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The construction of $\boldsymbol{\pi}_0$ in Axiomatic Cohesion

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Abstract

We study a construction suggested by Lawvere to rationalize, within a generalization of Axiomatic Cohesion, the classical construction of $\boldsymbol{\pi}_0$ as the image of a natural map to a product of discrete spaces. A particular case of this construction produces, out of a local and hyperconnected geometric morphism $p : \mathcal{E} \to \mathcal{S}$, an idempotent monad $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ such that, for every X in $\mathcal{E}$, $\boldsymbol{\pi}_0 X = 1$ if and only if $(p^*\Omega)^! : (p^*\Omega)^1 \to (p^*\Omega)^X$ is an isomorphism. For instance, if $\mathcal{E}$ is the topological topos (over $\mathcal{S} = \mathbf{Set}$), the $\boldsymbol{\pi}_0$-algebras coincide with the totally separated (sequential) spaces. To illustrate the connection with classical topology we show that the $\boldsymbol{\pi}_0$-algebras in the category of compactly generated Hausdorff spaces are exactly the totally separated ones. Also, in order to relate the construction above with the axioms for Cohesion we prove that, for a local and hyperconnected $p : \mathcal{E} \to \mathcal{S}$, p is pre-cohesive if and only if $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed. In this case, $p_! = p_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$ and the category of $\boldsymbol{\pi}_0$-algebras coincides with the subcategory $p^* : \mathcal{S} \to \mathcal{E}$.

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1 Introduction

It has already been argued convincingly (at least, since [11], but see also [10, 12, 13]) that the constrast between cohesion and non-cohesion can be expressed by geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ such that the adjunction $p^* \dashv p_*$ extends to a string of adjoints

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

satisfying certain properties including that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful and that the left-most adjoint $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. Intuitively, $\mathcal{E}$ is a category of spaces, $\mathcal{S}$ is a category of sets, $p^! : \mathcal{S} \to \mathcal{E}$ may be identified with the subcategory of codiscrete spaces, p_* sends a space to the associated set of points, $p^* : \mathcal{S} \to \mathcal{E}$ may be identified with the subcategory of discrete spaces, and $p_!$ sends a space to the corresponding set of pieces or connected components.

On the other hand, as observed in the Author Commentary of [11], there are examples of (Grothendieck) toposes $\mathcal{E}$, such as Johnstone's topological topos [6] and Lawvere's

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bornological topos (sheaves for finite coverings of countable sets), that are intuitively categories 'of spaces' (as opposed to 'generalized spaces') and yet the canonical geometric morphisms $p : \mathcal{E} \to \mathbf{Set}$ fail to be essential, so the 'pieces' leftmost adjoint does not exist.

During an email discussion in March 2013 related to the observation above, Lawvere wrote the following:

> I believe that the attempt to rationalize the classical set-theoretic topology within the conception of cohesion and its generalizations does include the construction of the classical zero dimensional pizero as the image of a natural map to a product of discrete spaces [...]. Namely, any X maps to the following semi-double-dualization: the internal function space 2ˆX contains its own discrete part, and restricting along this defines a forgetful map upon applying again 2ˆ( ). Does the codiscrete inclusion help in explaining this?

Notice that for a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ with counit $\beta : C = p^* p_* \to Id$, and any X in $\mathcal{E}$, restricting along $\beta_{2^X} : C(2^X) \to 2^X$ and pre-composing with the transposition of evaluation determines a canonical map

$$X \xrightarrow{\overline{ev}} 2^{(2^X)} \xrightarrow{2^\beta} 2^{C(2^X)}$$

from X to a power of 2. The epi/mono factorization of $X \to 2^{C(2^X)}$ may be denoted by

$$X \longrightarrow \boldsymbol{\pi}_0 X \longrightarrow 2^{C(2^X)}$$

as suggested in the quotation. Under mild conditions, this construction underlies an idempotent monad $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ with unit the epic $X \to \boldsymbol{\pi}_0 X$ above.

Our purpose here is to explore the construction suggested by Lawvere, illustrate its relation to classical topology and relate it to the axioms for Cohesion. The role of the codiscrete inclusion will be clarified in our discussion of local hyperconnected geometric morphisms.

In Section 2 we study a mild generalization of the construction above by considering an arbitrary object L (in a regular category) in place of the coproduct $2 = 1 + 1$. This will allow us to establish some simple facts that are specialized in Section 3 where we introduce the 'pizero' monad associated to an adjunction $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ where $\mathcal{S}$ is a topos and $\mathcal{E}$ is regular and cartesian closed. We also show in this section that a local and hyperconnected geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive if and only if $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed. For this, the necessary leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ is built as $p_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$.

In Section 4 we study the monad $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ in the context of a locally small, extensive, regular and cartesian closed category. This is applied in Section 5 to the particular case of the category of compactly generated Hausdorff spaces. We show that such a space is indecomposable if and only if $\boldsymbol{\pi}_0 X = 1$. We also show that $\boldsymbol{\pi}_0$-algebras are exactly the totally separated spaces. An analogous analysis of the quasi-topos of subsequential spaces is done in Section 6. It is well-known that this quasi-topos is the category of separated objects for the double-negation topology in the topological topos. The role of separated objects is clarified in Section 7 where we study the monad $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ induced by a local and hyperconnected geometric morphism $\mathcal{E} \to \mathcal{S}$.

In Section 8 we compare the construction of $\boldsymbol{\pi}_0$ with the proofs that locally connected geometric morphisms are essential.

2 Semi-double-dualization

Let $\mathcal{S}$ be a category and let $\mathcal{E}$ be a cartesian closed regular category. Let $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ be an adjunction with counit β, and denote the induced comonad on $\mathcal{E}$ by $C : \mathcal{E} \to \mathcal{E}$. Fix also an object L in $\mathcal{E}$.

For any X in $\mathcal{E}$, restricting along $\beta_{L^X} : C(L^X) \to L^X$ and pre-composing with the transposition of evaluation determines a canonical map

$$X \xrightarrow{\overline{ev}} L^{(L^X)} \xrightarrow{L^\beta} L^{C(L^X)}$$

from X to $L^{C(L^X)}$. The regular-epi/mono factorization of $X \to L^{C(L^X)}$ will be denoted by

$$X \xrightarrow{\lambda_X} \Lambda X \xrightarrow{\psi_X} L^{C(L^X)}$$

so that ΛX is the image of a natural map to a power of L.

If $f : X \to Y$ is a map in $\mathcal{E}$ then there exists a unique map $\Lambda f : \Lambda X \to \Lambda Y$ such that the following diagram commutes

$$\begin{array}{ccccccc}
X & \xrightarrow{\lambda_X} & \Lambda X & \longrightarrow & L^{C(L^X)} & \quad C(L^X) & \quad L^X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Lambda f} & & \downarrow{\scriptstyle L^{C(L^f)}} & \quad \uparrow{\scriptstyle C(L^f)} & \quad \uparrow{\scriptstyle L^f} \\
Y & \xrightarrow[\lambda_Y]{} & \Lambda Y & \longrightarrow & L^{C(L^Y)} & \quad C(L^Y) & \quad L^Y
\end{array}$$

and it is easy to check that the above determines a functor $\Lambda : \mathcal{E} \to \mathcal{E}$ and natural transformations $\lambda : Id_{\mathcal{E}} \to \Lambda$ and $\quad : \Lambda \to L^{C(L^{(-)})}$.

Lemma 2.1. The functor $\Lambda : \mathcal{E} \to \mathcal{E}$ preserves the terminal object, epimorphisms and regular epimorphisms.

Proof. Since $\lambda_1 : 1 \to \Lambda 1$ is a regular epimorphism it follows that $\Lambda 1$ is terminal. It also follows easily, from naturality and (regular-)epiness of λ, that $\Lambda : \mathcal{E} \to \mathcal{E}$ preserves epis and regular epis. Q.E.D.

The next result is a sufficient condition for Λ to produce a monomorphism. (Recall that the *support* of an object X is the image of the unique morphism $X \to 1$ and that X is *well-supported* if its support is 1, i.e. if the unique $X \to 1$ is a regular epimorphism.)

Lemma 2.2. For any map $f : X \to Y$ in $\mathcal{E}$, if $p_*(L^f) : p_*(L^Y) \to p_*(L^X)$ is an epimorphism then $\Lambda f : \Lambda X \to \Lambda Y$ is a monomorphism. In particular, for any object X in $\mathcal{E}$, if $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is epic then ΛX is the support of X; so, in this case, X is well-supported if and only if $\Lambda X = 1$.

Proof. The following diagram commutes

$$\begin{array}{ccc} \Lambda X & \longrightarrow & L^{p^*(p_*(L^X))} \\ {\scriptstyle \Lambda f}\downarrow & & \downarrow {\scriptstyle L^{p^*(p_*(L^f))}} \\ \Lambda Y & \longrightarrow & L^{p^*(p_*(L^Y))} \end{array}$$

and the horizontal maps are monic. If $p_*(L^f) : p_*(L^Y) \to p_*(L^X)$ is epic, the right vertical map is monic, so the left vertical map is also monic.

For the special case of $! : X \to 1$, if $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is epic then $\Lambda ! : \Lambda X \to \Lambda 1$ is monic and $\Lambda 1 = 1$ by Lemma 2.1. So we have the following regular-epi/mono factorization

$$X \xrightarrow{\lambda} \Lambda X \xrightarrow{\Lambda !} \Lambda 1 = 1$$

of the unique $X \to 1$. Q.E.D.

Although we are assuming that $\mathcal{E}$ has finite limits, we don't have a good description of the kernel of $\lambda : X \to \Lambda X$ but, in relation to this issue, the following will be useful.

