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Topological $*$-autonomous categories, revisited

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Abstract

Given an additive equational category with a closed symmetric monoidal structure and a potential dualizing object, we find sufficient conditions that the category of topological objects over that category has a good notion of full subcategories of strong and weakly topologized objects and show that each is equivalent to the chu category of the original category with respect to the dualizing object.

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1 Introduction

I met Peter in the summer of 1962 as I was leaving Penn on the way to Columbia and Peter was leaving Columbia for Penn. Among living mathematicians I know, I have known only two, Murray Gerstenhaber and Marta Bunge, longer. I met Bill, along with many other category theorists, at Oberwolfach in 1966. I have had many mathematical discussions with both of them and it is an honor to contribute to this volume.

This paper is an updated version of [Barr, 2006]. In the course of preparing some lectures on the subject, I discovered to my great chagrin that that paper was badly flawed. Several of the arguments had gaps or flaws. In the process of repairing them, I discovered that the main results were not only correct, but that better results were available. This updated paper is the result.

I would like to thank the referee who read the paper very carefully, but also found an embarrassingly large number of typos and minor errors, as well as mathematical confusions. Any remaining errors are, of course, mine.

[Mackey, 1945] introduced the notion of pairs of vector spaces, equipped with a bilinear pairing into the ground field. Needless to say, he did not view this as a category in 1945. He didn't even define mappings between pairs although it would have obvious what they should be. It is likely that he viewed this abstract duality as a replacement for the topology. See also [Mackey, 1946], the review of the latter paper by Dieudonné as well as Dieudonné's review of [Arens, 1947], for a clear expression of this point of view. In this paper we fully answer these questions.

[Barr, 2000] showed that the full subcategory of the category of (real or complex) topological vector spaces that consists of the Mackey spaces (defined in 2.6 below) is $*$-autonomous and equivalent to both the full subcategory of weakly topologized topological vector spaces and to the full subcategory of topological vector spaces topologized with the strong, or Mackey topology. This means, first, that those subcategories can, in principle at least, be studied without taking the topology into consideration. Second it implies that both of those categories are $*$-autonomous.

André Joyal raised the question whether there was a similar result for vector spaces over the field $\mathbf{Q}_p$ of p-adic rationals. This was mentioned as a motivation for [Barr, 2006], but oddly the

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actual question was not answered or even studied there.. Thinking about this question, I realized that there is a useful general theorem that answers this question for any locally compact field and also for locally compact abelian groups.

The results in this paper prove the following conclusion. Let K be a **spherically complete field** (defined below) and $|K|$ its underlying discrete field. Then the following five categories are equivalent:

1. $\mathrm{chu}(K\text{-Vect}, |K|)$ (Section 3)
2. The category $\mathcal{V}_w(K)$ of topological K-spaces topologized with the weak topology for all their continuous linear functionals into K.
3. The category $\mathcal{V}_s(K)$ of topological K-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into K.
4. The category $\mathcal{V}_w(|K|)$ of topological $|K|$-spaces topologized with the weak topology for all their continuous linear functionals into $|K|$.
5. The category $\mathcal{V}_s(|K|)$ of topological $|K|$-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into $|K|$.

A normed field is **spherically complete** if any family of closed balls with the finite intersection property has non-empty intersection. A locally compact field is spherically complete (so this answers Joyal's question since $\mathbf{Q}_p$, along with its finite extensions, is locally compact) and spherically complete is known to be strictly stronger than complete.

[Barr, 2006, Section 2] is a result on adjoint functors that is interesting and possibly new. The argument given there is flawed. Although the result is bypassed in the current paper, it seemed interesting enough to give a full proof of it. This appears as an appendix to this paper.

1.1 Terminology

We assume that all topological objects are Hausdorff. As we will see, each of the categories contains an object K with special properties. It will be convenient to call a morphism $V \longrightarrow K$ a **functional** on V. In the case of abelian groups, the word "character" would be more appropriate, but it is convenient to have one word. In a similar vein, we may refer to a mapping of topological abelian groups as "linear" to mean additive. We will be dealing with topological objects in categories of topological vector spaces and abelian groups. If V is such an object, we will denote by $|V|$ the underlying vector space or group.

