$.*

Proof. Let $x = x_1 + x_{12} + x_0$ and $\theta = \theta_1 + \theta_{12} + \theta_0$ be respectively the Peirce decomposition with respect to σ_l of an element x of V and an element θ of $-\Omega$. Then

$$\begin{aligned} L_\mu(\theta) &= \int_{V(e-\sigma_l,1)} \exp(\langle \theta, x_0\rangle)\mu_u(dx_0) \\ &= \int_{V(e-\sigma_l,1)} \exp(\langle \theta_0, x_0\rangle)\mu_u(dx_0) \\ &= \Delta_u^{e-\sigma_l}(-\theta_0^{-1}) \\ &= \Delta_1^{e-\sigma_l}(-\theta_0^{-1})^{u_1} \left(\frac{\Delta_2^{e-\sigma_l}(-\theta_0^{-1})}{\Delta_1^{e-\sigma_l}(-\theta_0^{-1})}\right)^{u_2} \dots \left(\frac{\Delta_{r-l}^{e-\sigma_l}(-\theta_0^{-1})}{\Delta_{r-l-1}^{e-\sigma_l}(-\theta_0^{-1})}\right)^{u_{r-l}} \end{aligned}$$

This according to Theorem 2.3 leads to

$$\begin{aligned} L_\mu(\theta) &= \left(\frac{\Delta_{l+1}(-\theta^{-1})}{\Delta_l(-\theta^{-1})}\right)^{u_1}\left(\frac{\Delta_{l+2}(-\theta^{-1})}{\Delta_{l+1}(-\theta^{-1})}\right)^{u_2}\cdots\left(\frac{\Delta_r(-\theta^{-1})}{\Delta_{r-1}(-\theta^{-1})}\right)^{u_{r-l}} \\ &= \Delta_l(-\theta^{-1})^{-u_1}\Delta_{l+1}(-\theta^{-1})^{u_1-u_2}\ldots\Delta_r(-\theta^{-1})^{u_{r-l}} \\ &= \Delta_s(-\theta^{-1}), \end{aligned}$$

where $s = (0, \ldots, 0, u_1, \ldots, u_{r-l}) \in \mathbb{R}_+^r$. $\square$

We come now to the construction of the Riesz measure R_s for any $s = (s_1, \ldots, s_r)$ in the set Ξ. From the definition of Ξ, if $s \in \Xi$, there exists $u = (u_1, \ldots, u_r) \in \mathbb{R}_+^r$ such that

$$s_1 = u_1 \quad \text{and} \quad s_i = u_i + \frac{d}{2}(\varepsilon(u_1) + \ldots + \varepsilon(u_{i-1})). \tag{3.2}$$

We will use $(u_1, \ldots, u_r)$, to construct a partition (A_i) of the set $\{1, \ldots, r\}$ such that, for all i, we have either $u_j = 0, \forall j \in A_i$ or $u_j > 0, \forall j \in A_i$. Such a partition is important in the description of the measure R_s.
Consider the sequences of integers $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ built as follows:
$i_1 = \inf\{p \geq 0 \ \ ; \ \ u_{p+1} \neq 0\}$,
$j_1 = \inf\{p \geq 0 \quad u_{i_1+p+1} = 0\ \}$,
$i_l = \inf\{p \geq i_{l-1} + j_{l-1}$ such that $u_{p+1} \neq 0\}$, $2 \leq l \leq k$,
$j_l = \inf\{p \geq 0$ such that $u_{i_l+p+1} = 0\ \}$, $1 \leq l \leq k$,
In this way, we get a partition of $u = (u_1, \ldots, u_r)$ in the form:

$$u = (\underbrace{0, \ldots, 0}_{i_1 \ \ terms}, \underbrace{u_{i_1+1}, \ldots, u_{i_1+j_1}}_{j_1 \ \ terms}, \ldots, \underbrace{0, \ldots, 0}, \underbrace{u_{i_l+1}, \ldots, u_{i_l+j_l}}_{j_l \ \ terms}, \ldots, \underbrace{0, \ldots, 0}, \underbrace{u_{i_k+1}, \ldots, u_{i_k+j_k}}_{j_k \ \ terms}, \ldots).$$

This partition of u leads to the following partition of the set $\{1, \ldots, r\}$ defined by

$I'_0 = \begin{cases} \emptyset & \text{if} \quad i_1 = 0 \\ \{1, \ldots, i_1\} & \text{if} \quad i_1 \neq 0 \end{cases}$

$I'_l = \{i_l + j_l + 1, \ldots, i_{l+1}\}$ if $1 \leq l \leq k-1$.

