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COVER TOPOLOGIES, SUBSPACES, AND QUOTIENTS FOR SOME SPACES OF VECTOR-VALUED FUNCTIONS

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Abstract. Let X be a completely regular Hausdorff space, and let $\mathcal{D}$ be a cover of X by C_b-embedded sets. Let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces (algebras), and let $\Gamma(\pi)$ be the section space of the bundle π. Denote by $\Gamma_b(\pi, \mathcal{D})$ the subspace of $\Gamma(\pi)$ consisting of sections which are bounded on each $D \in \mathcal{D}$. We construct a bundle $\rho' : \mathcal{F}' \to \beta X$ such that $\Gamma_b(\pi, \mathcal{D})$ is topologically and algebraically isomorphic to $\Gamma(\rho')$, and use this to study the subspaces (ideals) and quotients resulting from endowing $\Gamma_b(\pi, \mathcal{D})$ with the cover topology determined by $\mathcal{D}$.

1. Introduction

The present paper investigates the subspace and quotient structures of certain spaces and algebras of vector-valued functions. By using the theory of bundles of topological vector spaces, our work extends to more general spaces of vector-valued functions many of the results to be found in [1] and [2] regarding the structure of some subspaces (ideals) and quotients of $C(X)$, where X is a completely regular Hausdorff space.

Unless otherwise noted, X will denote a completely regular Hausdorff space, with Stone-Čech compactification βX. The scalar space, either $\mathbb{R}$ or $\mathbb{C}$, will be denoted by $\mathbb{K}$. As usual, $C_b(X)$ will denote the space of bounded, $\mathbb{K}$-valued functions on X. If Y is a topological space, and if $Z \subset W \subset Y$, it is clear what we mean if we say "W-closure of Z" or "Z is Y-closed". If $\alpha = \alpha_X : X \to \beta X$ is the

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canonical homeomorphism of X onto a dense subspace of βX, we will write, again as suits the context, $\alpha(X) = X \subset \beta X$. A set $D \subset X$ is said to be C_b-embedded if $D = \alpha(D) \subset \beta D = \overline{D} \subset \beta X$; equivalently, $D \subset X$ is C_b-embedded if each $f \in C_b(D)$ has an extension in $C_b(X)$. It is well-known that if X is completely regular, and if $K \subset X$ is compact, then K is C_b-embedded; if X is normal, then any closed set $C \subset X$ is C_b-embedded. If A is a commutative Banach algebra, $\Delta(A)$ will be the space of non-trivial $\mathbb{K}$-valued algebra homomorphisms on A; as usual, we can also identify $\Delta(A)$ as the space of closed maximal ideals in A. If $g : X \to Y$ is a function and $C \subset X$, then g_C is the restriction of g to C; if G is a set of functions $g : X \to Y$, then $G_C = \{g_C : g \in G\}$.

We will be concerned with certain subspaces and quotients of $\Gamma(\pi)$, the space of sections of the bundle of Banach spaces (= Banach bundle) $\pi : \mathcal{E} \to X$, and in particular, we will investigate such structures when $\pi : \mathcal{E} \to X$ is a bundle of Banach algebras (= Banach algebra bundle). For details of the development of such bundles, and of bundles of topological vector spaces in general, we refer the reader to [3]; further elaboration can be found in e.g. [8], [9], [4] and [5]. The essentials are the following; they can be found e.g. in [3], [5] or [6], but we repeat them here for convenience.

Consider the following situation: Let $\{E_x : x \in X\}$ be a collection of locally convex topological vector spaces over $\mathbb{K}$, indexed by X, and let the total space $\mathcal{E} = \dot{\bigcup} \{E_x : x \in X\}$ be their disjoint union, with $\pi : \mathcal{E} \to X$ the natural projection. We let $\mathcal{S}$ be a vector space of choice functions $\sigma : X \to \mathcal{E}$ (i.e. $\sigma(x) \in E_x$ for each $x \in X$) such that the following conditions hold:

C1) for each $x \in X$, $\phi_x(\mathcal{S}) = \{\sigma(x) : \sigma \in \mathcal{S}\} = E_x$ (in this case, $\mathcal{S}$ is said to be full; ϕ_x is the evaluation map at x);

C2) $\mathcal{S}$ is a $C_b(X)$-module;

C3) there is a collection $\{p_j^x : j \in \mathcal{J}\}$ of seminorms on each E_x such that for each $\sigma \in \mathcal{S}$ the numerical function $x \mapsto p_j^x(\sigma(x))$ is upper semicontinuous and bounded on X for each $j \in \mathcal{J}$;

C4) $\mathcal{S}$ is closed in the locally convex topology generated by

$$p_j(\sigma) = \sup_{x \in X} p_j^x(\sigma(x)) < \infty;$$

C5) for each $x \in X$, the relative topology on $E_x \subset \mathcal{E}$ is its original locally convex topology.

Under these conditions, there is a topology on $\mathcal{E}$ (the bundle topology) which makes $\mathcal{S}$ a subspace of the space $\Gamma(\pi)$ of all sections (section = continuous choice function) $\tau : X \to \mathcal{E}$. In this bundle topology, a neighborhood of $z \in E_x \subset \mathcal{E}$ is given by tubes of the form

$$T = T(U, z, \varepsilon) = \{z' \in \mathcal{E} \ : \pi(z') \in U \text{ and } p_j^{\pi(z')}(\sigma(\pi(z')) - z') < \varepsilon\},$$

where $U \subset X$ is a neighborhood of x, $\sigma \in \Gamma(\pi)$ with $\sigma(\pi(z)) = z$, $j \in \mathcal{J}$, and $\varepsilon > 0$. Especially, if $\mathcal{S}$ satisfies conditions C1)$-$C5), it is a subspace of $\Gamma_b(\pi)$, the space of bounded sections of the bundle $\pi : \mathcal{E} \to X$ (π, if there can be no confusion). Then the addition map from $\mathcal{E} \vee \mathcal{E}$ to $\mathcal{E}$, $(z, z') \mapsto z + z'$ is continuous, where $\mathcal{E} \vee \mathcal{E} = \{(z, z') \in \mathcal{E} \times \mathcal{E} : \pi(z) = \pi(z')\}$ is the fibered product of $\mathcal{E}$ with itself, and the multiplication map $C_b(X) \times \Gamma_b(\pi) \to \Gamma(\pi)$, $(f, \sigma) \mapsto f\sigma$ is jointly continuous when both $C_b(X)$ and $\Gamma_b(\pi)$ are given their sup-norms.

If each fiber E_x is a locally convex topological algebra and if $\mathcal{S}$ is an algebra, then π is a bundle of locally convex algebras. In this case, multiplication from $\mathcal{E} \vee \mathcal{E}$ to $\mathcal{E}$, $(z, z') \mapsto zz'$ is also continuous, and $\Gamma(\pi)$ is an algebra (evidently, commutative if and only if each fiber E_x is commutative).

If it happens that $\mathcal{J}$ is the singleton $\mathcal{J} = \{\|\bullet\|\}$, with $p^x_{\|\bullet\|}(\sigma(x)) = \|\sigma(x)\|$ for $\sigma \in \mathcal{S}$, we call $\pi : \mathcal{E} \to X$ a bundle of Banach spaces (algebras) (= Banach [algebra] bundle); in particular, $\Gamma_b(\pi)$ is a Banach space (Banach algebra) in the sup norm.