Lemma 2.3. For any $f, g : W \to X$ the following are equivalent:

1. The equality $\Lambda f = \Lambda g : \Lambda W \to \Lambda X$ holds.
2. The fork $W \overset{f}{\underset{g}{\rightrightarrows}} X \xrightarrow{\lambda} \Lambda X$ commutes.
3. The fork $W \times C(L^X) \overset{f\times\beta}{\underset{g\times\beta}{\rightrightarrows}} X \times L^X \xrightarrow{ev} L$ commutes.

   If W is terminal then the above are equivalent to:

4. The fork

$$p_*1 \times p_*(L^X) \overset{(p_*f)\times id}{\underset{(p_*g)\times id}{\rightrightarrows}} p_*X \times p_*(L^X) \xrightarrow{\cong} p_*(X \times L^X) \xrightarrow{p_*ev} p_*L$$

   commutes.

Proof. The first two items are equivalent because λ is a natural epimorphism. To prove that the second and third items are equivalent observe that, since $\psi : \Lambda X \to L^{C(L^X)}$ is monic, the diagram in the second item commutes if and only if the diagram below commutes

$$W \overset{f}{\underset{g}{\rightrightarrows}} X \xrightarrow{\overline{ev}} L^{(L^X)} \xrightarrow{L^\beta} L^{C(L^X)}$$

and, in turn, this holds if and only if the transpositions $W \times C(L^X) \to L$ are equal.

If W is terminal the fork in the third item commutes if and only if the next diagram

$$p^*(p_*(L^X)) \xrightarrow{\pi_1^{-1}} 1 \times p^*(p_*(L^X)) \underset{g\times\beta}{\overset{f\times\beta}{\rightrightarrows}} X \times L^X \xrightarrow{ev} L$$

commutes. By adjointness this is equivalent to the equality of the relevant transpositions $p_*(L^X) \to p_*L$. In other words, the fork in the third item commutes if and only if the top fork in the next diagram commutes

$$\begin{array}{ccccccccc}
p_*(L^X) & \xrightarrow{\alpha} & p_*(p^*(p_*(L^X))) & \xrightarrow{p_*\pi_1^{-1}} & p_*(1 \times C(L^X)) & \underset{p_*(g\times\beta)}{\overset{p_*(f\times\beta)}{\rightrightarrows}} & p_*(X \times L^X) & \xrightarrow{p_*ev} & p_*L \\
 & \searrow{\scriptstyle \pi_1^{-1}}\ \cong & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
 & & p_*1 \times p_*(L^X) & \xrightarrow{id\times\alpha} & p_*1 \times p_*(C(L^X)) & \underset{p_*g\times p_*\beta}{\overset{p_*f\times p_*\beta}{\rightrightarrows}} & p_*X \times p_*(L^X) & &
\end{array}$$

where α is the unit of $p^* \dashv p_*$. The rest of the diagram shows that the top fork is commutative if and only if the fork in the fourth item of the statement commutes. Q.E.D.

For any X in $\mathcal{E}$, the monomorphism $: \Lambda X \to L^{C(L^X)}$ may be transposed to a map $\Lambda X \times C(L^X) \to L$ and, again, transposed to a map that we denote by $\omega : C(L^X) \to L^{\Lambda X}$. Let us record this for future reference.

Lemma 2.4. The map $\omega : C(L^X) \to L^{\Lambda X}$ is the unique one that makes the following

$$\begin{array}{ccc}
C(L^X) \times \Lambda X & \xrightarrow{\omega\times id} & L^{\Lambda X} \times \Lambda X \\
\downarrow{\scriptstyle id\times\psi} & & \downarrow{\scriptstyle ev} \\
C(L^X) \times L^{C(L^X)} & \xrightarrow[ev]{} & L
\end{array}$$

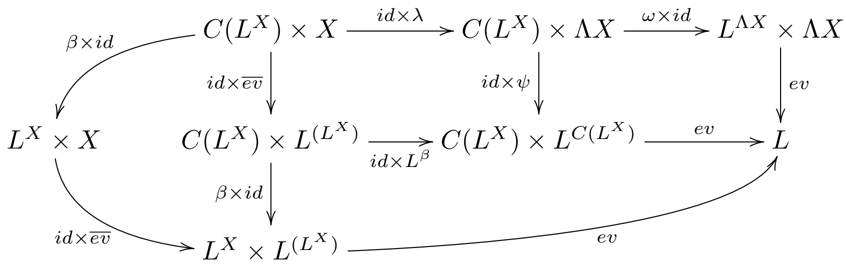
commute.

The next result is a convenient intermediate step for a later calculation.

Lemma 2.5. The following diagram commutes

$$\begin{array}{ccc}
C(L^X) \times X & \xrightarrow{\omega\times\lambda} & L^{\Lambda X} \times \Lambda X \\
\downarrow{\scriptstyle \beta\times id} & & \downarrow{\scriptstyle ev} \\
L^X \times X & \xrightarrow[ev]{} & L
\end{array}$$

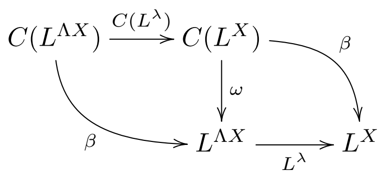
Proof. Just calculate using Lemma 2.4 and the definitions of λ and



together with the fact the $\overline{ev} : X \to L^{(L^X)}$ is the transposition of $ev : L^X \times X \to L$. Q.E.D.

We may now state the key fact about the map $\omega : C(L^X) \to L^{\Lambda X}$.

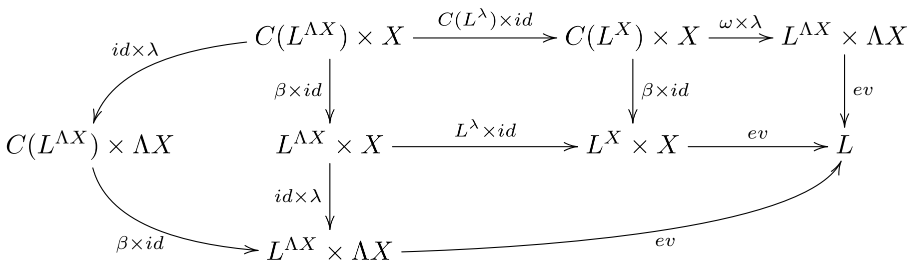
Lemma 2.6. The following triangles commute



Proof. We first consider the triangle in the left of the statement. Transpose the top-vertical composition and pre-compose with the epic $id \times \lambda : C(L^{\Lambda X}) \times X \to C(L^{\Lambda X}) \times \Lambda X$ to obtain the composite

$$C(L^{\Lambda X}) \times X \xrightarrow{id\times\lambda} C(L^{\Lambda X}) \times \Lambda X \xrightarrow{C(L^\lambda)\times id} C(L^X) \times \Lambda X \xrightarrow{\omega\times id} L^{\Lambda X} \times \Lambda X \xrightarrow{ev} L$$

which equals the top-right composite below, so we can calculate using Lemma 2.5

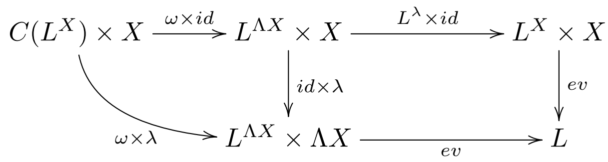


to arrive at the equality on the left below

$$ev(\omega \times id)(C(L^\lambda) \times id)(id \times \lambda) = ev(\beta \times id)(id \times \lambda) \quad ev(\omega \times id)(C(L^\lambda) \times id) = ev(\beta \times id)$$

and, since λ is epic, we may deduce the equality on the right above, which is the transposition of the left triangle in the statement.

Let us now consider the other triangle. Transpose and calculate

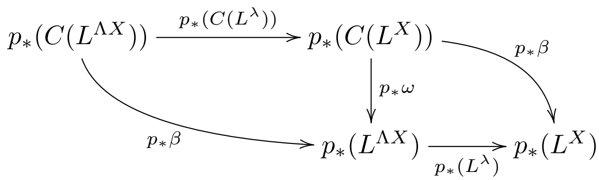


using Lemma 2.5 to complete the proof. Q.E.D.

We may now prove the main result of the section.

Proposition 2.7. If the left adjoint $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful then, for any X in $\mathcal{E}$, $p_*(L^\lambda) : p_*(L^{\Lambda X}) \to p_*(L^X)$ is an iso.

Proof. The assumption that p^* is fully faithful implies that $p_*\beta : p_*p^*p_* \to p_*$ is an iso. If we apply p_* to the diagram in Lemma 2.6 we obtain

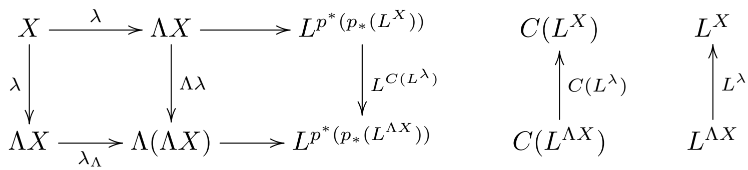


which implies (since $p_*\beta$ is an iso) that $p_*\omega$ is both a split mono and a split epi and so, an iso. Then the right triangle implies that $p_*(L^\lambda) : p_*(L^{\Lambda X}) \to p_*(L^X)$ also an iso. Q.E.D.

Notice that the conclusion of Proposition 2.7 is reminiscent of a property of finite-product preserving reflections. See, for example, Proposition A4.3.1 in [8].

Corollary 2.8. If the left adjoint $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful then the natural $\lambda : Id_{\mathcal{E}} \to \Lambda$ is the unit of an idempotent monad.