2 The strong and weak topologies

2.1 Blanket assumptions.

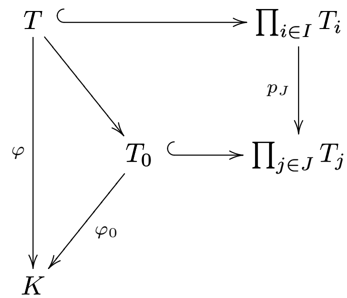
Throughout this section, we make the following assumptions.

1. $\mathcal{A}$ is an additive equational closed symmetric monoidal category and $\mathcal{T}$ is the category of topological $\mathcal{A}$-algebras.
2. K is a uniformly complete object of $\mathcal{T}$.
3. there is a neighbourhood U of 0 in K such that

(a) U contains no non-zero subobject;
(b) whenever T is an object of $\mathcal{T}$ and $\varphi : |T| \longrightarrow K$ is a morphism of the underlying discrete object such that $\varphi^{-1}(U)$ is open in T, then φ is continuous.

The lemma on which the entire theory depends is:

Lemma 2.1. Suppose there is an embedding $T \hookrightarrow \prod_{i\in I} T_i$ and there is a morphism $\varphi : T \longrightarrow K$. Then there is a finite subset $J \subseteq I$ and a commutative diagram



where p_J is the projection to the coordinates in J. Moreover, we can take T_0 closed in $\prod_{j\in J} T_j$.

Proof. Since $\varphi^{-1}(U)$ is a neighbourhood of 0 in T, it must be the meet with T of a neighbourhood of 0 in $\prod_{i\in I} T_i$. From the definition of the product topology, we must have a finite subset $J \subseteq I$ and neighbourhoods U_j of 0 in T_j such that

$$\varphi^{-1}(U) \supseteq T \cap \left(\prod_{j\in J} U_j \times \prod_{i\in I-J} T_i \right)$$

It follows that

$$U \supseteq \varphi \left(T \cap \left(\prod_{j\in J} 0 \times \prod_{i\in I-J} T_j \right) \right)$$

But the right hand side is a subobject and therefore must be 0. Now let

$$T_0 = \frac{T}{T \cap \left(\prod_{j\in J} 0 \times \prod_{i\in I-J} T_j \right)}$$

topologized not with the quotient topology, but with the coarser topology as a subspace of $\prod_{j\in J} T_0$ and let φ_0 be the induced map. It is immediate that $\varphi_0^{-1}(U) \supseteq \prod_{j\in J} U_j$, which is a neighbourhood of 0 in the induced topology and hence φ_0 is continuous. Finally, since K is complete, we can replace T_0 by its closure in $\prod_{j\in J} T_j$. ■

Theorem 2.2. Suppose $\mathcal{S}$ is a full subcategory of $\mathcal{T}$ that is closed under finite products and closed subobjects and that $K \in \mathcal{S}$ satisfies the assumptions in 2.1. If $\mathcal{V}$ is the closure of $\mathcal{S}$ under all products and all subobjects and K is injective in $\mathcal{S}$, then it is also injective in $\mathcal{V}$.

Proof. It is sufficient to show that if $V \subseteq \prod_{i\in I} S_i$ with each $S_i \in \mathcal{S}$, then every morphism $V \longrightarrow K$ extends to the product. But the object V_0 constructed in the preceding lemma is a closed suboject of $\prod_{j\in J} S_j$ so that $V_0 \in \mathcal{S}$ and the fact that K is injective in $\mathcal{S}$ completes the proof. ■

Definition 2.3. A bijective morphism $V \longrightarrow V'$ in $\mathcal{V}$ is called a **weak isomorphism** if the induced $\mathrm{Hom}(V', K) \longrightarrow \mathrm{Hom}(V, K)$ is a bijection.

Of course, a bijective morphism induces an injection so the only issue is whether the induced map is a surjection.

Proposition 2.4. A finite product of weak isomorphisms is a weak isomorphism.