The intuitive notion is that, if $\pi : \mathcal{E} \to X$ is a bundle of topological vector spaces, and if $\sigma \in \Gamma(\pi)$, then we can think of $\sigma(x)$ as moving continuously through the various spaces E_x as x moves continuously through X.

In the current paper, as we did in [6], we will restrict ourselves to starting with the case of Banach bundles. However, as in the papers [4] and [5], it is possible that some of the following results have analogues that hold in the context of bundles of locally convex vector spaces (resp. locally convex algebras).

The reader may wish to consult e.g. [3], [4] and [5] for various examples of Banach (algebra) bundles.

2. Stone-Čech bundles and restriction bundles

Consider now a Banach bundle $\pi : \mathcal{E} \to X$. We know that $\Gamma(\pi)$, the space of sections of π, is a $C_b(X)$-module under the pointwise operations, and that $\Gamma_b(\pi)$, the space of bounded sections of the bundle, is a Banach $C_b(X)$-module, when both spaces are given their sup norms. Note that $\Gamma_b(\pi)$ can also be considered as a $C(\beta X)$-module, in the following easy fashion: There is an isometric algebra isomorphism $\widehat{\ }: C_b(X) \to C(\beta X)$ such that for each $f \in C_b(X)$ and $x \in X$ we have $\widehat{f}(\alpha(x)) = f(x)$, where, again, $\alpha : X \to \beta X$ is the canonical map. We think of $\widehat{f}$ as the unique continuous extension of f from X to βX. We then use the mapping $\widehat{\ }$ to determine the action of $C(\beta X)$ on $\Gamma_b(\pi)$; namely for $f \in C_b(x)$ and $\sigma \in \Gamma_b(\pi)$ we define $\widehat{f} * \sigma$ by

$$\left(\widehat{f} * \sigma\right)(x) = \widehat{f}(\alpha(x))\sigma(x) = (f\sigma)(x),$$

i.e. $\widehat{f} * \sigma = f\sigma$. We note further that $\Gamma_b(\pi)$ is in this fashion a $C(\beta X)$-locally convex module: that is, if $\sigma \in \Gamma_b(\pi)$, and if $f, g \in C_b(X)$, with $\widehat{f} \cdot \widehat{g} = \widehat{fg} = 0$ (if and only if $fg = 0$), then

$$\left\|\widehat{f} * \sigma + \widehat{g} * \sigma\right\| = \|f\sigma + g\sigma\| = \max\{\|f\sigma\|, \|g\sigma\|\} = \max\{\left\|\widehat{f} * \sigma\right\|, \|\widehat{g} * \sigma\|\}.$$

Thus, from [8], there exists a "canonical bundle" $\rho : \mathcal{F} = \dot{\bigcup} \{F_y : y \in \beta X\} \to \beta X$ and an isometric $C(\beta X)$-isomorphism $\widetilde{\ } : \Gamma_b(\pi) \to \Gamma(\rho)$ which satisfies the equation

$$\widetilde{\left(\widehat{f} * \sigma\right)} = \widetilde{f\sigma} = \widehat{f} \cdot \widetilde{\sigma}.$$

For $y \in \beta X$ we set $I_y = \{\widehat{f} \in C(\beta X) : \widehat{f}(y) = 0\}$. In particular, for $x \in X$, we have $f \in I_x = \{g \in C_b(X) : g(x) = 0\}$ if and only if $\widehat{f} \in I_{\alpha(x)}$. The fibers F_y ($y \in \beta X$) of $\mathcal{F}$ then take the form $F_y \simeq \dfrac{\Gamma_b(\pi)}{I_y * \Gamma_b(\pi)}$, where $I_y * \Gamma_b(\pi)$ is the closure in $\Gamma_b(\pi)$ of the span of sections of the form $\widehat{f} * \sigma = f\sigma$, over all $\widehat{f} \in C(\beta X)$ with $\widehat{f} \in I_y$ and $\sigma \in \Gamma_b(\pi)$. Then $\widetilde{\sigma}(y) = \sigma + I_y * \Gamma_b(\pi)$. In particular, if $x \in X$, we identify $E_x \simeq \dfrac{\Gamma_b(\pi)}{I_{\alpha(x)} * \Gamma_b(\pi)} = \dfrac{\Gamma_b(\pi)}{I_x\Gamma_b(\pi)}$, and as appropriate in context we will write $\sigma(x)$ either as an explicit value in E_x or as the corresponding coset $\sigma + I_x\Gamma_b(\pi)$.

Define a map $\eta : \mathcal{E} \to \mathcal{F}$ as follows: If $z \in E_x \subset \mathcal{E}$ (so that $x \in X$), choose $\sigma \in \Gamma_b(\pi)$ such that $\sigma(\pi(z)) = \sigma(x) = z$. Set

$$\eta(z) = \widetilde{\sigma}(\pi(z)) = \widetilde{\sigma}(x) = \widetilde{\sigma} + I_{\alpha(x)} * \Gamma_b(\pi) = \sigma + I_x\Gamma_b(\pi) = \sigma(x).$$

Then ϕ is well-defined on $\mathcal{E}$, and $\eta(E_x) = F_{\alpha(x)}$.

Note, moreover, that for each σ, $\tau \in \Gamma_b(\pi)$ and $f \in C_b(X)$ we have $\|\sigma + f\tau\| = \left\|\widetilde{\sigma + \widehat{f} * \tau}\right\|$; this holds in particular for $f \in I_x \subset \Gamma_b(\pi)$. Since for $x \in X$ and $f \in I_x$ we have $f(x) = 0 = \widehat{f}(\alpha(x))$, this yields

$$\|\eta(z)\| = \|\sigma + I_x\Gamma_b(\pi)\| = \left\|\widetilde{\sigma} + I_{\alpha(x)} * \Gamma_b(\pi)\right\| = \|\sigma(x)\| = \|z\|.$$

Hence, η is an isometric isomorphism on fibers, and we have $E_x \simeq F_{\alpha(x)}$. With this identification established, we write $\mathcal{E} \subset \mathcal{F}$, and transfer the topology of $\mathcal{E}$ over into $\mathcal{F}$. We claim that η is then a homeomorphism of $\mathcal{E}$ (with its bundle topology) onto a dense subset of $\mathcal{F}$ (with its bundle topology).

To see the density claim, let $y \in \beta X \setminus X$, and let $z \in F_y$. Since the bundle $\rho : \mathcal{F} \to \beta X$ is full (because βX is compact; see [3, Cor. 2.10]), and recalling that $\Gamma_b(\pi) \simeq \Gamma(\rho)$, there exists $\sigma \in \Gamma_b(\pi)$ such that $\widetilde{\sigma}(y) = z$. Then for a βX-neighborhood U of y, and $\varepsilon > 0$, the set

$$T = T(U, \widetilde{\sigma}, \varepsilon) = \{z' \in \mathcal{F} : \rho(z') \in U \text{ and } \|z' - \widetilde{\sigma}(\rho(z'))\| < \varepsilon\}$$

is a basic open neighborhood of z in $\mathcal{F}$. But X is dense in βX, so we can choose $x \in X$ with $x \in U$. We then have $\sigma(x) = \widetilde{\sigma}(x) = \widetilde{\sigma}(\rho(\widetilde{\sigma}(x)))$, so that $z' = \sigma(x) \in T$. That is, T contains an element of $\mathcal{E}$, so that $\mathcal{E}$ is dense in $\mathcal{F}$. We call $\rho : \mathcal{F} \to \beta X$ the Stone-Čech bundle associated with $\pi : \mathcal{E} \to X$. That η is injective and continuous is clear. The density argument itself shows that the topology of $\mathcal{F}$ relativized to $\mathcal{E}$ is the original bundle topology on $\mathcal{E}$, in that the tube T in $\mathcal{F}$ contains a tube in $\mathcal{E}$ around $\sigma(x) = \widetilde{\sigma}(x)$, where $x \in U \cap X$.