Proof. Lemma 2.1 implies that $\Lambda\lambda$ is a regular epimorphism. Also, the following diagram commutes



by definition of the functor $\Lambda : \mathcal{E} \to \mathcal{E}$. The right inner square, together with Proposition 2.7, implies that $\Lambda\lambda : \Lambda X \to \Lambda(\Lambda X)$ is also monic, so it is an isomorphism. The left inner square above implies that $\lambda_\Lambda = \Lambda\lambda : \Lambda X \to \Lambda(\Lambda X)$. So Λ is an idempotent monad with unit λ and multiplication λ_Λ^{-1}. Q.E.D.

Let us introduce a piece of terminology.

Definition 2.9. The monad described in Corollary 2.8 will be called the *semi-double-dualization monad* (determined by the fully faithful $p^* : \mathcal{S} \to \mathcal{E}$ and the object L).

The next result provides a characterization of the objects in $\mathcal{E}$ whose associated free Λ-algebra is terminal.

Corollary 2.10. Let $p^* : \mathcal{S} \to \mathcal{E}$ be full and faithful. For any X in $\mathcal{E}$, $\Lambda X = 1$ if and only if X is well-supported and $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is an isomorphism in $\mathcal{S}$.

Proof. If $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is an isomorphism in $\mathcal{S}$ then, by Lemma 2.2, X is well-supported if and only if $\Lambda X = 1$. So, to complete the proof, we need only show that if $\Lambda X = 1$ then $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is an isomorphism. Let us start with a more general observation. For every $f : X \to Y$ in $\mathcal{E}$ the following diagram commutes

$$\begin{array}{ccc} p_*(L^{\Lambda Y}) & \xrightarrow{p_*(L^{\lambda})} & p_*(L^Y) \\ {\scriptstyle p_*(L^{\Lambda f})}\downarrow & & \downarrow{\scriptstyle p_*(L^f)} \\ p_*(L^{\Lambda X}) & \xrightarrow[p_*(L^{\lambda})]{} & p_*(L^X) \end{array}$$

in $\mathcal{S}$. By Proposition 2.7 the horizontal maps are isomorphisms, so the vertical maps are isomorphic in the arrow category $\mathcal{E}^{\to}$. In particular, if we let $Y = 1$ and $f = \,! : X \to 1$ then $\Lambda f = \Lambda ! = \,! : \Lambda X \to \Lambda 1 = 1$ by Lemma 2.1 and we have the diagram below

$$\begin{array}{ccc} p_*(L^{1}) & \xrightarrow[=]{p_*(L^{\lambda})} & p_*(L^1) \\ {\scriptstyle p_*(L^{!})}\downarrow & & \downarrow{\scriptstyle p_*(L^!)} \\ p_*(L^{\Lambda X}) & \xrightarrow[p_*(L^{\lambda})]{} & p_*(L^X) \end{array}$$

showing that, if $\Lambda X = 1$ then the right vertical map is an isomorphism. Q.E.D.

Before discussing the particular case of semi-double-dualization monad that we are mainly interested in, it seems relevant to notice that $\mathcal{S}$ may be degenerate. In this case, for every X in $\mathcal{E}$, $CX = p^*(p_*X) = p^*0 = 0$. So $L^{C(L^X)} = L^0 = 1$ and ΛX coincides with the support of X. This is not the kind of example we have in mind. In order to positively exclude it we introduce the hypothesis that p^* preserves the terminal object. Notice that in this case, any point $1 \to p_*X$ is uniquely determined by a point $1 \to X$.

Corollary 2.11. Let $p^* : \mathcal{S} \to \mathcal{E}$ be fully faithful and preserve 1. Then, for any X in $\mathcal{E}$, the following are equivalent:

1. $\Lambda X = 1$.

2. $p_*(\Lambda X)$ is well-supported and $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is epic in $\mathcal{S}$.

3. X is well-supported and $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is an isomorphism in $\mathcal{S}$.

Proof. The first and last items are equivalent by Corollary 2.10. The first item easily implies that $p_*(\Lambda X) = 1$, so $p_*(\Lambda X)$ is well-supported, and the third item trivially implies that $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is epic. So, to complete the proof, it is enough to show that the second item implies the first.

If $p_*(L^!) : p_*(L^1) \to p_*(L^X)$ is epic in $\mathcal{S}$ then ΛX is subterminal in $\mathcal{E}$ by Lemma 2.2. Then $p_*(\Lambda X)$ is subterminal in $\mathcal{S}$. So, if it is also well-supported, then $p_*(\Lambda X) = 1$ and the point $1 \to p_*(\Lambda X)$ transposes to a point of ΛX. Altogether, ΛX is subterminal and has a point. Q.E.D.

3 The construction of $\boldsymbol{\pi}_0$

Fix a topos $\mathcal{S}$ with subobject classifier denoted by $\top : 1 \to \mathit{2}$. (To avoid a possible confusion we stress that we are not assuming that $\mathcal{S}$ is Boolean. The present notation is suggestive and allows us to avoid subindices such as that in $\Omega_{\mathcal{S}}$.)

Let $\mathcal{E}$ be a cartesian closed regular category and let $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ be an adjunction with fully faithful $p^* : \mathcal{S} \to \mathcal{E}$ that also preserves 1. As before, the counit of this adjunction will be denoted by β and the induced comonad by $C : \mathcal{E} \to \mathcal{E}$. The intuition is that $\mathcal{E}$ is a category of spaces and that $p^* : \mathcal{S} \to \mathcal{E}$ is the full subcategory of discrete spaces. For this reason we say that an object X in $\mathcal{E}$ is *discrete* if the counit $\beta_X : CX = p^*(p_* X) \to X$ is an isomorphism.

The discrete object $p^*\mathit{2}$ will be denoted by Υ.

We have all the ingredients necessary for the construction of a particular case of the monads introduced in Definition 2.9.

Definition 3.1. Let $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ be the semi-double-dualization monad on $\mathcal{E}$ determined by the adjunction $p^* \dashv p_*$ and the object $L = \Upsilon = p^*\mathit{2}$. The unit of this monad will be denoted by $\varsigma_X : X \to \pi_0 X$.

We have slightly changed the notation in order to emphasize this particular case of Corollary 2.8, so let us stress that the unit $\varsigma_X : X \to \pi_0 X$ (denoted by λ in the general case) makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{\overline{ev}} & \Upsilon^{(\Upsilon^X)} \\ \downarrow{\scriptstyle \varsigma} & & \downarrow{\scriptstyle \Upsilon^{\beta}} \\ \boldsymbol{\pi}_0 X & \xrightarrow[\psi_X]{} & \Upsilon^{C(\Upsilon^X)} \end{array}$$

and that the left and bottom maps form the regular-epi/mono factorization of the top-right composite.

Following the intuition that $p^* : \mathcal{S} \to \mathcal{E}$ is the full subcategory determined by the 'discrete spaces' among all spaces, $\Upsilon = p^*\mathit{2}$ is the discrete space of truth values, so that Υ^X is the space of clopens of X and $X \to \Upsilon^{C(\Upsilon^X)}$ sends $x \in X$ to the set of clopen neighborhoods of x. Hence, $\varsigma : X \to \boldsymbol{\pi}_0 X$ identifies x and x' if and only if they have the same clopen neighborhoods. So we may picture $\boldsymbol{\pi}_0 X$ as the space of 'pieces' or '(quasi-)components' of X. Let us also emphasize Corollary 2.11 by restating it using the new notation.

Corollary 3.2 (2.11 restated)**.** For any X in $\mathcal{E}$, the following are equivalent:

1. $\boldsymbol{\pi}_0 X = 1$
2. $p_*(\boldsymbol{\pi}_0 X)$ is well-supported and $p_*(\Upsilon^!) : p_*(\Upsilon^1) \to p_*(\Upsilon^X)$ is epic in $\mathcal{S}$.
3. X is well-supported and $p_*(\Upsilon^!) : p_*(\Upsilon^1) \to p_*(\Upsilon^X)$ is an isomorphism in $\mathcal{S}$.

Roughly speaking, $\boldsymbol{\pi}_0 X = 1$ if and only if X is well-supported and the set of clopens of X equals $p_*(\Upsilon) = p_*(p^* \mathit{2}) = \mathit{2}$. This intuition, supported by the corollary above, seems robust enough to justify the following piece of terminology.

Definition 3.3. An object X in $\mathcal{E}$ is *connected* if it satisfies the equivalent conditions of Corollary 3.2.

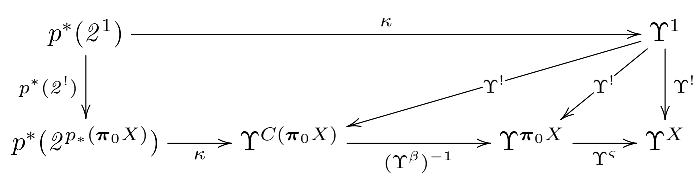
It is natural to wonder about other alternatives. For example, what about $p_*(\boldsymbol{\pi}_0 X) = 1$ or $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ being an isomorphism? Certainly, X connected implies $p_*(\boldsymbol{\pi}_0 X) = 1$ and we will have more to say about this but, for the moment, consider the following.

Proposition 3.4. Assume that the fully faithful $p^* : \mathcal{S} \to \mathcal{E}$ preserves finite products and that subterminals in $\mathcal{E}$ are discrete. For every X in $\mathcal{E}$, if $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism then X is connected.

Proof. If $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism then so is $p_*(\Upsilon^!) : p_*(\Upsilon^1) \to p_*(\Upsilon^X)$. Therefore, we need only prove that $p_*(\boldsymbol{\pi}_0 X)$ is well-supported.