Proof. Assume that J is a finite set and for each $j \in J$, $V_j \longrightarrow V'_j$ is a weak isomorphism. Then since finite products are the same as finite sums in an additive category, we have

$$\mathrm{Hom}\left(\prod V'_j, K\right) \cong \mathrm{Hom}\left(\sum V'_j, K\right) \cong \prod \mathrm{Hom}(V'_j, K)$$

$$\cong \prod \mathrm{Hom}(V_j, K) \cong \mathrm{Hom}\left(\sum V_j, K\right) \cong \mathrm{Hom}\left(\prod V_j, k\right) \qquad \blacksquare$$

Theorem 2.5. Assume the conditions of Theorem 2.2 and also suppose that for every object of $\mathcal{S}$, and therefore of $\mathcal{V}$, there are enough functionals to separate points. Then for every object V of $\mathcal{V}$, there are weak isomorphisms $\tau V \longrightarrow V \longrightarrow \sigma V$ with the property that σV has the coarsest topology that has the same functionals as V and τV has the finest topology that has the same functionals as V.

Proof. The argument for σ is standard. Simply retopologize V as a subspace of $K^{\mathrm{Hom}(V,K)}$.

Let $\{V_i \longrightarrow V\}$ range over the isomorphism classes of weak isomorphisms to V. We define τV as the pullback in

$$\begin{array}{ccc} \tau V & \longrightarrow & \prod V_i \\ \downarrow & & \downarrow \\ V & \longrightarrow & V^I \end{array}$$

The bottom map is the diagonal and is a topological embedding so that the top map is also a topological embedding. We must show that every functional on τV is continuous on V. Let φ be a functional on τV. From injectivity, it extends to a functional ψ on $\prod V_i$. By Lemma 2.1, there is a finite subset $J \subseteq I$ and a functional ψ_0 on $\prod_{j \in J} V_j$ such that is the composite $\prod_{i \in I} V_i \longrightarrow \prod_{j \in J} V_j \xrightarrow{\psi_0} K$. Thus we have the commutative diagram

$$\begin{array}{ccccccc} \tau V & \longrightarrow & \prod_{i \in I} V_i & \longrightarrow & \prod_{j \in I} V_j & & \\ & & & & & \searrow^{\psi_0} & \\ \downarrow & & \downarrow & & \downarrow & & K \\ & & & & & \nearrow & \\ V & \longrightarrow & V^I & \longrightarrow & V^J & & \end{array}$$

The dashed arrow exists because of Proposition 2.4, which completes the proof. $\blacksquare$

Remark 2.6. We will call the topologies on σV and τV the **weak** and **strong** topologies, respectively. They are the coarsest and finest topology that have the same underlying $\mathscr{A}$ structure and the same functionals as V. The strong topology is also called the **Mackey topology**.

Proposition 2.7. Weak isomorphisms are stable under pullback.

Proof. Suppose that

$$\begin{array}{ccc} W' & \longrightarrow & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ V' & \longrightarrow & V \end{array}$$

and the bottom arrow is a weak isomorphism. Clearly, $W' \longrightarrow W$ is a bijection, so we need only show that $\mathrm{Hom}(W,K) \longrightarrow \mathrm{Hom}(W',K)$ is surjective.

I claim that $W' \subseteq W \times V'$ with the induced topology. Let us define W'' to be the subobject $W \times_V V'$ with the induced topology. Since $W' \longrightarrow W$ and $W' \longrightarrow V$ are continuous, the topology on W' is at least as fine as that of W''. On the other hand, we do have $W'' \longrightarrow W$ and $W'' \longrightarrow V'$ with the same map to V so that we have $W'' \longrightarrow W'$, so that the topology on W'' is at least as fine as that of W'. Then we have a commutative diagram

$$\begin{array}{ccc} W' & \longrightarrow & W \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},f)} \\ W \times V' & \longrightarrow & W \times V \end{array}$$

Apply $\mathrm{Hom}(-,K)$ and use the injectivity of K to get:

$$\begin{array}{ccc} \mathrm{Hom}(W',K) & \longleftarrow & \mathrm{Hom}(W,K) \\ \big\uparrow & & \big\uparrow \\ \mathrm{Hom}(W,K) \times \mathrm{Hom}(V',K) & \xleftarrow{\ \cong\ } & \mathrm{Hom}(W,K) \times \mathrm{Hom}(V,K) \end{array}$$

The bottom arrow is a bijection and the left hand arrow is a surjection, which implies that the top arrow is a surjection. ∎

Proposition 2.8. σ and τ are functors on $\mathscr{V}$.