The following summarizes this discussion; it formalizes a remark stated without proof after Corollary 3.2 in [8].

Theorem 2.1. *Suppose that $\pi : \mathcal{E} \to X$ and $\Gamma_b(\pi)$ are as generally given, and let $\alpha : X \to \beta X$ be the homeomorphism which takes X onto a dense subset of βX. Then there is a bundle of Banach spaces $\rho : \mathcal{F} \to \beta X$, and a homeomorphism η*

of $\mathcal{E}$ onto a dense subset of $\mathcal{F}$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{E} & \stackrel{\eta}{\rightarrow} & \mathcal{F} \\ \pi \downarrow & & \downarrow \rho \\ X & \stackrel{\alpha}{\rightarrow} & \beta X \end{array}$$

In particular, for each $x \in X$, the fiber E_x over $x \in X$ in π is isometrically isomorphic as a Banach space to $F_{\alpha(x)}$, the fiber over $\alpha(x) = x$ in ρ. Using this identification of fibers, there is also a $C(\beta X)$-isometric isomorphism $\widetilde{\ }: \Gamma_b(\pi) \rightarrow \Gamma(\rho)$ such that for each $\sigma \in \Gamma_b(\pi)$, and $x \in X$, we have $\widetilde{\sigma}(\alpha(x)) = \sigma(x)$. Thus defined, $\widetilde{\sigma}$ can be thought of as the unique continuous extension of σ from X to βX.

Corollary 2.2. *Suppose $\pi : \mathcal{E} \rightarrow X$ is a bundle of commutative Banach algebras E_x. Then the maximal ideal space $\Delta(\Gamma_b(\pi))$ can be identified with the disjoint union $\dot{\bigcup} \{\Delta(F_y) : y \in \beta X\}$ of the maximal ideal spaces of the fibers F_y, where $\rho : \mathcal{F} \rightarrow \beta X$ is the Stone-Čech bundle associated with $\pi : \mathcal{E} \rightarrow X$. If $H \in \Delta(\Gamma_b(\pi))$, then the identification is given by $H \longleftrightarrow h \circ \phi_y$ for some $y \in \beta X$ and $h \in \Delta(F_y)$; we have $H(\sigma) = h(\widetilde{\sigma}(y))$.*

Proof. Note that $\Gamma_b(\pi)$ and $\Gamma(\rho)$ are isometrically isomorphic, where $\rho : \mathcal{F} \rightarrow \beta X$ is the Stone-Čech bundle for π, and apply [9, Corollary 5]. □

Remark 2.3. Note that if $\pi : \mathcal{E} \rightarrow X$ is a bundle of Banach algebras, so also is its Stone-Čech bundle $\rho : \mathcal{F} \rightarrow \beta X$. In particular, if $\sigma, \tau \in \Gamma_b(\pi)$ then $\widetilde{\sigma} \cdot \widetilde{\tau} = \widetilde{\sigma\tau} \in \Gamma(\rho)$; since multiplication is continuous from $\mathcal{E} \vee \mathcal{E}$ to $\mathcal{E}$, a standard density argument shows that it is also continuous from $\mathcal{F} \vee \mathcal{F}$ to $\mathcal{F}$.

Some examples may serve to illustrate the point.

Example 2.4.

(2.4.1) Let X be infinite, and let $\pi_1 : \mathcal{E}_1 = \dot{\bigcup}_{x \in X} \mathbb{K} = X \times \mathbb{K} \rightarrow X$ be the trivial bundle, where $X \times \mathbb{K}$ is given its product topology, and whose section space $\Gamma(\pi)$ can be identified with $C(X)$; of course, $\Gamma_b(\pi)$ can then be identified with $C_b(X)$. Then $\rho : \mathcal{F} = \dot{\bigcup}_{y \in \beta X} \mathbb{K} \rightarrow \beta X$ is the trivial bundle, and $\Gamma(\rho)$ can be identified with $C(\beta X)$; the unique extension property from $\Gamma_b(\pi)$ to $\Gamma(\rho)$ is then just the usual Stone-Čech extension of bounded continuous functions on X to continuous functions on βX.

(2.4.2) Let $X = \mathbb{N}$, and let $\pi_0 : \mathcal{E}_0 = \dot{\bigcup}_{x \in X} \mathbb{K} \rightarrow X$ be the "spiky" line bundle with fibers $\mathbb{K}$ (see [8, Section 3]), whose section space $\Gamma_b(\pi_0)$ can be identified with c_0, the closure in the sup-norm of the $\mathbb{K}$-valued functions on X with finite support. Then $\rho : \mathcal{F} \rightarrow \beta X$ has fibers $F_y = \mathbb{K}$ if $y \in X$, and $F_y = \{0\}$ if $y \in \beta X \setminus X$; the unique extension of $\sigma \in \Gamma_b(\pi_0)$ is defined by $\widetilde{\sigma}(y) = \sigma(y)$ if $y \in X$, and $\widetilde{\sigma}(y) = 0$ otherwise. To see this, note that we can regard c_0 as a subspace of l^∞, the space of bounded functions on X; and for $\sigma \in \Gamma_b(\pi_0)$, its image $\widetilde{\sigma}$ in $\Gamma(\rho)$ is "in" $C(\beta X)$. Since X is dense in βX, for $y \in \beta X \setminus X$ we choose a net $(x_\lambda) \subset X$ such that $x_\lambda \to y$ in βX. Then $\widetilde{\sigma}(x_\lambda) \to \widetilde{\sigma}(y)$ in $\mathbb{K}$, and since for any neighborhood of $y \in \beta X$ and each $\varepsilon > 0$ there are only finitely many x_λ such that $|\widetilde{\sigma}(x_\lambda)| > \varepsilon$, it follows that $\widetilde{\sigma}(y) = 0$.

(2.4.3) We can actually do somewhat better than in Example 2.4.2: given infinite X, let $\{E_x : x \in X\}$ be a collection of non-trivial spaces indexed by X, and consider the spiky bundle $\pi_0' : \mathcal{E}_0' = \dot{\bigcup}_{x\in X} E_x \to X$ with fibers E_x whose section space is the sup-norm closure of the choice functions $\sigma : X \to \mathcal{E}_0'$ with finite support. Let $x \in X$ and $y \in \beta X \setminus X$, choose $z \in E_x$. Let U and V be disjoint closed neighborhoods in βX of x and y, respectively. Let $\sigma \in \Gamma_b(\pi_0')$ be defined by $\sigma(x) = z$ and $\sigma(x') = 0$ if $x' \neq x$. Let $\widehat{a}, \widehat{b} \in C(\beta X)$ be supported on U and V, respectively, with $\widehat{a}(x) = \widehat{b}(y) = 1$, where $a, b \in C_b(X)$. Then $\alpha^{-1}(U) = U'$ is a closed neighborhood of x in X, and $\alpha^{-1}(V) = V'$ is a closed set in X, disjoint from U'. We have $\sigma = a\sigma = (a+b)\sigma$, so that $\widetilde{\sigma} = \widetilde{a\sigma} = \widetilde{a\sigma} + \widetilde{b\sigma}$, forcing $\widetilde{b\sigma} = 0$, and thus $\widetilde{b\sigma}(y) = 0$. Since $\widehat{b}(y) = 1$, we have $\widetilde{\sigma}(y) = 0$; and so since $y \in \beta X \setminus X$ was arbitrary, we have $\widetilde{\sigma}(y) = 0$ everywhere on $\beta X \setminus X$. But now, if $\tau \in \Gamma_b(\pi_0')$, we can write $\tau = \sum_n \sigma_n$ for some collection $(\sigma_n) \subset \Gamma_b(\pi_0')$ where each σ_n has singleton support in X, so that we have $\widetilde{\tau}(y) = 0 = \sum_n \widetilde{\sigma_n}(y)$ for all $y \in \beta X \setminus X$. Hence $E_y = \{0\}$ for each $y \in \beta X \setminus X$, and so the Stone-Čech bundle for the spiky bundle π_0' has fibers E_x for $x \in X \subset \beta X$, and $E_y = \{0\}$ on $\beta X \setminus X$.