Lemma 2.2 implies that $\boldsymbol{\pi}_0 X$ is subterminal so, by hypothesis, $\boldsymbol{\pi}_0 X$ is discrete. That is $\beta : C(\boldsymbol{\pi}_0 X) \to \boldsymbol{\pi}_0 X$ is an isomorphism. Also by hypothesis, $p^* : \mathcal{S} \to \mathcal{E}$ preserves finite products. Let us denote the canonical comparison map by $\kappa_{A,B} = \kappa : p^*(B^A) \to (p^* B)^{p^* A}$.

To prove that $p_*(\boldsymbol{\pi}_0 X)$ is well-supported notice that since $\beta : C(\boldsymbol{\pi}_0 X) \to \boldsymbol{\pi}_0 X$ is an isomorphism then so is $\Upsilon^\beta : \Upsilon^{\boldsymbol{\pi}_0 X} \to \Upsilon^{C(\boldsymbol{\pi}_0 X)}$. Hence, we can consider the following diagram



which commutes because the inner polygons do. The top map is an iso. Hence, if the right vertical map is monic, then so is the left vertical map and, since $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful, $\mathit{2}^! : \mathit{2}^1 \to \mathit{2}^{p_*(\boldsymbol{\pi}_0 X)}$ is monic in $\mathcal{S}$. Now recall that for any $f : A \to B$ in $\mathcal{S}$, $\mathit{2}^f : \mathit{2}^B \to \mathit{2}^A$ is monic if and only if f is epic (see, for example, Exercise IV.6 in [15]). In particular, if the left vertical map is monic then $! : p_*(\boldsymbol{\pi}_0 X) \to 1$ is epic. Q.E.D.

We will later see that, under further hypotheses, X is connected if and only if the map $\Upsilon^{!} : \Upsilon^{1} \to \Upsilon^{X}$ is an isomorphism. For the moment let us turn our attention to $\boldsymbol{\pi}_0$-algebras. Notice that an object X is a $\boldsymbol{\pi}_0$-algebra if and only if $\varsigma : X \to \pi_0 X$ is an isomorphism if and only if the canonical composite $X \to \Upsilon^{C(\Upsilon^X)}$ is monic.

Following the intuition about points and pieces, an object X such that $\varsigma : X \to \boldsymbol{\pi}_0 X$ is an isomorphism (i.e. a $\boldsymbol{\pi}_0$-algebra) may be thought of as being 'totally separated'. This idea suggests that discrete spaces are $\boldsymbol{\pi}_0$-algebras. We prove (in Proposition 3.6 below) that this holds under reasonable hypotheses.

Lemma 3.5. For any object A in $\mathcal{S}$, the transposition $\overline{ev} : A \to \mathit{2}^{(\mathit{2}^A)}$ of the evaluation $ev : A \times \mathit{2}^A \to \mathit{2}$ is monic.

Proof. The composite

$$A \xrightarrow{\overline{ev}} \mathit{2}^{(\mathit{2}^A)} \xrightarrow{\mathit{2}^{\{\}}} \mathit{2}^A$$

equals the monic 'singleton' map $\{\} : A \to \mathit{2}^A$. (See Lemma IV.1.1 in [15].) Q.E.D.

If $p^* : \mathcal{S} \to \mathcal{E}$ is assumed to preserve finite products, there is a natural canonical iso $\delta = \delta_{A,Y} : (p_* Y)^A \to p_*(Y^{p^*A})$. Indeed, it is the unique one such that the following diagram commutes

$$\begin{array}{ccccc}
p^*((p_*Y)^A) \times p^*A & \xrightarrow{(p^*\delta)\times id} & p^*(p_*(Y^{p^*A})) \times p^*A & \xrightarrow{\beta\times id} & Y^{p^*A} \times p^*A \\
{\scriptstyle \langle p^*\pi_0, p^*\pi_1\rangle}\uparrow\cong & & & & \downarrow{\scriptstyle ev} \\
p^*((p_*Y)^A \times A) & \xrightarrow[p^*ev]{} & p^*(p_*A) & \xrightarrow[\beta]{} & A
\end{array}$$

where the left vertical iso is given by the assumption that p^* preserves finite products.

Proposition 3.6. If $p^* : \mathcal{S} \to \mathcal{E}$ preserves finite limits and β is monic then, for every A in $\mathcal{S}$, the unit $\varsigma : p^*A \to \boldsymbol{\pi}_0(p^*A)$ is an isomorphism.

Proof. It is enough to show that the canonical $p^*A \xrightarrow{\overline{ev}} \Upsilon^{(\Upsilon^{p^*A})} \xrightarrow{\Upsilon^{\beta}} \Upsilon^{p^*(p_*(\Upsilon^{p^*A}))}$ is monic. Using the diagram before the statement, it is straightforward to check that the following diagram commutes

$$\begin{array}{ccccccccc}
p^*A & \xrightarrow{\overline{ev}} & \Upsilon^{(\Upsilon^{p^*A})} & \xrightarrow{\Upsilon^{\beta}} & \Upsilon^{p^*(p_*(\Upsilon^{p^*A}))} & \xrightarrow[\cong]{\Upsilon^{p^*\delta}} & \Upsilon^{p^*((p_*\Upsilon)^A)} & \xrightarrow{=} & \Upsilon^{p^*((p_*(p^*\mathit{2}))^A)} \\
{\scriptstyle p^*\overline{ev}}\downarrow & & & & & & & & \cong\downarrow{\scriptstyle \Upsilon^{p^*(\alpha^A)}} \\
p^*(\mathit{2}^{(\mathit{2}^A)}) & & \xrightarrow[p^*(\alpha^{(\mathit{2}^A)})]{} & & p^*((p_*\Upsilon)^{(\mathit{2}^A)}) & \xrightarrow[p^*\delta]{} & p^*(p_*(\Upsilon^{p^*(\mathit{2}^A)})) & \xrightarrow[\beta]{} & \Upsilon^{p^*(\mathit{2}^A)}
\end{array}$$

where α is the (iso) unit of $p^* \dashv p_*$. Since p^* preserves monos, the left vertical map $p^*\overline{ev}$ is monic by Lemma 3.5. Since β is monic by hypothesis, the left-bottom composite is monic. Q.E.D.

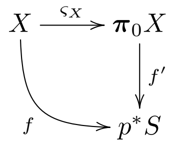
In other words, discrete objects are $\boldsymbol{\pi}_0$-algebras. (The referee noticed that the assumption that β is monic is equivalent to saying that the coreflective subcategory $p^* : \mathcal{S} \to \mathcal{E}$ is closed under (regular) quotients, and that this immediately yields the fact that $\boldsymbol{\pi}_0(p^* A)$ is discrete.) The next result shows that the composite $p_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$ is very close to being a left adjoint to $p^* : \mathcal{S} \to \mathcal{E}$.

Corollary 3.7. Assume that $p^* : \mathcal{S} \to \mathcal{E}$ preserves finite limits and that β is mono. For every X in $\mathcal{E}$, if $\boldsymbol{\pi}_0 X$ is discrete then the composite

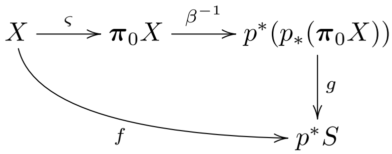
$$X \xrightarrow{\varsigma} \boldsymbol{\pi}_0 X \xrightarrow{\beta^{-1}} p^*(p_*(\boldsymbol{\pi}_0 X))$$

is universal from X to $p^* : \mathcal{S} \to \mathcal{E}$.

Proof. Let $f : X \to p^* S$ be a map in $\mathcal{E}$. Since $\boldsymbol{\pi}_0 X$ is the free $\boldsymbol{\pi}_0$-algebra determined by X and $p^* S$ is a $\boldsymbol{\pi}_0$-algebra by Proposition 3.6, there is a unique map $f' : \boldsymbol{\pi}_0 X \to p^* S$ such that the triangle below commutes.



As $\boldsymbol{\pi}_0 X$ is discrete by hypothesis, there exists a unique $g : p^*(p_*(\boldsymbol{\pi}_0 X)) \to p^* S$ making the following triangle



commute. Since $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful, this $g : p^*(p_*(\boldsymbol{\pi}_0 X)) \to p^* S$ equals $p^* \overline{f}$ for a unique $\overline{f} : p_*(\boldsymbol{\pi}_0 X) \to S$ in $\mathcal{S}$. Q.E.D.

Assume from now that $\mathcal{E}$ is a topos, just as $\mathcal{S}$.

Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *hyperconnected* if and only if $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful and the counit $\beta : p^* p_* \to Id_{\mathcal{E}}$ is monic. In this case, $p^* : \mathcal{S} \to \mathcal{E}$ is closed under subobjects in $\mathcal{E}$.

Corollary 3.8. Let $p : \mathcal{E} \to \mathcal{S}$ be a hyperconnected geometric morphism. If $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed then the following hold:

1. An object is discrete if and only if it is a $\boldsymbol{\pi}_0$-algebra.
2. The functor p^* has a left adjoint $p_! = p_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$.
3. X is connected if and only if $p_! X = 1$.

Proof. If an object is discrete then it is a $\boldsymbol{\pi}_0$-algebra by Proposition 3.6. For the converse consider a $\boldsymbol{\pi}_0$-algebra $\boldsymbol{\pi}_0 X$ where X is some object in $\mathcal{E}$. The very definition of $\boldsymbol{\pi}_0 X$ shows that it is a subobject of an object of the form $(p^* 2)^{p^* A}$ for some A in $\mathcal{S}$. Since $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed, $\boldsymbol{\pi}_0 X$ is a subobject of a discrete object. Since p is hyperconnected, $\boldsymbol{\pi}_0 X$ is discrete.