Proof. For σ, this is easy. If $f : W \longrightarrow V$ is a morphism, the induced $\sigma f : \sigma W \longrightarrow \sigma V$ will be continuous if and only if its composite with every functional on V is a functional on W, which obviously holds.

To see that τ is a functor, suppose $f : W \longrightarrow V$ is a morphism. Form the pullback

$$\begin{array}{ccc} W' & \longrightarrow & W \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \tau V & \longrightarrow & V \end{array}$$

It is a weak isomorphism by the preceding proposition. Thus we get $\tau W \longrightarrow W' \longrightarrow \tau V$. ∎

Proposition 2.9. If $V \longrightarrow V'$ is a weak isomorphism, then $\sigma V \longrightarrow \sigma V'$ and $\tau V \longrightarrow \tau V'$ are isomorphisms.

Proof. For σ, this is obvious. Clearly, $\tau V \longrightarrow V \longrightarrow \tau V'$ is also a weak isomorphism so that τV is one of the factors in the computation of $\tau V'$ and then $\tau V' \longrightarrow \tau V$ is a continuous bijection, while the other direction is evident. ∎

Corollary 2.10. Both σ and τ are idempotent, while $\sigma\tau \cong \sigma$ and $\tau\sigma \cong \tau$. ∎

Proposition 2.11. For any $V, V' \in \mathcal{V}$, we have $\mathrm{Hom}(\sigma V, \sigma V') \cong \mathrm{Hom}(\tau V, \tau V')$.

Proof. It is easiest to assume that the underlying objects $|V| = |\sigma V| = |\tau V|$ and similarly for V'. Then for any $f : V \longrightarrow V'$, we also have that $|f| = |\sigma f| = |\tau f|$. Thus the two composition of the two maps below

$$\mathrm{Hom}(\sigma V, \sigma V') \longrightarrow \mathrm{Hom}(\tau\sigma V, \tau\sigma V') = \mathrm{Hom}(\tau V, \tau V')$$

and

$$\mathrm{Hom}(\tau V, \tau V') \longrightarrow \mathrm{Hom}(\sigma\tau V, \sigma\tau V') \cong \mathrm{Hom}(\sigma V, \sigma V')$$

give the identity in each direction. ∎

Let $\mathcal{V}_w \subseteq \mathcal{V}$ and $\mathcal{V}_s \subseteq \mathcal{V}$ denote the full subcategories of weak and strong objects, respectively. Then as an immediate corollary to the preceding, we have:

Theorem 2.12. $\tau : \mathcal{V}_w \longrightarrow \mathcal{V}_s$ and $\sigma : \mathcal{V}_s \longrightarrow \mathcal{V}_w$ determine inverse equivalences of categories. ∎

3 Chu and chu

Now we add to the assumptions on $\mathcal{A}$ that it be a symmetric monoidal closed category in which the underlying set of $A \multimap B$ is $\mathrm{Hom}(A, B)$. We denote by $\mathcal{E}$ and $\mathcal{M}$ the classes of surjections and injections, respectively.