Note that the original bundles π_0 and π_1 of Examples 2.4.1 and 2.4.2 are both continuously normed, that is $x \mapsto \|\sigma(x)\|$ is continuous for each section σ of the respective bundle. It is also the case in both examples (in Example 2.4.2 because each point of $\mathbb{N}$ is isolated in $\beta\mathbb{N}$), that each section $\widetilde{\sigma}$ is continuously normed in its Stone-Čech bundle. Is it always the case that if $\pi : \mathcal{E} \to X$ is continuously normed, then so is its Stone-Čech bundle?

Now, let $\pi : \mathcal{E} \to X$ be a Banach bundle, and let $D \subset X$ be C_b-embedded. We set $\mathcal{E}_D = \dot{\bigcup} \{E_x : x \in D\}$, and give $\mathcal{E}_D$ its relative topology from $\mathcal{E}$. Evidently, if $\sigma \in \Gamma(\pi)$, then $\sigma_D : D \to \mathcal{E}_D$ is a continuous choice function when both D and $\mathcal{E}_D$ have their relative topologies. Since $\pi : \mathcal{E} \to X$ is a full bundle, then $\{\sigma_D : \sigma \in \Gamma(\pi)\}$ is a full set of sections for $\mathcal{E}_D$, and the map $x \mapsto \|\sigma_D(x)\|$ is upper semicontinuous on D.

That there is a bundle $\pi_D : \mathcal{E}_D \to D$ such that $\{\sigma_D : \sigma \in \Gamma(\pi)\} = [\Gamma(\pi)]_D \subset \Gamma(\pi_D)$ follows from [3, Proposition 5. 11] and the following:

Lemma 2.5. *Let $D \subset X$ be C_b-embedded. Then $[\Gamma_b(\pi)]_D$ is a $C_b(D)$-module.*

Proof. Let $f \in C_b(D)$, and let f^* be a bounded extension of f to all of X. If $\sigma \in [\Gamma_b(\pi)]_D$, then there exists $\tau \in \Gamma_b(\pi)$ such that $\sigma = \tau_D$. We then have $f\sigma = (f^*)_D\tau_D = (f^*\tau)_D \in [\Gamma_b(\pi)]_D$. □

Call π_D the restriction bundle (to D) of $\pi : \mathcal{E} \to X$. In particular, if $K \subset X$ happens to be compact, we have $[\Gamma(\pi)]_K = \Gamma(\pi_K)$ ([3, Theorem. 5.8]). A similar situation obtains if $D \subset X$ is C_b-embedded.

Definition 2.6. Let $\pi : \mathcal{E} \to X$ be a Banach bundle, and let $D \subset X$. Say that D is $\Gamma_b(\pi)$-embedded if each bounded section of the restriction bundle $\pi_D : \mathcal{E}_D \to D$ can be extended to a section in $\Gamma_b(\pi)$; that is, if $\sigma \in \Gamma_b(\pi_D)$ then there exists $\tau \in \Gamma_b(\pi)$ such that $\sigma = \tau_D$. Say that $D \subset X$ is Γ_b-embedded if D is $\Gamma_b(\pi)$-embedded for each Banach bundle $\pi : \mathcal{E} \to X$.

Theorem 2.7. *Let $\pi : \mathcal{E} \to X$ be a Banach bundle and let $D \subset X$. Then D is C_b-embedded if and only if it is Γ_b-embedded.*

Proof. Let $D \subset X$ be C_b-embedded, and let $\pi : \mathcal{E} \to X$ be a Banach bundle, with Stone-Čech bundle $\rho : \mathcal{F} \to \beta X$. Let $\sigma \in \Gamma_b(\pi_D)$, and let $\widetilde{\sigma}$ be its unique extension in $\Gamma(\rho_{\beta D})$, where $\rho_{\beta D} : \mathcal{F}_{\beta D} \to \beta D$ is the Stone-Čech bundle for π_D; $\rho_{\beta D}$ is also the restriction of ρ to βD, because βD is compact in βX. Now, by the Tietze theorem for bundles (see [3, Corollary 4.5] or [7, Lemma 3.3]), each section $\rho \in \Gamma(\rho_{\beta D})$ can be extended in a norm-preserving fashion to $\tau^* \in \Gamma(\rho)$. Thus, if we start with $\sigma \in \Gamma_b(\pi_D)$, then σ can be extended uniquely to $\widetilde{\sigma} \in \Gamma(\rho_{\beta D})$, and $\widetilde{\sigma}$ itself can be extended to $\tau \in \Gamma(\rho)$. We then have

$$(\tau_X)_D = \tau_D = (\widetilde{\sigma})_D = \sigma,$$

so that τ_X extends σ to all of X.

For the reverse direction, simply note that to say $D \subset X$ is C_b-embedded is equivalent to saying that D is $\Gamma_b(\pi_1)$-embedded with respect to the trivial bundle $\pi_1 : \mathcal{E}_1 = \dot{\bigcup}_{x\in X} \mathbb{K} \to X$, whose section space can be identified with $C(X)$. □

Corollary 2.8. *Let $D \subset X$ be C_b-embedded. Then $[\Gamma_b(\pi)]_D = \Gamma_b(\pi_D)$.*

Proof. From the preceding discussion, we need only show that $\Gamma_b(\pi_D) \subset [\Gamma_b(\pi)]_D$. But if $\sigma \in \Gamma_b(\pi_D)$, then σ has a bounded extension τ to all of X; i.e. $\sigma = \tau_D \in [\Gamma_b(\pi)]_D$. □

Remark 2.9. If $D \subset X$ is C_b-embedded, and if $\sigma \in \Gamma_b(\pi_D)$, let τ and ω be extensions in $\Gamma_b(\pi)$ of σ. Both τ and ω have unique extensions τ^* and ω^* in $\Gamma(\rho)$, where $\rho : \mathcal{F} \to \beta D$ is the Stone-Čech bundle for π. Then $\tau^*_{\beta D}$ and $\omega^*_{\beta D}$ are extensions of τ and ω to βD, so that $\tau^*_{\beta D} = \omega^*_{\beta D}$. Thus, if τ and ω are any two bounded continuous extensions of σ to X, their continuous extensions τ^* and ω^* to βX must agree on βD.