Corollary 3.7 implies that $p_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$ is left adjoint to p^*.

If X is connected then, of course, $p_! X = p_*(\boldsymbol{\pi}_0 X) = 1$. On the other hand, if $p_! X = 1$ then $p^*(p_! X) = p^*(p_*(\boldsymbol{\pi}_0 X)) = 1$. Since $\beta : p^*(p_*(\boldsymbol{\pi}_0 X)) \to \boldsymbol{\pi}_0 X$ is an iso because $\boldsymbol{\pi}_0 X$ is discrete, $\boldsymbol{\pi}_0 X = 1$. Q.E.D.

The referee observed that the functor $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ need not preserve finite products and provided the following example. For a non-trivial group G, the canonical geometric morphism $p : [G, \mathbf{Set}] \to \mathbf{Set}$ is connected atomic, and so hyperconnected and locally connected (and hence essential, see C3.5 in [8]). The leftmost adjoint $\boldsymbol{\pi}_0 = p_! : [G, \mathbf{Set}] \to \mathbf{Set}$ sends a G-set to its set of orbits, so $\boldsymbol{\pi}_0 G = 1$ but $\boldsymbol{\pi}_0(G \times G) \neq 1$. In contrast, consider the following.

Corollary 3.9. Let $p : \mathcal{E} \to \mathcal{S}$ be a hyperconnected geometric morphism. Then, the fully faithful $p^* : \mathcal{S} \to \mathcal{E}$ is an exponential ideal if and only if it has a finite-product preserving left adjoint.

Proof. Part of this result is well-known. Indeed, if the left adjoint $p_! : \mathcal{E} \to \mathcal{S}$ exists then, $p^* : \mathcal{S} \to \mathcal{E}$ is an exponential ideal if and only if $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. (See, for instance, Proposition A4.3.1 in [8]). So all we need to prove is that: if $p^* : \mathcal{S} \to \mathcal{E}$ is an exponential ideal then it has a left adjoint. This follows from Corollary 3.8, because if the full subcategory $p^* : \mathcal{S} \to \mathcal{E}$ is an exponential ideal then the inclusion is cartesian closed. Q.E.D.

Let us recall two concepts from standard topos theory. A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *essential* if $p^* : \mathcal{S} \to \mathcal{E}$ has a left adjoint $p_! : \mathcal{E} \to \mathcal{S}$. The morphism p is *local* if $p_* : \mathcal{E} \to \mathcal{S}$ has a fully faithful right adjoint $p^! : \mathcal{S} \to \mathcal{E}$. Recall also [18, 17, 13, 16] that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to a string of adjoints

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \;\dashv\; \uparrow p^* \;\dashv\; \downarrow p_* \;\dashv\; \uparrow p^! \\ \mathcal{S} \end{array}$$

where $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, the canonical natural transformation $p_* \to p_!$ is epic (Nullstellensatz) and $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. Notice that pre-cohesive geometric morphisms are local and essential by definition.

A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *locally connected* if p^* has an $\mathcal{S}$-indexed left adjoint. In particular, locally connected geometric morphisms are essential. Less trivially, it holds that if p is locally connected then p^* is cartesian closed. (See C3.3 in [8] and also Section 8 below.)

Although not stated in this way, one of the main results in [9] is a characterization of the locally connected geometric morphisms that are pre-cohesive. Indeed, a locally connected geometric morphism is pre-cohesive if and only if it is local and hyperconnected. Part of this result does not need local connectedness. For instance, Lemma 3.1(i) shows that if $p : \mathcal{E} \to \mathcal{S}$ is local then, p is hyperconnected if and only if the canonical natural transformation $\varphi : p^* \to p^!$ is monic; which, as stated in [12], is equivalent to $\theta : p_* \to p_!$ being epic when $p_!$ exists. See also [13]. In particular, pre-cohesive geometric morphisms are hyperconnected. Another consequence of the work in [9] that does not need local connectedness is the following.

Proposition 3.10. Let $p : \mathcal{E} \to \mathcal{S}$ be local, hyperconnected and essential. If $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed then $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

Proof. Since p^* is full and faithful, the unit $1 \to p_!(p^*1) = p_!1$ is an iso and so $p_!$ preserves the terminal object. Proposition 2.7 in [9] proves that $p_!$ preserves binary products. (We stress that Proposition 2.7 in [9] does not need local connectedness. It only requires the string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$, the Nullstellensatz, and cartesian closure of p^*.) Q.E.D.

As we have already mentioned, pre-cohesive geometric morphisms are local and hyperconnected. On the other hand, I still do not know if pre-cohesive implies locally connected so it still makes sense to characterize pre-cohesive geometric morphisms without assuming local connectedness. The construction of $\boldsymbol{\pi}_0$ suggests a characterization among local hyperconnected ones.

Corollary 3.11. Let $p : \mathcal{E} \to \mathcal{S}$ be a local and hyperconnected geometric morphism. Then $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive if and only if $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed.

Proof. If p is pre-cohesive then $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed by Corollary A1.5.9 in [8]. On the other hand, if p is hyperconnected and local, and $p^* : \mathcal{S} \to \mathcal{E}$ is cartesian closed then Corollary 3.8 implies that p^* has a left adjoint $p_! : \mathcal{E} \to \mathcal{S}$. So we have a string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ with fully faithful $p^! : \mathcal{S} \to \mathcal{E}$ (because p is local) and satisfying the Nullstellensatz (because p is hyperconnected). Proposition 3.10 is then applicable so that $p_!$ preserves finite products, completing the proof that p is pre-cohesive. Q.E.D.

Further information about the connection between local-connectedness and pre-cohesion may be found in [13].

4 Total separation and indecomposability

The notion of totally separated topological space admits the following elementary generalization. Let $\mathcal{E}$ be an extensive category with terminal object. As usual, let $2 = 1 + 1$.

Definition 4.1. Let X be an object in $\mathcal{E}$. A pair of points $x, x' : 1 \to X$ in $\mathcal{E}$ is said to be *inseparable* if, for every $f : X \to 2$, $fx = fx'$. The object X is *totally separated* if, for every inseparable $x, x' : 1 \to X$, $x = x'$. That is, inseparable points are equal.

For example, let **Top** be the category of topological spaces. An object X in **Top** is totally separated in the sense of Definition 4.1 if and only if it is totally separated in the classical sense, that is: whenever x and x' are distinct points of X, there is a clopen subset of X containing x but not y. (See, e.g., II.4.1 in [7].) We want to compare totally separated objects (in the sense of Definition 4.1) with $\boldsymbol{\pi}_0$-algebras.

1. Let $\mathcal{E}$ be a locally small, extensive, regular, and cartesian closed category.
2. Let $p_* : \mathcal{E} \to \mathbf{Set}$ have a fully faithful left adjoint $p^* : \mathbf{Set} \to \mathcal{E}$ that preserves terminal object. (It follows that $p_* = \mathcal{E}(1, \_) : \mathcal{E} \to \mathbf{Set}$ and $\Upsilon = p^*2 = 2$ in $\mathcal{E}$.)

Let $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ be the associated idempotent monad of Definition 3.1. The unit of $\boldsymbol{\pi}_0$ is denoted by $\varsigma_X : X \to \boldsymbol{\pi}_0 X$ for X in $\mathcal{E}$. The main thing to realize in this section is that inseparability is detected by $\boldsymbol{\pi}_0$.

Lemma 4.2. For every $x, x' : 1 \to X$ in $\mathcal{E}$, $\varsigma x = \varsigma x' : 1 \to \boldsymbol{\pi}_0 X$ if and only if, x and x' are inseparable.

Proof. By Lemma 2.3, $\varsigma x = \varsigma x'$ if and only if the top-fork in the diagram below

$$
\begin{array}{ccccccc}
p_*1 \times p_*(2^X) & \underset{(p_*x')\times id}{\overset{(p_*x)\times id}{\rightrightarrows}} & p_*X \times p_*(2^X) & \xrightarrow{\cong} & p_*(X \times 2^X) & \xrightarrow{p_*ev} & p_*2 \\
\cong\downarrow & & \downarrow\cong & & & & \downarrow = \\
\mathcal{E}(1,1) \times \mathcal{E}(X,2) & \underset{\mathcal{E}(1,x')\times id}{\overset{\mathcal{E}(1,x)\times id}{\rightrightarrows}} & \mathcal{E}(1,X) \times \mathcal{E}(X,2) & & \xrightarrow[\circ]{\qquad\qquad} & & \mathcal{E}(1,2)
\end{array}
$$

commutes. The rest of the diagram commutes because the composite on the left below

$$
p_*X \times p_*(2^X) \xrightarrow{\cong} p_*(X \times 2^X) \xrightarrow{p_*ev} p_*2 \qquad\qquad \mathcal{E}(1,X) \times \mathcal{E}(X,2) \xrightarrow{\circ} \mathcal{E}(1,2)
$$

is canonically isomorphic to the composition of maps indicated on the right above. So the top fork (in the first diagram) commutes if and only if the bottom fork does. In turn, the bottom fork commutes if and only if for every $f : X \to 2$, $fx = fx'$. That is, if and only if x and x' are inseparable. Q.E.D.

We may characterize total separation in terms of $\boldsymbol{\pi}_0$ as follows.

Proposition 4.3. For every object X in $\mathcal{E}$, X is totally separated if and only if the map $p_*\varsigma_X : p_*X \to p_*(\boldsymbol{\pi}_0 X)$ is monic (in $\mathbf{Set}$).