We briefly review the categories $\mathrm{Chu}(\mathcal{A}, K)$ and $\mathrm{chu}(\mathcal{A}, K)$. See [Barr, 1998] for details. The first has as objects pairs (A, X) of objects of $\mathcal{A}$ equipped with a "pairing" $\langle -, - \rangle : A \otimes X \longrightarrow K$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of a map $f : A \longrightarrow B$ and a map $g : Y \longrightarrow X$ such that

$$\begin{array}{ccc} A \otimes Y & \xrightarrow{f \otimes Y} & B \otimes Y \\ {\scriptstyle A \otimes g}\big\downarrow & & \big\downarrow{\scriptstyle \langle -,- \rangle} \\ A \otimes X & \xrightarrow[\langle -,- \rangle]{} & K \end{array}$$

commutes. This diagram says that $\langle fa, y\rangle = \langle a, gy\rangle$ for all $a \in A$ and $y \in Y$. The set of arrows can be enriched over $\mathcal{A}$ by internalizing its definition as follows. Note first that the map $A \otimes X \longrightarrow K$ induces, by exponential transpose, a map $X \longrightarrow A \multimap K$. This gives a map $Y \multimap X \longrightarrow Y \multimap (A \multimap K) \cong (A \otimes Y) \multimap K$. There is a similarly defined arrow $A \multimap B \longrightarrow (A \otimes Y) \multimap K$. Define $[(A, X), (B, Y)]$ so that

$$\begin{array}{ccc} [(A,X),(B,Y)] & \longrightarrow & A \multimap B \\ \downarrow & & \downarrow \\ Y \multimap X & \longrightarrow & (A \otimes Y) \multimap K \end{array}$$

is a pullback. Then define

$$(A, X) \multimap (B, Y) = ([(A, X), (B, Y)], A \otimes Y)$$

with $\langle (f, g), a \otimes y\rangle = \langle fa, y\rangle = \langle a, gy\rangle$ and

$$(A, X) \otimes (B, Y) = (A \otimes B, [(A, X), (Y, B)])$$

with pairing $\langle a \otimes b, (f, g)\rangle = \langle b, fa\rangle = \langle a, gb\rangle$. The duality is given by $(A, X)^* = (X, A) \cong (A, X) \multimap (K, \top)$ where $\top$ is the tensor unit of $\mathcal{A}$. Incidentally, the tensor unit of $\mathrm{Chu}(\mathcal{A}, K)$ is $(\top, K)$.

The category $\mathrm{Chu}(\mathcal{A}, K)$ is *-autonomous. It is also complete and cocomplete when $\mathcal{A}$ is. The limit of a diagram is calculated using the limit of the first coordinate and the colimit of the second. The full subcategory $\mathrm{chu}(\mathcal{A}, K) \subseteq \mathrm{Chu}(\mathcal{A}, K)$ consists of those objects (A, X) for which the two transposes of $A \otimes X \longrightarrow K$ are injective homomorphisms. When $A \rightarrowtail X \multimap K$, the pair is called separated and when $X \rightarrowtail A \multimap K$, it is called extensional. In the general case, one must choose a factorization system $(\mathcal{E}, \mathcal{M})$ and assume that the arrows in $\mathcal{E}$ are epic and that $\mathcal{M}$ is stable under $\multimap$, but here these conditions are clear. Let us denote by $\mathrm{Chu}_s(\mathcal{A}, K)$ the full subcategory of separated pairs and by $\mathrm{Chu}_e(\mathcal{A}, K)$ the full subcategory of extensional pairs.

The inclusion $\mathrm{Chu}_s(\mathcal{A}, K) \hookrightarrow \mathrm{Chu}(\mathcal{A}, K)$ has a left adjoint S and the inclusion $\mathrm{Chu}_e(\mathcal{A}, K) \hookrightarrow \mathrm{Chu}(\mathcal{A}, K)$ has a right adjoint E. Moreover, S takes an extensional pair into an extensional one and E does the dual. In addition, when (A, X) and (B, Y) are separated and extensional, $(A, X) \multimap (B, Y)$ is separated but not necessarily extensional and, dually, $(A, X) \otimes (B, Y)$ is extensional, but not necessarily separated. Thus we must apply the reflector to the internal hom and the coreflector to the tensor, but everything works out and $\mathrm{chu}(\mathcal{A}, K)$ is also ∗-autonomous. See [Barr, 1998] for details.