3. The cover topology on $\Gamma_b(\pi, \mathcal{D})$

Now, let $\mathcal{D}$ be a cover of X, let $\pi : \mathcal{E} \to X$ be a Banach bundle and set

$$\Gamma_b(\pi, \mathcal{D}) = \{\sigma \in \Gamma(\pi) : \sigma \text{ is bounded on each } D \in \mathcal{D}\}.$$

Clearly we have

$$\Gamma_b(\pi) \subset \Gamma_b(\pi, \mathcal{D}) \subset \Gamma(\pi).$$

For $D \in \mathcal{D}$, define the seminorm p_D on $\Gamma_b(\pi, \mathcal{D})$ by $p_D(\sigma) = \sup_{x\in D} \|\sigma(x)\|$, and let $\mathfrak{t}_{\mathcal{D}}$ be the topology on $\Gamma_b(\pi, \mathcal{D})$ generated by the seminorms p_D. We call $\mathfrak{t}_{\mathcal{D}}$ the cover-topology on $\Gamma_b(\pi, \mathcal{D})$ determined by $\mathcal{D}$, or the $\mathcal{D}$-cover topology, and note that on $\Gamma_b(\pi)$ we have $\mathfrak{t}_p \prec \mathfrak{t}_{\beta,\mathcal{D}} \prec \mathfrak{t}_{\mathcal{D}} \prec \mathfrak{t}_u$, where $\mathfrak{t}_p$ and $\mathfrak{t}_u$ are the topologies of pointwise and uniform convergence, respectively, $\mathfrak{t}_{\beta,\mathcal{D}}$ is the cover-strict topology studied in [6], and where $\mathfrak{t}_1 \prec \mathfrak{t}_2$ means that convergence with respect to $\mathfrak{t}_2$ implies convergence with respect to $\mathfrak{t}_1$. In the $\mathcal{D}$-cover topology, subbasic neighborhoods of $\sigma \in \Gamma_b(\pi, \mathcal{D})$ are of the form

$$N = N(D, \sigma, \varepsilon) = \{\tau \in \Gamma_b(\pi, \mathcal{D}) : p_D(\sigma - \tau) = \sup_{x \in D} \|\sigma(x) - \tau(x)\| < \varepsilon\}$$

as $D \in \mathcal{D}$ and $\varepsilon > 0$ vary. Briefly, $\mathfrak{t}_{\mathcal{D}}$ is the topology of uniform convergence on the elements of $\mathcal{D}$.

Remark 3.1. If $\mathcal{D}$ is a compact cover, then $\mathfrak{t}_{\mathcal{D}} = \mathfrak{t}_{\beta,\mathcal{D}}$ on $\Gamma_b(\pi, \mathcal{D})$.

Proof. A $\mathfrak{t}_{\beta,\mathcal{D}}$-subbasic neighborhood around $\sigma \in \Gamma_b(\pi, \mathcal{D})$ is of the form

$$N'(D, \sigma, v, \varepsilon) = \{\tau \in \Gamma_b(\pi, \mathcal{D}) : \sup_{x \in D} v(x) \|\sigma(x) - \tau(x)\| < \varepsilon\},$$

where $D \in \mathcal{D}$, $\varepsilon > 0$, and v is a non-negative upper semicontinuous function (a weight) on X which disappears at infinity. (See [6] for this definition.) Since χ_D, the characteristic function of D, is such a weight, we have $N(D, \sigma, \varepsilon) = N'(D, \sigma, \chi_D, \varepsilon)$. On the other hand, it is easily checked that $N(D, \sigma, \frac{\varepsilon}{\|v\| + 1}) \subset N'(D, \sigma, v, \varepsilon)$. □

We now prove some completeness and density results for $\Gamma_b(\pi, \mathcal{D})$ in its $\mathfrak{t}_{\mathcal{D}}$-topology under certain conditions on $\mathcal{D}$. Again from [6] we have the following:

Definition 3.2. Let $\mathcal{D}$ be a cover of X. Say that $\mathcal{D}$ is sufficiently open if given $x \in X$ there exists $D \in \mathcal{D}$ and an X-neighborhood U of x such that $x \in U \subset D$. Say that $\mathcal{D}$ is sufficiently locally compact if given $x \in X$ there exists $D \in \mathcal{D}$ and an X-neighborhood U of x such that U is D-compact (and hence X-compact) and $x \in U \subset D$.

Noting that a sufficiently locally compact cover is sufficiently open, the following is the $\mathfrak{t}_{\mathcal{D}}$-topology version of Propositions 2.4 and 2.5 of [6].

Theorem 3.3. *Let $\mathcal{D}$ be a sufficiently open cover of X. Let $\pi : \mathcal{E} \to X$ be a Banach bundle. Then $\Gamma_b(\pi, \mathcal{D})$ is $\mathfrak{t}_{\mathcal{D}}$-complete.*

Proof. Let (σ_λ) be a $\mathfrak{t}_{\mathcal{D}}$-Cauchy net in $\Gamma_b(\pi, \mathcal{D})$, and let $\mathcal{D}$ be a sufficiently open cover of X. Given $\varepsilon > 0$ and $D \in \mathcal{D}$, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$ then $p_D(\sigma_\lambda - \sigma_{\lambda'}) = \sup_{y \in D} \|\sigma_\lambda(y) - \sigma_{\lambda'}(y)\| < \varepsilon$. Thus, (σ_λ) is uniformly Cauchy on D, and hence converges pointwise on D, and hence on all of X. Then this pointwise limit τ of (σ_λ) is also bounded on each D because (σ_λ) is uniformly Cauchy on D, and each σ_λ is itself bounded on D. We claim that τ is continuous on all of X.

If not, suppose that τ is discontinuous at $x \in X$, and choose $D \in \mathcal{D}$ and an X-open U such that $x \in U \subset D$. From the definition of continuity, and the definition of the topology in $\mathcal{E}$, there exist $\varepsilon > 0$, $\xi \in \Gamma(\pi)$ with $\xi(x) = \tau(x)$ and ξ bounded on a neighborhood $V \subset U$, and a net $(x_\mu) \subset V$ such that $x_\mu \to x$ but such that for no μ do we have $\tau(x_\mu) \in T = T(V, \xi(x), \varepsilon) = \{z' \in \mathcal{E} : \pi(z') \in V$ and $\|\tau(\pi(z')) - \xi(\pi(z'))\| < \varepsilon\}$. That is, $x_\mu \to x$, but $\|\tau(x_\mu)) - \xi(x_\mu)\| \geq \varepsilon$.

Since (σ_λ) is $\mathfrak{t}_\mathcal{D}$-Cauchy, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$ we have

$$\sup_{x_\mu \in U} \|\sigma_\lambda(x_\mu) - \sigma_{\lambda'}(x_\mu)\| \leq \sup_{y \in D} \|\sigma_\lambda(y) - \sigma_{\lambda'}(y)\| = p_D(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon/2.$$

Setting $\lambda' = \lambda_0$ and passing to the limit in λ, we thus have $\|\tau(x_\mu) - \sigma_{\lambda_0}(x_\mu)\| \leq \varepsilon/2$ for all $x_\mu \in V$. But $\sigma_{\lambda_0} \in \Gamma_b(\pi, \mathcal{D}) \subset \Gamma(\pi)$ and $x_\mu \to x$, so that for sufficiently large μ we have $\sigma_{\lambda_0}(x_\mu) \in T(V, \xi(x), \varepsilon/2)$; this is a contradiction, since $\|\tau(x_\mu) - \sigma_{\lambda_0}(x_\mu)\| \leq \varepsilon/2$ implies that $\|\sigma_{\lambda_0}(x_\mu) - \xi(x_\mu)\| > \varepsilon/2$, and hence that $\sigma_{\lambda_0}(x_\mu) \notin T(V, \xi(x), \varepsilon/2)$. □

Assume now that $\mathcal{D}$ is a cover of X by C_b-embedded sets.