$I'_k = \begin{cases} \emptyset & \text{if} \quad i_k + j_k = r \\ \{i_k + j_k + 1, \ldots, r\} & \text{if} \quad i_k + j_k < r \end{cases}$

and

$I_l = \{i_l + 1, \ldots, i_l + j_l\}$ if $1 \leq l \leq k$.

Thus we have that

$$\bigcup_{1 \leq p \leq k} I_p = \{i \ \ ; \ \ u_i \neq 0\} \quad \text{and} \quad \bigcup_{0 \leq p \leq k} I'_p = \{i \ \ ; \ \ u_i = 0\}.$$

In conclusion, for an element s in Ξ, we associate $u = (u_1, \ldots, u_r)$, k in $\{1, \ldots, r\}$, and the partition of the set $\{1, \ldots, r\}$ defined above. We also define for $1 \leq l \leq k$,

$$u^{(l)} = \left(u_{i_l+1}, u_{i_l+2} + \frac{d}{2}, \ldots, u_{i_l+j_l} + \frac{d}{2}(j_l - 1)\right),$$

which is in $\mathbb{R}^{j_l}$ and the element of $\mathbb{R}^r$

$$s^{(l)} = \left(\underbrace{0, \ldots, 0}_{i_l \ \ terms}, u^{(l)}, \frac{d}{2}j_l, \ldots, \frac{d}{2}j_l\right), \tag{3.3}$$

which can be written as

$$s^{(l)} = (\alpha_1^{(l)}, \dots, \alpha_r^{(l)})$$

with

$$\begin{cases} \alpha_p^{(l)} = 0 & if \quad 1 \leq p \leq i_l \\ \alpha_{i_l+p}^{(l)} = u_{i_l+p} + \frac{d}{2}(p-1) & if \quad 1 \leq p \leq j_l \\ \alpha_p^{(l)} = \frac{d}{2} j_l & if \quad i_l + j_l + 1 \leq p \leq r. \end{cases}$$

The last term disappears if $i_l + j_l = r$.

Proposition 3.5. *With the previous notations, for any s in Ξ, we have*

$$s = \sum_{1 \leq l \leq k} s^{(l)}.$$

Proof. Recall that the corresponding vector $u = (u_1, \dots, u_r)$ to a given s in Ξ is such that

$$s_1 = u_1 \quad \text{and} \quad s_i = u_i + \frac{d}{2}(\varepsilon(u_1) + \ldots + \varepsilon(u_{i-1}))$$

Given m in $\{1, \dots, r\}$, we distinguish between four cases according to its position in the elements I'_0, I'_l, I'_k and I_l of the partition of $\{1, \dots, r\}$.

If $m \in I'_0$, then $s_m = 0$ and $\alpha_m^{(l)} = 0$, for $1 \leq l \leq k$, since $m \leq i_1 \leq i_l$ so that we have

$$s_m = \sum_{1 \leq l \leq k} \alpha_m^{(l)}$$

.

If $m \in I_l$ with $1 \leq l \leq k$, then $i_l + 1 \leq m \leq i_l + j_l$, . It follows that

$$\begin{cases} \alpha_m^{(p)} = \frac{d}{2} j_p & if \quad 1 \leq p \leq l-1, \quad since \;\; i_p + j_p \leq i_{l-1} + j_{l-1} < i_l < m \\ \alpha_m^{(p)} = u_m + \frac{d}{2}(m - i_l - 1) & if \quad p = l, \quad since \;\; 1 \leq m - i_l \leq j_l \\ \alpha_m^{(p)} = 0 & if \quad l+1 \leq p \leq k, \quad since \;\; m \leq i_l + j_l < i_{l+1} < i_p. \end{cases}$$

Therefore

$$\sum_{1 \leq p \leq k} \alpha_m^{(p)} = u_m + \frac{d}{2}(m - i_l - 1 + j_1 + \ldots + j_{l-1}) = s_m$$

.