In the chu category, one sees immediately that in a map $(f, g) : (A, X) \longrightarrow (B, Y)$, f and g determine each other uniquely. So a map could just as well be described as an $f : A \longrightarrow B$ such that $x.\tilde{y} \in X$ for every $y \in Y$. Here $\tilde{y} : B \longrightarrow K$ is the evaluation at $y \in Y$ of the exponential transpose $Y \longrightarrow B \multimap K$.

Although the situation in the category of abelian groups is as described, in the case of vector spaces over a field, the hom and tensor of two separated extensional pairs turns out to be separated and extensional already ([Barr, 1996]).

4 The main theorem

Theorem 4.1. Assume the hypotheses of Theorem 2.5 and also assume that the canonical map $I \longrightarrow K \multimap K$ is an isomorphism. Then the categories of weak spaces and strong spaces are equivalent to each other and to $\mathrm{chu}(\mathcal{A}, K)$ and are thus $*$-autonomous.

Proof. For the first claim see Theorem 7.7 below. Now define $F : \mathcal{V} \longrightarrow \mathrm{chu}$ by $F(V) = (|V|, \mathrm{Hom}(V, K))$ with evaluation as pairing. We first define the right adjoint R of F. Let $R(A, X)$ be the object A, topologized as a subobject of K^X. Since it is already inside a power of K, it has the weak topology. Let $f : |V| \longrightarrow A$ be a homomorphism such that for all $x \in X$, $\tilde{x}.f \in \mathrm{Hom}(V, K)$. This just means that the composite $V \longrightarrow R(A, X) \longrightarrow K^X \xrightarrow{\pi_x} K$ is continuous for all $x \in X$, exactly what is required for the map into $R(A, X)$ to be continuous. The uniqueness of f is clear and this establishes the right adjunction.

We next claim that $FR \cong \mathrm{Id}$. That is equivalent to showing that $\mathrm{Hom}(R(A, X), K) = X$. Suppose $\varphi : R(A, X) \longrightarrow K$ is a functional. By injectivity, it extends to a $\psi : K^X \longrightarrow K$. It follows from 2.1, there is a finite set of elements $x_1, \ldots, x_n \in X$ and morphisms $\theta_1, \ldots, \theta_n$ such that ψ factors as $K^X \longrightarrow K^n \xrightarrow{(\theta_1,\ldots,\theta_n)} K$. Applied to $R(A, X)$, this means that $\varphi(a) = \langle \theta_1 x_1, a\rangle + \cdots + \langle \theta_n x_n, a\rangle$. But the $\theta_i \in I$ and the tensor products are over I so that the pairing is a homomorphism $A \otimes_I X \longrightarrow K$. This means that $\varphi(a) = \langle \theta_1 x_1 + \cdots \theta_n x_n, a\rangle$ and $\theta_n x_1 + \cdots + \theta_n x_n \in X$.

Finally, we claim that $RF = S$, the left adjoint of the inclusion $\mathcal{V}_w \subseteq \mathcal{V}$. If $V \in \mathcal{V}$, then $RFV = R(|V|, \mathrm{Hom}(V, K))$ which is just V with the weak topology it inherits from $K^{\mathrm{Hom}(V,K)}$, exactly the definition of SV. It follows that $F|\mathcal{V}_w$ is an equivalence.

Since $\mathcal{V}_w$ and $\mathcal{V}_s$ are equivalent to a $*$-autonomous category, they are $*$-autonomous. ■

The fact that the categories of weak and Mackey spaces are equivalent was shown, for the case of Banach spaces in [Dunford & Schwartz, 1958, Theorem 15, p. 422]. Presumably the general case has also been long known, but I am not aware of a reference.

5 Examples

Example 1. Locally compact abelian groups.

Let $\mathcal{V}$ the category of topological abelian groups that are topological subgroups of products of locally compact abelian groups. The object K in this case is the circle group $\mathbf{R}/\mathbf{Z}$. A simple representation of this group is as the closed interval $[-1/2, 1/2]$ with the endpoints identified and addition mod 1. The group is compact. Let U be the open interval $(-1/3, 1/3)$. It is easy to see that any non-zero point in that interval, added to itself sufficiently often, eventually escapes that neighborhood so that U contains no non-zero subgroup. It is well-known that the endomorphism group of the circle is $\mathbf{Z}$.