Proposition 3.4. *Let $\pi : \mathcal{E} \to X$ be as usual, and let $\mathcal{D}$ be a cover of X by C_b-embedded sets which is closed under finite unions. Then $\Gamma_b(\pi)$ is $\mathfrak{t}_\mathcal{D}$-dense in $\Gamma_b(\pi, \mathcal{D})$. In particular, this holds if 1) $\mathcal{D}$ is a compact cover, or if 2) $\mathcal{D}$ is a closed cover and X is normal.*

Proof. If $D_1, \ldots, D_n \in \mathcal{D}$, then $D = D_1 \cup \ldots \cup D_n$ is also C_b-embedded. Hence, for $\sigma \in \Gamma(\pi, \mathcal{D})$, σ is bounded on D, and thus σ_D, the restriction of σ to D, has a bounded extension $\sigma^* \in \Gamma_b(\pi)$. For $\varepsilon > 0$, setting $N = \{\tau \in \Gamma_b(\pi) : \|\sigma_{D_k}(x) - \tau_{D_k}(x)\| < \varepsilon$ for each $x \in D_k$, $k = 1, \ldots n\}$, we see that N is a typical $\mathfrak{t}_\mathcal{D}$-neighborhood of σ, and evidently $\sigma^* \in N$. □

Corollary 3.5. *Let $\mathcal{D}$ be a cover of X by C_b-embedded sets which is closed under finite unions. Suppose also that $M \subset \Gamma_b(\pi, \mathcal{D})$ is a $C_b(X)$-submodule of $\Gamma_b(\pi)$ such that M_D is (sup-norm) dense in $\Gamma_b(\pi_D)$ for each $D \in \mathcal{D}$. Then M is $\mathfrak{t}_\mathcal{D}$-dense in $\Gamma_b(\pi, \mathcal{D})$.*

This naturally leads to the question of when M_D might be dense in $\Gamma_b(\pi_D)$ for each $D \in \mathcal{D}$, or, equivalently, when $\widetilde{M_D}$ is dense in $\Gamma(\rho_{\beta D})$ for each D. If $\mathcal{D}$ is a compact cover, then of course for each $D \in \mathcal{D}$ we have $D = \beta D$.

Corollary 3.6. *Suppose that $\mathcal{D}$ is a compact cover of X which is closed under finite unions. Let M be a $C_b(X)$-submodule of $\Gamma_b(\pi, \mathcal{D})$ such that M_x is dense in E_x for each $x \in X$. Then M is $\mathfrak{t}_\mathcal{D}$-dense in $\Gamma_b(\pi, \mathcal{D})$.*

Proof. Let $D \in \mathcal{D}$. Under the given circumstance, M_D is a $C_b(D)$-submodule of the (Banach space) $\Gamma_b(\pi_D)$. By the Stone-Weierstrass theorem for section spaces of Banach bundles over compact bases ([3, Corollary 4.3]), M_D is norm-dense in $\Gamma_b(\pi_D)$. □

The following example shows that we cannot in the above Corollary replace "compact cover closed under finite unions" with "C_b-embedded cover closed under finite unions" and obtain our density result.

Example 3.7. Consider the trivial bundle $\pi_1 : \mathcal{E} = \dot{\bigcup}_{x \in \mathbb{N}} \mathbb{K} \to \mathbb{N}$ whose section space $\Gamma(\pi_1)$ can be identified with the set of all $\mathbb{K}$-valued sequences, and let $\mathcal{D}$ = $\{\mathbb{N}\}$. Then $\Gamma_b(\pi_1) = C_b(\mathbb{N}) = \ell^\infty \simeq C(\beta\mathbb{N})$. We have $M = c_0 = c_0(\mathbb{N})$ is a closed $C_b(\mathbb{N})$-submodule of $\Gamma_b(\pi_1)$, and $M_x = \mathbb{K}$ for each $x \in \mathbb{N}$, but clearly M is not dense in $\Gamma_b(\pi_1)$, even though $\mathbb{N}$ is C_b-embedded in itself.

This then raises the following: Suppose that $\pi : \mathcal{E} \to X$ is a Banach bundle and that $M \subset \Gamma_b(\pi)$ is a $C_b(X)$-submodule such that M_x is dense in E_x for each $x \in X$. Are there conditions apart from compactness of X sufficient to imply that M is dense in $\Gamma_b(\pi)$?

The following positive result is rather special, but we state it for the record. We refer the reader to [3] for definitions.

Proposition 3.8. *Suppose that $\pi : \mathcal{E} \to X$ is a Banach bundle and that $\mathcal{D}$ is a cover of X by locally paracompact and C_b-embedded sets. Let $M \subset \Gamma_b(\pi)$ be a subspace such that M_D is fully additive and fiberwise dense in $\Gamma_b(\pi_D)$ for each $D \in \mathcal{D}$. Then M is $\mathfrak{t}_{\mathcal{D}}$-dense in $\Gamma_b(\pi)$.*

Proof. By [3, Theorem 4.2], M_D is dense in $\Gamma_b(\pi_D)$ for each D, and the result follows from above. □

4. A bundle construction, and its application to ideals and quotients in $\Gamma_b(\pi, \mathcal{D})$

In the paper [6] we obtained ideal and quotient results about $\Gamma_b(\pi, \mathcal{D})$ in its $\mathfrak{t}_{\beta,\mathcal{D}}$-topology directly from the definition of the topology, essentially because of the presence of compact sets X in that definition. In our current situation, we do not necessarily have compact sets, and it would be useful to have them. We guarantee their existence (but lose some generality) by letting $\mathcal{D}$ be a cover of X by C_b-embedded sets and then constructing a bundle $\rho' : \mathcal{F}' \to \beta X$ from the restriction bundles $\rho_{\beta D} : \mathcal{F}_{\beta D} \to \beta D$ as D ranges over $\mathcal{D}$.

Letting $\mathcal{D}$ be such a C_b-embedded cover, note first that if $D \in \mathcal{D}$, then with our construction of $\pi_D : \mathcal{E}_D \to D$, both D and $\mathcal{E}_D$ have their relative topologies, which we can transfer homeomorphically into βX and $\mathcal{F}$, respectively. We can then talk about the closures of D and $\mathcal{E}_D$ in βX and $\mathcal{F}$. In particular, if $\sigma \in \Gamma_b(\pi_D)$, and $\tilde{}D$ is the isometric map from $\Gamma_b(\pi_D)$ to $\Gamma(\rho_{\beta D})$ (where again for $D \in \mathcal{D}$, $\rho_{\beta D} : \mathcal{F}_{\beta D} = \dot{\bigcup} \{F_y : y \in \beta D\} \to \beta D$ is the Stone-Čech bundle for π), then $\mathcal{E}_D$ is dense in $\mathcal{F}_{\beta D}$, and $\mathcal{F}_{\beta D}$ carries its relative topology from $\mathcal{F}$. If $D_1, D_2 \in \mathcal{D}$, and $\sigma \in \Gamma_b(\pi, \mathcal{D})$, let $\widetilde{\sigma}^{D_1}$ and $\widetilde{\sigma}^{D_2}$ be the extensions of σ_{D_1} and σ_{D_2} to βD_1 and βD_2, respectively. Since these extensions are unique, for $y \in \beta D_1 \cap \beta D_2$, we have $\widetilde{\sigma}^{D_1}(y) = \widetilde{\sigma}^{D_2}(y)$.