If $m \in I'_l$, with $1 \leq l \leq k-1$, then $i_l + j_l + 1 \leq m \leq i_{l+1}$. Il follows that

$$\begin{cases} \alpha_m^{(p)} = \frac{d}{2} j_p & if \quad 1 \leq p \leq l, \quad since \;\; i_1 + j_1 < \ldots < i_l + j_l < m \\ \alpha_m^{(p)} = 0 & if \quad l+1 \leq p \leq k, \quad since \;\; m \leq i_{l+1} < i_{l+2} < \ldots < i_k. \end{cases}$$

As $u_m = 0$, we obtain

$$\sum_{1 \leq p \leq k} \alpha_m^{(p)} = \frac{d}{2}(j_1 + \ldots + j_l) = s_m$$

.

If $m \in I'_k$, then $i_k + j_k + 1 \leq m \leq r$. Since $i_1 + j_1 < \ldots < i_k + j_k < m$, it follows that

$$\alpha_m^{(p)} = \frac{d}{2} j_p \quad 1 \leq p \leq k.$$

Thus

$$\sum_{1\le p\le k}\alpha_m^{(p)}=\frac{d}{2}(j_1+\ldots+j_k)=s_m$$

. ∎

To continue our description of the Riesz measures, we require some further notations. For s in Ξ, and $1\le l\le k$, where k is the integer corresponding to s defined above, we set

$$\overline{c}_{i_l}=c_{i_l+1}+\ldots+c_r$$

$$\overline{c}_{i_l,j_l}=c_{i_l+1}+\ldots+c_{i_l+j_l}$$

$\overline{c}_{i_l,j_l}$ is an idempotent of rank j_l in $V(\overline{c}_{i_l},1)$.
Let $\widehat{V}(\overline{c}_{i_l,j_l},1)$ and $\widehat{V}(\overline{c}_{i_l,j_l},\frac{1}{2})$ be the subspaces of $V(\overline{c}_{i_l},1)$ corresponding to the eigenvalues 1 and $\frac{1}{2}$, and let $\widehat{\Omega}_{\overline{c}_{i_l,j_l}}$ be the symmetric cone associated to $\widehat{V}(\overline{c}_{i_l,j_l},1)$.
Consider the map

$$\alpha:\widehat{\Omega}_{\overline{c}_{i_l,j_l}}\times\widehat{V}(\overline{c}_{i_l,j_l},\tfrac{1}{2})\rightarrow V(\overline{c}_{i_l},1)\ \ ;\ \ (x,v)\longmapsto x+2v\sqrt{x}+(\overline{c}_{i_l}-\overline{c}_{i_l,j_l})v^2,$$

and let i be the canonical injection of $V(\overline{c}_{i_l},1)$ into V.
We now define the measure

$$\gamma_{u^{(l)}}(dx,dv)=\frac{\Delta_{u^{(l)}}^{\overline{c}_{i_l,j_l}}(x)(\det(x))^{-1-(j_l-1)\frac{d}{2}}}{(2\pi)^{j_l(r-i_l-j_l)\frac{d}{2}}\Gamma_{\widehat{\Omega}_{\overline{c}_{i_l,j_l}}}(u^{(l)})}\mathbf{1}_{\widehat{\Omega}_{\overline{c}_{i_l,j_l}}\times\widehat{V}(\overline{c}_{i_l,j_l},\frac{1}{2})}(x,v)dxdv,$$

and we denote $\mu_{u^{(l)}}$ the image of $\gamma_{u^{(l)}}$ by the map $i\circ\alpha$.
We are now ready to state and prove our main result.

Theorem 3.6. *For all s in Ξ, we have*

$$R_s=\mu_{u^{(1)}}\star\ldots\star\mu_{u^{(k)}},$$

where $\star$ is the convolution product.