If $f : G \longrightarrow K$ is a homomorphism such that $T = f^{-1}(U)$ is open in G, let $T = T_0, T_1, \ldots, T_n, \ldots$ be a sequence of open balanced neighborhoods of 0 in G such that $T_{i+1} + T_{i+1} \subseteq T_i$ for all i. Let $U_i = (-2^{-i}/3, 2^{-i}/3) \subseteq K$. Then $f(T_0) \subseteq U_0$ and if we assume by induction that $f(T_i) \subseteq U_i$, we show that $f(T_{i+1}) \subseteq U_{i+1}$. For if not, there will be an element $x \in U_{i+1}$ such that $|f(x)| > 2^{-(i+1)/3}$ and then $|f(x) + f(x)| > 2^{-i}/3$, a contradiction. This shows that $f^{-1}(U_i) \supseteq T_i$ for all i. Since the U_i form a neighborhood base at 0 in the circle, this shows that f is continuous.

We take for $\mathcal{S}$ the category of locally compact abelian groups. The fact that K is injective in $\mathcal{S}$ follows directly from the Pontrjagin duality theorem. A result [Glicksberg, 1962, Theorem

1.1] says that every locally compact group is strongly topologized. Thus both categories of weakly topologized and strongly topologized groups that are subobjects of products of locally compact abelian groups are equivalent to chu($\mathcal{Ab}$, $|K|$) and thus are *-autonomous.

Note that by Glickberg's theorem, the strong category includes the locally compact abelian groups and the duality extends that of Pontrjagin. Unfortunately, there is less there than meets the eye because what makes the duality on locally compact abelian groups so powerful is the existence of Haar measure. That allows you to dualize not only homomorphisms but measurable functions to the circle and gives rise to harmonic analysis.

Example 2. Vector spaces over a locally compact field.

Let K be a locally compact field. Locally compact fields have been classified, see [Pontrjagin, 1968] or [Weil, 1967]. Besides the discrete fields and the real and complex numbers, they come in two varieties. The first are finite algebraic extensions of the field $\mathbf{Q}_p$, which is the completion of the rational field in the p-adic norm. The second are finite algebraic extensions of the field $\mathbf{S}_p$, which is the completion in t-adic norm of the field $\mathbf{Z}_p\{t\}$ of Laurent series over the field of p elements. Notice that all these locally compact fields are normed.

We take for $\mathcal{S}$ the category of normed linear K-spaces, except in the case that K is discrete, we require also that the spaces have the discrete norm. We know that K is injective in the discrete case. The injectivity of K in the real or complex case is just the Hahn-Banach theorem, which has been generalized to ultrametric fields according to the following, found in [Robert, 2000].

An ultrametric is a metric for which the ultratriangle inequality, $\|x + y\| \leq \|x\| \vee \|y\|$, holds. This is obviously true for p-adic and t-adic norms. Spherically complete means that the meet of any descending sequence of non-empty closed balls is non-empty. This is known to be satisfied by locally compact ultrametric spaces.

Theorem 5.1 (Ingelton)**.** Let K be a spherically complete ultrametric field, E a K-normed space, and V a subspace of E. For every bounded linear functional φ defined on V, there exists a bounded linear functional ψ defined on E whose restriction to V is φ and such that $\|\varphi\| = \|\psi\|$.

Regardless of the topology on a field K (assuming it is topological field), K is its own endomorphism ring.

In the non-discrete case, we take for the neighborhood U of 0, the open ball of radius 1. It obviously contains no non-zero subspace. The proof that every $f : V \longrightarrow K$, for which $f^{-1}(U)$ is open, is continuous can be carried out just as in the first example.