Given the above data, consider the set $\mathcal{F}' = \dot{\bigcup}_{D \in \mathcal{D}} \{F'_y : y \in \beta D\}$, where $F'_y = F_y$ if $y \in \beta D$ for some $D \in \mathcal{D}$, and $F'_y = \{0\}$ if $y \in \beta X \setminus \bigcup \{\beta D : D \in \mathcal{D}\}$. Let $\rho' : \mathcal{F}' \to \beta X$ be the natural projection. For $\sigma \in \Gamma_b(\pi, \mathcal{D})$, define the selection $\psi(\sigma) : \beta X \to \mathcal{F}'$ by

$$[\psi(\sigma)](y) = \begin{cases} 0 \in F'_y, \text{ if } y \in \beta X \setminus \bigcup\{\beta D : D \in \mathcal{D}\} \\ \\ \widetilde{r_D(\sigma)}^D (y) \in F'_y = F_y, \text{ if } y \in \beta D. \end{cases},$$

where $r_D : \Gamma_b(\pi, \mathcal{D}) \to \Gamma_b(\pi, D)$ is the restriction map. (The preceding paragraph shows that F'_y and $[\psi(\sigma)](y)$ are well-defined, should $y \in \beta D_1 \cap \beta D_2$.)

Then $M = \psi(\Gamma_b(\pi,\mathcal{D}))$ is a collection of selections from βX to $\mathcal{F}'$; clearly, M is a $C_b(X) \simeq C(\beta X)$-module. It is also evident that $M_y = \{[\psi(\sigma)](y) : \sigma \in \Gamma_b(\pi,\mathcal{D})\} = F'_y$ for each $y \in \beta X$. For $\sigma \in \Gamma_b(\pi,\mathcal{D})$ and $y \in \beta X$, define

$$\widetilde{p^y_{\beta D}}(\psi(\sigma)) = \chi_{\beta D}(y) \left\| [\psi(\sigma)](y) \right\|,$$

where $\chi_{\beta D}$ is the characteristic function of $\beta D \subset \beta X$. Then $y \mapsto \widetilde{p^y_{\beta D}}([\psi(\sigma)](y))$ is upper semicontinuous on βX (because $\chi_{\beta D}$ is upper semicontinuous on βX, and because $\psi(\sigma)_{\beta D} = \widetilde{\sigma}_{\beta D}$ so that $y \mapsto \|[\psi(\sigma)](y)\|$ is upper semicontinuous on βD). The $\widetilde{p^y_{\beta D}}$ $(D \in \mathcal{D})$ generate the norm topology on F'_y. Now, for $\sigma \in \Gamma_b(\pi,\mathcal{D})$ and $D \in \mathcal{D}$, define seminorms $\widetilde{p_{\beta D}}$ on $\psi(\Gamma_b(\pi,\mathcal{D}))$ by

$$\begin{aligned}\widetilde{p_{\beta D}}(\psi(\sigma)) &= \sup_{y\in\beta X} \chi_{\beta D}(y) \left\|[\psi(\sigma)](y)\right\| = \sup_{y\in\beta D} \left\|[\psi(\sigma)](y)\right\| \\ &= \sup_{y\in\beta D} \widetilde{p^y_{\beta D}}(\psi(\sigma)) = \sup_{y\in\beta D} \|\sigma(y)\| \\ &= p_D(\sigma).\end{aligned}$$

We now observe that the selections $\psi(\Gamma_b(\pi,\mathcal{D}))$, together with the seminorms $\{\widetilde{p_{\beta D}} : D \in \mathcal{D}\}$ satisfy conditions C1) - C5) on βX, and therefore determine a bundle topology on $\rho' : \mathcal{F}' \to \beta X$ (so that ρ' is a bundle of topological vector spaces). The only condition which might be at issue for this assertion is C4), and to show this we need only note that the equality $p_D(\sigma) = \widetilde{p_{\beta D}}(\psi(\sigma))$ for each $D \in \mathcal{D}$ establishes a topological and algebraic isomorphism between $\Gamma_b(\pi,\mathcal{D})$ and $\psi(\Gamma_b(\pi,\mathcal{D}))$.

This combined with the compactness of βX and the fullness of the space of selections $\psi(\Gamma_b(\pi,\mathcal{D}))$ allow us to apply [3, Propositions 4.2 and 5.11] to obtain:

Theorem 4.1. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces, and $\mathcal{D}$ a cover of X by C_b-embedded sets. Then there is a bundle $\rho' : \mathcal{F}' \to \beta X$ of topological vector spaces such that $\Gamma_b(\pi,\mathcal{D})$ in its $\mathfrak{t}_{\mathcal{D}}$-topology is topologically and algebraically isomorphic to $\Gamma(\rho')$. The topology on $\Gamma(\rho')$ generated by the seminorms $\widetilde{\rho_{\beta D}}$ $(D \in \mathcal{D})$ is the $\mathcal{D}'$-cover topology defined by the compact cover*

$$\mathcal{D}' = \{\beta D : D \in \mathcal{D}\} \cup \{\{y\} : y \in \beta X \setminus \bigcup\{\beta D : D \in \mathcal{D}\}\}$$

of βX.

We now use this bundle to study ideals and quotients in $\Gamma_b(\pi,\mathcal{D})$ when π is a bundle of commutative Banach algebras. Our first goal is to specify $\Delta(\Gamma_b(\pi,\mathcal{D}))$, the space of continuous non-trivial $\mathbb{K}$-valued multiplicative homomorphisms on $\Gamma_b(\pi,\mathcal{D})$ with its $\mathcal{D}$-cover topology.

Remark 4.2. Suppose that $\pi : \mathcal{E} \to X$ is a bundle of commutative Banach algebras. If $D \in \mathcal{D}$, $y \in \beta D$, $\phi_y : \Gamma(\rho') \to F'_y$ is the evaluation map, and $h \in \Delta(F'_y)$, then $H = h \circ \phi_y \circ \psi \in \Delta(\Gamma_b(\pi,\mathcal{D}))$ when $\Gamma_b(\pi,\mathcal{D})$ is given its $\mathcal{D}$-cover topology.

Proof. Each of the maps making up H is a non-trivial algebra homomorphism. □

The next result then provides the converse to the above remark.

Theorem 4.3. *Let $\pi : \mathcal{E} \to X$ be a bundle of commutative Banach algebras, with Stone-Čech bundle $\rho : \mathcal{F} \to \beta X$, and let $\mathcal{D}$ be a cover of X by C_b-embedded sets. Let $\rho' : \mathcal{F}' \to \beta X$ be the bundle constructed above. Let $H \in \Delta(\Gamma_b(\pi, \mathcal{D}))$, where $\Gamma_b(\pi, \mathcal{D})$ is given its $\mathcal{D}$-cover topology. Then there exist $D \in \mathcal{D}$, $y \in \beta D$, and $h \in \Delta(F'_y)$ such that $H = h \circ \phi_y \circ \psi$. As a consequence, we may identify $\Delta(\Gamma_b(\pi, \mathcal{D}))$ as a point set with $\bigcup\{\Delta(F_y) : y \in \bigcup_{D\in\mathcal{D}} \beta D\}$. Analogously to Corollary 2, if $H \in \Delta(\Gamma_b(\pi, \mathcal{D}))$, then $H \longleftrightarrow h \circ \phi_y \circ \psi$, for some choice of $D \in \mathcal{D}$, $y \in \bigcup_{D\in\mathcal{D}} \beta D$, and $h \in \Delta(F_y)$.*