Proof. The map $p_*\varsigma : p_*X \to p_*(\boldsymbol{\pi}_0 X)$ is a monomorphism in $\mathbf{Set}$ if and only if the function $\mathcal{E}(X, \varsigma) : \mathcal{E}(1, X) \to \mathcal{E}(1, \boldsymbol{\pi}_0 X)$ is injective. That is, if and only if, for every pair of points $x, x' : 1 \to X$, $\varsigma x = \varsigma x'$ implies $x = x'$. So it is enough to prove that for every $x, x' : 1 \to X$, $\varsigma x = \varsigma x'$ if and only if x and x' are inseparable; but this is Lemma 4.2. Q.E.D.

We may now relate total separation and $\boldsymbol{\pi}_0$-algebras.

Corollary 4.4. If $p_* : \mathcal{E} \to \mathbf{Set}$ is faithful then, an object in $\mathcal{E}$ is a $\boldsymbol{\pi}_0$-algebra if and only if it is totally separated.

Proof. Since p_* is faithful, $\varsigma : X \to \boldsymbol{\pi}_0 X$ is monic if and only if $p_*\varsigma : p_*X \to p_*(\boldsymbol{\pi}_0 X)$ is monic. Q.E.D.

Recall that in an extensive category an object is called *indecomposable* if it has exactly two complemented subobjects. Hence, if the extensive category has a terminal object then an object X is indecomposable if and only if there are exactly two maps $X \to 1 + 1 = 2$. In particular, 1 is indecomposable if and only if the function $2 \to \mathcal{E}(1, 2)$ determined by the injections $1 \to 2$ is an isomorphism.

As we are assuming that $p^* : \mathcal{S} \to \mathcal{E}$ is full and faithful and preserves 1, the unit of $p^* \dashv p_*$ is an iso and $p^*2 = 2$. Therefore $2 \to p_*(p^*2) = \mathcal{E}(1, 2)$ is an iso. That is, under our current hypotheses, 1 is indecomposable in $\mathcal{E}$.

Corollary 4.5. For every object X in $\mathcal{E}$, X is indecomposable if and only if the map $p_*(2^!) : p_*(2^1) \to p_*(2^X)$ is an iso in **Set**.

Proof. Consider the unique map $! : X \to 1$ and the induced $\mathcal{E}(!, 2) : \mathcal{E}(1, 2) \to \mathcal{E}(X, 2)$. Since 1 is indecomposable then, by definition, X is indecomposable if and only if the function $\mathcal{E}(!, 2) : \mathcal{E}(1, 2) \to \mathcal{E}(X, 2)$ is an isomorphism. This is equivalent to $p_*(2^!) : p_*(2^1) \to p_*(2^X)$ being an isomorphism. Q.E.D.

5 Compactly generated Hausdorff spaces

Let **CGHaus** be the category with objects all compactly generated Hausdorff spaces, and with arrows all continuous maps between them. Theorem VII.8.3 in [14] shows that **CGHaus** is cartesian closed. It is also an extensive category, and it is regular [2]. Moreover, regular epimorphisms in **CGHaus** are exactly the topological quotients.

The faithful functor $p_* = \mathbf{CGHaus}(1, \_) : \mathbf{CGHaus} \to \mathbf{Set}$ that sends each space to its underlying set of points has a left adjoint $p^* : \mathbf{Set} \to \mathbf{CGHaus}$ that assigns to each set the corresponding discrete space. The unit of $p^* \dashv p_*$ will be denoted by β and the induced comonad by $C : \mathbf{CGHaus} \to \mathbf{CGHaus}$. For any X in **CGHaus**, $\beta_X : CX \to X$ is simply the canonical (epic) inclusion of the discrete space determined by X.

The functor $p^* : \mathbf{Set} \to \mathbf{CGHaus}$ preserves finite limits so, in this example, we have that $\Upsilon = p^*2 = 2 \in \mathbf{CGHaus}$. Let $\boldsymbol{\pi}_0 : \mathbf{CGHaus} \to \mathbf{CGHaus}$ denote the induced monad (Definition 3.1).

Corollary 5.1. The $\boldsymbol{\pi}_0$-algebras in **CGHaus** are exactly the totally separated spaces there.

Proof. Follows from Corollary 4.4. Q.E.D.

Exponential transpositions in **CGHaus** are calculated as in **Set** so, for every X in **CGHaus**, the canonical composite

$$X \xrightarrow{\overline{ev}} 2^{(2^X)} \xrightarrow{2^\beta} 2^{C(2^X)}$$

sends $x \in X$ to the function $ev_x : C(2^X) \to 2$ defined by $ev_x(f : X \to 2) = fx \in 2$. Alternatively, if we identity $f : X \to 2$ with the clopen arising by pulling back $\top : 1 \to 2$ then we may say that the composite $X \to 2^{C(2^X)}$ sends x to the set of clopens of X containing x. It is then clear, that $\varsigma : X \to \boldsymbol{\pi}_0 X$ is the topological quotient that equates two points if and only if they have the same clopen neighborhoods.

Corollary 5.2. For any X in **CGHaus** the following conditions are equivalent.

1. $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism.
2. X is connected.
3. $p_*(\boldsymbol{\pi}_0 X) = 1$.
4. X is indecomposable.

Proof. The first item implies the second by Proposition 3.4. The second item trivially implies the third. The third implies the second because, in the present context, p_* reflects terminal object. Corollary 4.5 and the fact that indecomposables are well-supported in **CGHaus** imply that the second and fourth items are equivalent. So, to complete the proof, we need to check that the second item implies the first. For this, assume that $p_*(\Upsilon^!) : p_*(\Upsilon^1) \to p_*(\Upsilon^X)$ is an isomorphism. Then Υ^X is finite, as well as Hausdorff, so it is discrete. Hence, we have that the top and vertical maps in the commutative diagram below

$$\begin{array}{ccc} p^*(p_*(\Upsilon^1)) & \xrightarrow{p^*(p_*(\Upsilon^!))} & p^*(p_*(\Upsilon^X)) \\ \beta \downarrow & & \downarrow \beta \\ \Upsilon^1 & \xrightarrow[\Upsilon^!]{} & \Upsilon^X \end{array}$$

are isomorphisms. It follows that the bottom map is also an iso. Q.E.D.

6 Subsequential spaces

Let $\mathbb{N}^+$ be the one-point compactification of the discrete space of natural numbers. A *subsequential space* consists of a set X together with a distinguished family of functions $\mathbb{N}^+ \to X$ called *convergent sequences* in X, satisfying:

1. for every $x \in X$, the constant sequence (x) converges to x;
2. if (x_n) converges to x, then so does every subsequence of (x_n);
3. if (x_n) is a sequence and x a point such that every subsequence of (x_n) contains a subsequence converging to x, then (x_n) converges to x.

A function between subsequential spaces is said to be continuous if it preserves convergent sequences. We write **sSeq** for the category of subsequential spaces and continuous maps between them.

The category **sSeq** is a quasi-topos [6] so it is cartesian closed and regular. It is straightforward to prove that it is also extensive and the functor $p_* = \mathbf{sSeq}(1, \_) : \mathbf{sSeq} \to \mathbf{Set}$ is clearly faithful. Moreover it has a fully faithful left adjoint $p^* : \mathbf{Set} \to \mathbf{sSeq}$ that preserves finite limits. So we have the induced $\boldsymbol{\pi}_0 : \mathbf{sSeq} \to \mathbf{sSeq}$ and we will characterize the indecomposable objects and the $\boldsymbol{\pi}_0$-algebras in **sSeq**. Before we do this it is convenient to recall the category of sequential spaces and its relation to **sSeq**.

Let X be a topological space. A subset $U \subseteq X$ is *sequentially open* if every sequence in X converging to a point of U is eventually in U. The space X is called *sequential* if every sequentially open subset of X is open [4]. Let $\mathbf{Seq} \to \mathbf{Top}$ be the full subcategory determined by sequential spaces. Lemma 2.2 in [6] proves that this inclusion has a right adjoint which sends an X in $\mathbf{Top}$ to the sequential space with the same underlying set of points and the sequentially open sets of X as opens. It follows that $\mathbf{Seq} \to \mathbf{Top}$ preserves coequalizers (see also Proposition 1.2 in [4]).

As observed in Lemma 2.3 loc. cit., the standard definition of convergent sequence makes a sequential space into a subsequential space and this assignment extends to a fully faithful inclusion $\mathbf{Seq} \to \mathbf{sSeq}$ that has a left adjoint.

A subsequential space is *sequentially Hausdorff* if every sequence converges to at most one point. Theorem 10.4 in [5] proves that the inclusion $\mathbf{Seq} \to \mathbf{sSeq}$ restricts to an equivalence between sequentially Hausdorff subsequential spaces and sequentially Hausdorff sequential spaces.

Lemma 6.1. Let Y in $\mathbf{sSeq}$ be sequentially Hausdorff. Then every subobject of Y is sequentially Hausdorff and, for every X in $\mathbf{sSeq}$, Y^X is sequentially Hausdorff.

Proof. Let $m : X \to Y$ be mono in $\mathbf{sSeq}$. Assume that the sequence (x_n) converges to x and to x' in X. Then (mx_n) converves to mx and to mx' in Y, so $mx = mx'$. Since m is injective, $x = x'$.

As recalled in [5], the underlying set of Y^X is $\mathbf{sSeq}(X, Y)$ and a sequence (f_n) in Y^X converges to f if and only if whenever (x_n) converges to x in X then $(f_n x_n)$ converges to fx in Y. To prove the second part of the statement, suppose that (f_n) converges to f and f' in Y^X then, for every $x \in X$, $(f_n x)$ converges to fx and to $f'x$. Since Y is sequentially Hausdorff, $fx = f'x$. Q.E.D.

We may now characterize the $\boldsymbol{\pi}_0$-algebras in $\mathbf{sSeq}$.