Proof. We need to show that the Laplace transform of $\mu_{u^{(1)}}\star\ldots\star\mu_{u^{(k)}}$ defined in an element θ of $-\Omega$ is equal to $\Delta_s(-\theta^{-1})$.
For $1\le l\le k$, let $\theta=\theta_1+\theta_{12}+\theta_0$ be the Peirce decomposition of θ with respect to $\overline{c}_{i_l}$. If we denote $\mu'_{u^{(l)}}$ the image of $\gamma_{u^{(l)}}$ by the map α, then according to Theorem 3.2, we have that

$$L_{\mu'_{u^{(l)}}}(-\theta_0)=\Delta_{s'^{(l)}}^{\overline{c}_{i_l}}(-\theta_0^{-1}),$$

where $s'^{(l)}=(u^{(l)},\frac{d}{2}j_l,\ldots,\frac{d}{2}j_l)\in\mathbb{R}^{r-i_l}$. On the other hand, as $\mu_{u^{(l)}}$ is the image of $\mu'_{u^{(l)}}$ by the canonical injection of $V(\overline{c}_{i_l},1)$ into V, Theorem 3.4 implies that

$$L_{\mu_{u^{(l)}}}(\theta_0)=\Delta_{s^{(l)}}(-\theta^{-1}),$$

where $s^{(l)} = (0, \dots 0, s'^{(l)}) \in \mathbb{R}^r$.
Therefore the Laplace transform of $\mu_{u^{(1)}} \star \dots \star \mu_{u^{(k)}}$ in $\theta \in -\Omega$ is

$$\begin{aligned} L_{\mu_{u^{(1)}} \star \dots \star \mu_{u^{(k)}}}(\theta) &= \prod_{1 \le l \le k} L_{\mu_{u^{(l)}}}(\theta) \\ &= \prod_{1 \le l \le k} \Delta_{s^{(l)}}(-\theta^{-1}) \\ &= \Delta_{\sum\limits_{1 \le l \le k} s^{(l)}}(-\theta^{-1}) \\ &= \Delta_s(-\theta^{-1}), \end{aligned}$$

which is the desired result □

Corollary 3.7. (1) *The measure $\mu_{u^{(l)}}$ is supported by the set*

$$J'_{u^{(l)}} = \{x \in V(\overline{c}_{i_l}, 1) \ \text{ such that } \ x \in \overline{\Omega} \quad \text{and} \ \ rank x = j_l\}$$

(2) *The measure R_s is supported by the set*

$$J'_{u^{(1)}} + \dots + J'_{u^{(k)}} \subseteq V(\overline{c}_1, 1) \cap \overline{\Omega}$$

Proof. (1) Follows from Corollary 3.4.
(2) It suffices to observe that $V(\overline{c}_1, 1) \supset V(\overline{c}_2, 1) \supset \dots \supset V(\overline{c}_k, 1)$

□

Remark 3.8. (1) When s is in Ξ such that $s_i > (i-1)\frac{d}{2}$, $1 \le i \le r$, then the integer k corresponding to s is equal to 1. In this case $R_s = \mu_{u^{(1)}}$, it is concentrated on Ω.
(2) When s is in $\Xi \backslash \prod_{i=1}^{r}](i-1)\frac{d}{2}, +\infty,$, then the integer k corresponding to s is strictly greater than 1 and $j_1 + \dots + j_k < r$. The measure R_s is in this case concentrated on $J'_{u^{(1)}} + \dots + J'_{u^{(k)}}$ whose the elements are of rank less than or equal to $j_1 + \dots + j_k$. As $j_1 + \dots + j_k < r$, R_s is supported by the boundary $\partial\Omega$ of the symmetric cone Ω.

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[1]Department of Mathematics, Faculty of Sciences Sfax, P. O. Box 1171, Sfax 3000, Tunisia.

E-mail address: Abdelhamid.Hassairi@fss.rnu.tn

[2]Department of Computer Science and Mathematics, ENIS, Route Soukra, Sfax 3000, Tunisia.

E-mail address: sallouha.lajmi@enis.rnu.tn