Proof. Give $\Gamma_b(\pi, \mathcal{D})$ its $\mathcal{D}$-cover topology, and let $H \in \Delta(\Gamma_b(\pi, \mathcal{D}))$. Then $H' = H \circ \psi^{-1} \in \Delta(\Gamma(\rho'))$, where ψ is the topological isomorphism from $\Gamma_b(\pi, \mathcal{D})$ to $\Gamma(\rho')$. Because $\mathcal{D}'$ is a compact cover of βX, from [9, Corolary 5], we have $H' = h \circ \phi_y$ for some $y \in \beta X$ and $h \in \Delta(F'_y)$; since $\Delta(F'_y) = \varnothing$ for $y \in \beta X \setminus \bigcup\{\beta D : D \in \mathcal{D}\}$, we may thus write $H' = h \circ \phi_y$ for some $y \in \bigcup\{\beta D : D \in \mathcal{D}\}$ and $h \in \Delta(F'_y) = \Delta(F_y)$. Hence

$$H' = h \circ \phi_y = H \circ \psi^{-1},$$

or

$$H' \circ \psi = H = h \circ \phi_y \circ \psi.$$

□

As a consequence of the isomorphism $\psi : \Gamma_b(\pi, \mathcal{D}) \to \Gamma(\rho')$, we obtain a complete description of the structure of the $\mathfrak{t}_{\mathcal{D}}$-closed ideals in $\Gamma_b(\pi, \mathcal{D})$ in certain situations. Namely,

Corollary 4.4. *Given a bundle of Banach algebras $\pi : \mathcal{E} \to X$ and a cover $\mathcal{D}$ of X by C_b-embedded sets, the algebra $\Gamma_b(\pi, \mathcal{D})$ satisfies spectral synthesis (i.e. each $\mathfrak{t}_{\mathcal{D}}$-closed ideal in $\Gamma_b(\pi, \mathcal{D})$ is the intersection of the $\mathfrak{t}_{\mathcal{D}}$-closed maximal ideals which contain it) if and only if each fiber $F'_y \subset \mathcal{F}'$ (where $\rho' : \mathcal{F}' \to \beta X$ is the bundle constructed above) satisfies spectral synthesis.*

Proof. Since βX is compact and $\mathcal{D}'$ is a compact cover of βX, the $\mathcal{D}'$-cover topology $\mathfrak{t}_{\mathcal{D}'}$ on $\Gamma(\rho')$ is also the $\mathcal{D}'$-strict cover topology $\mathfrak{t}_{\beta,\mathcal{D}'}$; it follows that $\Gamma(\rho')$ satisfies spectral synthesis if and only if each fiber F'_y satisfies spectral synthesis; see [6, Theorem 3.8]. Hence a $\mathfrak{t}_{\mathcal{D}'}$-closed ideal $I' \subset \Gamma(\rho')$ can be written as the intersection of the closed maximal ideals (in $\Gamma(\rho')$) which contain it. But then our result follows by the topological isomorphism of $\Gamma_b(\pi, \mathcal{D})$ and $\Gamma(\rho')$, and the consequent correspondence of closed ideals. □

As in [6], however, it is too much to hope that $\mathfrak{t}_{\mathcal{D}}$-closed ideals in $\Gamma_b(\pi, \mathcal{D})$ correspond on a one-to-one basis with closed sets in X; see the example in the cited paper. That such a result does hold in $C_b(X, \mathcal{D})$ (see [1, Theorem 3.4]) is a consequence of the fibers in the trivial bundle $\pi_1 : \mathcal{E}_1 = X \times \mathbb{K} \to X$ being one-dimensional over $\mathbb{K}$.

We now investigate the quotients of $\Gamma_b(\pi, \mathcal{D})$. Given our characterization above of $\Gamma_b(\pi, \mathcal{D})$ as the space of sections $\Gamma(\rho')$, where $\rho' : \mathcal{F}' \to \beta X$ is the bundle constructed above, this is not difficult.

We let $I \subset \Gamma_b(\pi, \mathcal{D})$ be a $\mathfrak{t}_{\mathcal{D}}$-closed ideal. Then $I' = \psi(I) \subset \Gamma(\rho')$ is a $\mathfrak{t}_{\mathcal{D}'}$-closed ideal. For $y \in \beta X$, let $I'_y = \{[\psi(\sigma)](y) : \sigma \in \Gamma_b(\pi, \mathcal{D})\}$. Then because βX is compact and $\mathfrak{t}_{\mathcal{D}'} = \mathfrak{t}_{\beta,\mathcal{D}'}$, we have $I' = \{\psi(\sigma) \in \Gamma(\rho') : [\psi(\sigma)](y) \in \overline{I'_y}$ for all $y \in \beta X$, as σ ranges over $\Gamma_b(\pi, \mathcal{D})\}$. For $\sigma \in \Gamma_b(\pi, \mathcal{D})$ and $y \in \beta X$, set $\widehat{[\psi(\sigma)]}(y) = \widehat{[\psi(\sigma)]} + \frac{F'_y}{\overline{I'_y}}$. Translating the language of [6, Prop. 4.1] to the current situation, we then have the following:

Theorem 4.5. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, $\mathcal{D}$ a C_b-embedded cover of X, and $I \subset \Gamma_b(\pi, \mathcal{D})$ a $\mathfrak{t}_{\mathcal{D}}$-closed ideal. Let $\rho' : \mathcal{F}' \to \beta X$ be the bundle constructed earlier, and let $I' \subset \Gamma(\rho')$ be the $\mathfrak{t}'_{\mathcal{D}}$-closed ideal corresponding to I. Then there is a bundle of Banach algebras $\rho_{I'} : \mathcal{G}' \to \beta X$, with fibers $G'_y = \frac{F'_y}{\overline{I'_y}}$ such that the map $\widehat{\ } : \frac{\Gamma(\rho')}{I'} \to \Gamma_b(\rho'_{I'}, \mathcal{D}')$ defined for $\sigma \in \Gamma_b(\pi, \mathcal{D})$ by*

$$[\psi(\sigma)] + I' \mapsto [y \mapsto \widehat{[\psi(\sigma)]}(y) = \widehat{[\psi(\sigma)]} + \frac{F'_y}{\overline{I'_y}}]$$

is an injective and continuous $C(\beta X)$-homomorphism whose image is dense in $\Gamma(\rho'_{I'}, \mathcal{D}')$.

Then, restricting to X, we have

Corollary 4.6. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, $\mathcal{D}$ a C_b-embedded cover of X, and $I \subset \Gamma_b(\pi, \mathcal{D})$ a $\mathfrak{t}_{\mathcal{D}}$-closed ideal. Then there is a bundle $\pi_I : \mathcal{G} \to X$ of quotients $G_x = \frac{E_x}{\overline{I_x}}$ such that the map $\widehat{\ } : \frac{\Gamma_b(\pi, \mathcal{D})}{I} \to \Gamma_b(\pi_I, \mathcal{D})$ defined for each $\sigma \in \Gamma_b(\pi, \mathcal{D})$ by*

$$\sigma + I \mapsto [x \mapsto \widehat{\sigma}(x) = \widehat{\sigma} + \frac{E_x}{I_x}]$$

is an injective and continuous $C_b(X)$-homomorphism whose image is dense in $\Gamma(\pi_I, \mathcal{D})$.

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