Proposition 6.2. An object in $\mathbf{sSeq}$ is a $\boldsymbol{\pi}_0$-algebra if and only if it is a totally separated sequential space.

Proof. First observe that, since $\boldsymbol{\pi}_0 X \to 2^{C(2^X)}$ is monic, Lemma 6.1 implies that $\boldsymbol{\pi}_0 X$ is sequentially Hausdorff, because 2 is. Also, as in the case of $\mathbf{CGHaus}$, we may apply Corollary 4.4 to conclude that an object X in $\mathbf{sSeq}$ is a $\boldsymbol{\pi}_0$-algebra if and only if X is totally separated in the extensive category $\mathbf{sSeq}$. Q.E.D.

The same argument in Corollary 5.2 proves the following.

Corollary 6.3. For any X in $\mathcal{E}$ the following conditions are equivalent.

1. $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism.

2. X is connected.

3. $p_*(\boldsymbol{\pi}_0 X) = 1$.

4. X is indecomposable.

It may be interesting to characterize the objects X in **CGHaus** or **sSeq** such that $\boldsymbol{\pi}_0 X$ is discrete.

7 Local maps and separated objects

Let $c : \mathcal{S} \to \mathcal{E}$ be a subtopos. A monomorphism m in $\mathcal{E}$ is *dense* (with respect to the subtopos c) if $c^* m$ is an isomorphism in $\mathcal{S}$. An object X in $\mathcal{E}$ is *separated* if for every span

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ {\scriptstyle m}\downarrow & & \\ Y & & \end{array}$$

with m a dense monomorphism, there exists at most one $g : Y \to X$ such that $gm = f$.

It is well-known that the full subcategory $\mathrm{Sep}\mathcal{E} \to \mathcal{E}$ of separated objects is closed under subobjects, reflective and that the left adjoint preserves finite products and monomorphisms. (See [3] and the comments before and after A4.4.6 in [8].) Either directly or as a corollary of product-preservation of the left adjoint, it follows that $\mathrm{Sep}\mathcal{E} \to \mathcal{E}$ is an exponential ideal.

Assume from now on that $p : \mathcal{E} \to \mathcal{S}$ is a hyperconnected and local geometric morphism. We will consider separated objects with respect to the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$. As before, we denote the (monic) counit of $p^* \dashv p_*$ by β. Notice that, since p^* is fully faithful, $\beta_X : p^*(p_* X) \to X$ is dense for every X in $\mathcal{E}$.

Denote the unit of $p^* \dashv p_*$ by α, the unit of $p_* \dashv p^!$ by η and its counit by ε. The following diagram commutes

$$\begin{array}{ccc} p^* & \xrightarrow{\eta} & p^! p_* p^* \\ {\scriptstyle p^*\varepsilon^{-1}}\downarrow & & \downarrow{\scriptstyle p^!\alpha^{-1}} \\ p^* p_* p^! & \xrightarrow[\beta]{} & p^! \end{array}$$

and we denote the composite by $\varphi : p^* \to p^!$ as in [12] and [9]. The natural φ is monic if and only if β is monic. Therefore, since p is hyperconnected, φ is monic.

Let $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ be the idempotent 'pizero' monad determined by the hyperconnected and local $p : \mathcal{E} \to \mathcal{S}$.

Lemma 7.1. Discrete objects and $\boldsymbol{\pi}_0$-algebras are separated in $\mathcal{E}$.

Proof. Let A be an object in $\mathcal{S}$. The monomorphism $\varphi_A : p^* A \to p^! A$ shows that $p^* A$ is a subobject of a sheaf so discrete objects are indeed separated. In particular, $\Upsilon = p^* \mathit{2}$ is separated and, as separated objects form an exponential ideal, $\Upsilon^{C(\Upsilon^X)}$ is separated for any X in $\mathcal{E}$. The monomorphism $\boldsymbol{\pi}_0 X \to \Upsilon^{C(\Upsilon^X)}$ implies that $\boldsymbol{\pi}_0 X$ is separated for any X. Q.E.D.

Let M be the idempotent monad on the topos $\mathcal{E}$ determined by the reflective subcategory $\mathrm{Sep}\mathcal{E} \to \mathcal{E}$, and denote its unit by ρ.

Proposition 7.2. For any X in $\mathcal{E}$, $\boldsymbol{\pi}_0 \rho : \boldsymbol{\pi}_0 X \to \boldsymbol{\pi}_0 (MX)$ is an isomorphism.

Proof. Consider the naturality diagram

$$\begin{array}{ccc} X & \xrightarrow{\varsigma} & \boldsymbol{\pi}_0 X \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_0 \rho} \\ MX & \xrightarrow[\varsigma]{} & \boldsymbol{\pi}_0(MX) \end{array}$$

and notice that, since $\boldsymbol{\pi}_0$-algebras are separated by Lemma 7.1, the left-bottom composite has the same universal property as the top map. It follows that the right map is an iso. Q.E.D.

In other words, every object X in $\mathcal{E}$ has the same $\boldsymbol{\pi}_0$ as its separated reflection.

Lemma 7.3. If $m : Y \to X$ is a dense mono and Z is separated then $Z^m : Z^X \to Z^Y$ is monic.

Proof. If the fork below commutes

$$W \underset{g}{\overset{f}{\rightrightarrows}} Z^X \xrightarrow{Z^m} Z^Y$$

then, by transposition, the left-bottom composites of the commutative diagram below are equal

$$\begin{array}{ccccc} W \times Y & \underset{g \times id}{\overset{f \times id}{\rightrightarrows}} & Z^X \times Y & \xrightarrow{Z^m \times id} & Z^Y \times Y \\ {\scriptstyle id \times m}\downarrow & & {\scriptstyle id \times m}\downarrow & & \downarrow{\scriptstyle ev} \\ W \times X & \underset{g \times id}{\overset{f \times id}{\rightrightarrows}} & Z^X \times X & \xrightarrow[ev]{} & Z \end{array}$$

and, since Z is separated and the left vertical map is a dense monomorphism, the bottom fork in the diagram above commutes; which means that $f = g$. Q.E.D.

Notice that Lemma 7.3 is a statement about separated objects for an arbitrary subtopos. That is, the subtopos does not need to be the center of a local geometric morphism. On the other hand, the application of the lemma in the next result involves the dense and monic counit of the local hyperconnected p.

Lemma 7.4. For every X in $\mathcal{E}$ the map $\Upsilon^{\beta} : \Upsilon^{(\Upsilon^X)} \to \Upsilon^{C(\Upsilon^X)}$ is monic. Therefore, there exists a unique (monic) map $\nu : \boldsymbol{\pi}_0 X \to \Upsilon^{(\Upsilon^X)}$ such that the inner triangles below commute

$$\begin{array}{ccc} X & \xrightarrow{\varsigma} & \boldsymbol{\pi}_0 X \\ {\scriptstyle \overline{ev}}\downarrow & \swarrow{\scriptstyle \nu} & \downarrow \\ \Upsilon^{(\Upsilon^X)} & \xrightarrow[\Upsilon^{\beta}]{} & \Upsilon^{C(\Upsilon^X)} \end{array}$$

in $\mathcal{E}$.

Proof. As p is hyperconnected, the counit β of $p^* \dashv p_*$ is monic and $p_*\beta$ is an isomorphism. So, for every X in $\mathcal{E}$, $\Upsilon^\beta : \Upsilon^{(\Upsilon^X)} \to \Upsilon^{C(\Upsilon^X)}$ is monic by Lemma 7.3. The map $\nu : \boldsymbol{\pi}_0 X \to \Upsilon^{(\Upsilon^X)}$ exists by orthogonality and it is monic because ψ is. Q.E.D.

In other words, $\boldsymbol{\pi}_0 X$ may be calculated as the image of $\overline{ev} : X \to \Upsilon^{(\Upsilon^X)}$. The referee noticed that this implies that the subcategory of $\boldsymbol{\pi}_0$-algebras coincides with that of subobjects of powers of Υ. Moreover, the referee's observation leads to the following.

Proposition 7.5. The functor $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ preserves finite products. Therefore, the map $S^\varsigma : S^{\boldsymbol{\pi}_0 X} \to S^X$ is an isomorphism for any X in $\mathcal{E}$ and any $\boldsymbol{\pi}_0$-algebra S. In particular, $\Upsilon^\varsigma : \Upsilon^{\boldsymbol{\pi}_0 X} \to \Upsilon^X$ is an isomorphism.

Proof. First let us expand on the referee's comment. Fix an object L in $\mathcal{E}$ and consider the subcategory $\mathcal{L} \to \mathcal{E}$ determined by the objects that appear as the domain of a subobject of a power of L. The inclusion $\mathcal{L} \to \mathcal{E}$ has a left adjoint which sends X in $\mathcal{E}$ to the image of $\overline{ev} : X \to L^{(L^X)}$. To prove the universal property consider a map $f : X \to Y$ and a monomorphism $m : Y \to L^Z$. Interchanging X and Z in the composite $mf : X \to L^Z$ determines a map $g : Z \to L^X$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\overline{ev}} & L^{(L^X)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle L^g} \\ Y & \xrightarrow[m]{} & L^Z \end{array}$$

commutes. Since the underlying category is regular and m is monic, orthogonality implies that f factors uniquely through the image of $\overline{ev} : X \to L^{(L^X)}$. This proves that $\mathcal{L} \to \mathcal{E}$ is reflective. Moreover, notice that if $m : Y \to L^Z$ is a monomorphism then so is the map $m^W : Y^W \to (L^Z)^W \cong L^{Z \times W}$ for any W in $\mathcal{E}$. In other words, the subcategory $\mathcal{L} \to \mathcal{E}$ is an exponential ideal.

In particular, we may consider the object $L = \Upsilon$. Lemma 7.4 implies that the resulting exponential ideal $\mathcal{L} \to \mathcal{E}$ coincides with the category of $\boldsymbol{\pi}_0$-algebras. It follows that the reflection $\boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{E}$ preserves finite products and that the unit $\varsigma : X \to \boldsymbol{\pi}_0 X$ induces an iso $S^\varsigma : S^{\boldsymbol{\pi}_0 X} \to S^X$ for every $\boldsymbol{\pi}_0$-algebra S. See, for example, Proposition A4.3.1 in [8].

Proposition 3.6 implies that $\Upsilon = p^* \mathit{2}$ is a $\boldsymbol{\pi}_0$-algebra so the proof is complete. Q.E.D.

It follows that finite products of connected objects are connected. Also, connected objects in the present hyperconnected and local context have a nice characterization. In order to prove it we need the following auxiliary fact about separated objects.

Lemma 7.6. The functor $p_* : \mathcal{E} \to \mathcal{S}$ is faithful with respect to morphisms with separated codomain. So, if Y is separated and $p_* Y = 1$ then $Y = 1$.

Proof. Let Y be separated and let $f, g : Z \to Y$ be such that $p_* f = p_* g$. Then

$$f\beta = \beta(p^*(p_* f)) = \beta(p^*(p_* g)) = g\beta$$

and, since β is dense and Y is separated, $f = g$.

If Y is separated and $p_* Y$ is subterminal then Y is subterminal by the first part of the result. If $p_* Y = 1$ then Y also has a point. Q.E.D.

Notice the similarity between Corollaries 5.2 and 6.3 and the following one.

Corollary 7.7. For any X in $\mathcal{E}$ the following are equivalent:

1. $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism.

2. X is connected.

3. $p_*(\boldsymbol{\pi}_0 X) = 1$.

Proof. The first item implies the second by Proposition 3.4 and the second easily implies the third. To prove that the third implies the second assume that $p_*(\boldsymbol{\pi}_0 X) = 1$. By Lemma 7.1, $\boldsymbol{\pi}_0 X$ is separated. So $\boldsymbol{\pi}_0 X = 1$ by Lemma 7.6. To prove that the second implies the first assume that $\boldsymbol{\pi}_0 X = 1$. Then $\Upsilon^! : \Upsilon^1 \to \Upsilon^X$ is an isomorphism by Proposition 7.5. Q.E.D.

Proposition 7.2 and Lemma 7.4 together imply that $\boldsymbol{\pi}_0 X$ may be calculated as the image of $\overline{ev} : MX \to \Upsilon^{(\Upsilon^{MX})}$; but more is true: we may calculate this image in the category Sep$\mathcal{E}$, thanks to the following result which is probably folklore.

Lemma 7.8. The functor Sep$\mathcal{E} \to \mathcal{E}$ preserves regular epimorphisms.

Proof. Let $e : X \to Y$ be a regular epimorphism in Sep$\mathcal{E}$. Let $e = md$ be the (regular-)epi/mono factorization of e in $\mathcal{E}$, say with $m : U \to Y$. Since Sep$\mathcal{E} \to \mathcal{E}$ is closed under subobjects, m is in Sep$\mathcal{E}$. Then $d : X \to U$ is also in Sep$\mathcal{E}$ and the left adjoint $\mathcal{E} \to$ Sep$\mathcal{E}$ sends it to a regular epimorphism. So $e = md$ is a regular-epi/mono factorization of e in Sep$\mathcal{E}$. It follows that m is an isomorphism in Sep$\mathcal{E}$ because quasi-toposes are regular. So m is an isomorphism in $\mathcal{E}$. That is, e is a (regular-)epimorphism in $\mathcal{E}$. Q.E.D.

(The referee confirmed that Lemma 7.8 is folklore and observed that it is a general fact about regular-epi-reflective subcategories of regular categories, since they are closed under arbitrary subobjects.)

We may conclude that the $\boldsymbol{\pi}_0$-algebras in $\mathcal{E}$ coincide with those in Sep$\mathcal{E}$. Rather than introducing notation for the restriction of p to Sep$\mathcal{E}$, and for the resulting $\boldsymbol{\pi}_0$ on Sep$\mathcal{E}$, we illustrate the idea with a concrete example.

The *topological topos*, denoted here by $\mathcal{J}$, is the topos of sheaves for the canonical topology on the monoid of continuous endofunctions of the one-point compactification of the discrete space of natural numbers [6]. It follows from the results loc. cit. that the canonical geometric morphism $p : \mathcal{E} \to \mathbf{Set}$ is hyperconnected, local, and moreover, the subtopos $p_* \dashv p^! : \mathbf{Set} \to \mathcal{J}$ coincides with the subtopos of $\neg\neg$-sheaves. Proposition 3.6 loc. cit. shows that the subcategory of $\neg\neg$-separated objects in $\mathcal{E}$ is equivalent to the category of subsequential spaces.

Corollary 7.9. An object X in $\mathcal{J}$ is a $\boldsymbol{\pi}_0$-algebra if and only if X is a totally separated sequential space.

Proof. By Proposition 7.2, $\boldsymbol{\pi}_0 X = \boldsymbol{\pi}_0(MX)$ and, since $\mathrm{Sep}\mathcal{E} \to \mathcal{E}$ is an exponential ideal, $\overline{ev} : MX \to \Upsilon^{\Upsilon^{MX}}$ is a map in $\mathrm{Sep}\mathcal{E}$. By Lemma 7.8 (and Lemma 7.4), its image in $\mathrm{Sep}\mathcal{E}$ coincides with $\boldsymbol{\pi}_0(MX)$. Moreover, $MX \to \boldsymbol{\pi}_0(MX)$ is an iso in $\mathrm{Sep}\mathcal{E}$ if and only if it is an iso in $\mathcal{E}$. The result then follows from Proposition 6.2. Q.E.D.

8 Locally connected geometric morphisms

Corollary 3.8 is reminiscent of the fact that locally connected morphisms are essential.

Theorem 8.1 (Barr-Paré [1]). For a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ the following are equivalent:

1. f^* has an $\mathcal{S}$-indexed left adjoint (denoted by $f_! : \mathcal{E} \to \mathcal{S}$).
2. f^* preserves $\prod_a$ for each map a in $\mathcal{S}$.
3. For each A in $\mathcal{S}$, $(f/A)^* : \mathcal{S}/A \to \mathcal{E}/(f^*A)$ is cartesian closed.

A geometric morphism satisfying the equivalent conditions of Theorem 8.1 is called *locally connected*.

Any proof of Theorem 8.1 must produce, from items 2 or 3, a left adjoint to f^*. For example, the proof (of Proposition C3.3.1) in [8] observes that item 2 is just the assertion that f^* is continuous as an $\mathcal{S}$-indexed functor, so the existence of $f_!$ follows from an Indexed Adjoint Functor Theorem. On the other hand, the proof in [1] invokes the following result.

Theorem 8.2 (Butler - see [1]). Assume that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Phi} & \mathcal{B} \\ & {\scriptstyle F}\nwarrow & \uparrow{\scriptstyle G} \\ & & \mathcal{C} \end{array}$$

$F \dashv U$, $G \dashv V$, $U : \mathcal{A} \to \mathcal{C}$ is monadic, Φ preserves coequalizers of U-split pairs. Then Φ has a right adjoint $\Psi : \mathcal{B} \to \mathcal{A}$.

We will not reproduce the proof of Theorem 8.2 here. Suffice it to say that Ψ is defined as the coequalizer

$$FUFV \underset{F\alpha}{\overset{\varepsilon_{FV}}{\rightrightarrows}} FV \longrightarrow \Psi$$

where ε is the counit of $F \dashv U$ and α is an explicit map that may be constructed under the hypothesis of the theorem.

Let $\top : 1 \to \mathit{2}$ be the subobject classifier of $\mathcal{S}$ and let $\Upsilon = p^*\mathit{2}$. Barr and Paré apply Theorem 8.2 to the following diagram

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{op}} & \xrightarrow{(f^*)^{\mathrm{op}}} & \mathcal{E}^{\mathrm{op}} \\ & {\scriptstyle \mathit{2}^{(\text{-})}}\nwarrow & \uparrow{\scriptstyle (f^*\mathit{2})^{(f^*\text{-})} = \Upsilon^{(f^*\text{-})}} \\ & & \mathcal{S} \end{array}$$

so that $UX = \mathit{2}^X$ and $VX = f_*(\Upsilon^X)$. Hence, the right adjoint to $(f^*)^{\mathrm{op}}$, i.e. the left adjoint to f^*, is defined by an equalizer

$$f_! X \longrightarrow \mathit{2}^{f_*(\Upsilon^X)} \rightrightarrows \mathit{2}^{\mathit{2}^{\mathit{2}^{f_*(\Upsilon^X)}}}$$

in $\mathcal{S}$. In contrast, $\boldsymbol{\pi}_0 X$ appears as a subobject

$$\boldsymbol{\pi}_0 X \rightarrowtail \Upsilon^{f^*(f_*(\Upsilon^X))}$$

in $\mathcal{E}$. By Corollary 3.8 we may conclude that: if f is connected and locally connected then $f_! = f_* \boldsymbol{\pi}_0 : \mathcal{E} \to \mathcal{S}$.

Altogether, we have mentioned three constructions. The two 'classical' ones are the Barr-Paré application of Butler's Theorem on right adjoints to functors (over some base) whose domain is monadic, and Johnstone's application of the Indexed Adjoint Functor Theorem. In contrast, the simple elementary construction of $\boldsymbol{\pi}_0$ does not produce a left adjoint in general, but it involves no indexing and has a direct geometric intuition.

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