Notice that if K is non-discrete, then what we have established is that both $\mathcal{V}_s$ and $\mathcal{V}_w$ are equivalent to chu(Vect-$|K|, |K|$). But exactly the same considerations show that the same is true if we ignore the topology on K and use the discrete norm. The category $\mathcal{S}$ will now be the category of discrete finite-dimensional $|K|$-vector spaces. Its product and subobject closure will consist of spaces that are mostly not discrete, but there are still full subcategories of weakly and strongly topologized spaces within this category and they are also equivalent to chu(Vect-$|K|, |K|$).

Thus, these categories really do not depend on the topologies. Another interpretation is that this demonstrates that, for these spaces, the space of functionals replaces the topology, which was arguably Mackey's original intention.

Example 3. Modules over a self injective cogenerator.

If we examine the considerations that are used in vector spaces over a field, it is clear that what is used is that a field is both an injective module over itself and a cogenerator in the category of vector spaces. Let K be such a commutative ring, topologized discretely and let $\mathcal{T}$ be the category of topological K-modules, $\mathcal{S}$ be the full subcategory of submodules of finite powers of K with the discrete topology, and $\mathcal{V}$ the limit closure of $\mathcal{S}$. Then $\operatorname{chu}(\mathrm{Mod}_K, K)$ is equivalent to each of the categories $\mathcal{V}_s$ and $\mathcal{V}_w$ of topological K-modules that are strongly and weakly topologized, respectively, with respect to their continuous linear functionals into K.

We now show that there is a class of commutative rings with that property. Let k be a field and $K = k[x]/(x^n)$. When $n = 2$, this is called the ring of dual numbers over k.

Proposition 5.2. K is self injective.

We base this proof on the following well-known fact:

Lemma 5.3. Let k be a commutative ring, K is a k-algebra, Q an injective k-module, and P a flat right K-module then $\mathrm{Hom}_k(K, Q)$ is an injective K-module.

The K-module structure on the Hom set is given by $(rf)(a) = f(ar)$ for $r \in K$ and $a \in P$.
Proof. Suppose $A \rightarrowtail B$ is an injective homomorphism of K-modules. Then we have

$$\begin{array}{ccc}
\mathrm{Hom}_R(B, \mathrm{Hom}_k(P, Q)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_k(P, Q)) \\
\cong \big\downarrow & & \big\downarrow \cong \\
\mathrm{Hom}_k(P \otimes_R B, Q) & \rightarrowtail & \mathrm{Hom}_k(P \otimes_R A, Q)
\end{array}$$

and the flatness of P, combined with the injectivity of Q, force the bottom arrow to be a surjection. ■

Proof of 5.2. From the lemma it follows that $\mathrm{Hom}_k(K, k)$ is K-injective. We claim that, as K-modules, $\mathrm{Hom}_k(K, k) \cong K$. To see this, we map $f : K \longrightarrow \mathrm{Hom}_k(K, k)$. Since these are vector spaces over k, we begin with a k-linear map and show it is K-linear. A k-basis for K is given by $1, x, \ldots, x^{n-1}$. We define $f(x^i) : K \longrightarrow k$ for $0 \leq\leq n-1$ by $f(x^i)(x^j) = \delta_{i+j,n}$ (the Kronecker δ). For this to be K-linear, we must show that $f(xx^i) = xf(x^i)$. But

$$f(xx^i)(x^j) = f(x^{i+1}(x^j)) = \delta_{i+1+j,n} = f(x^i)(x^{j+1}) = (xf(x^i))(x^j)$$

Clearly, the $f(x^i)$, for $0 \leq i \leq n$ are linearly independent and so f is an isomorphism. ■

Proposition 5.4. K is a cogenerator in the category of K-modules.

Proof. Using the injectivity, it suffices to show that every cyclic module can be embedded into K. Suppose M is a cyclic module with generator m. Let i be the first power for which $x^i m = 0$. I claim that $m, xm, \ldots, x^{i-1}m$ are linearly independent over k. If not, suppose that $\lambda_0 m + \lambda_1 xm + \cdots \lambda_{i-1}x^{i-1}m = 0$ with not all coefficients 0. Let λ_j be the first non-zero coefficient, so that $\lambda_j x^j + \cdots + \lambda_{i-1}x^{i-1}m = 0$. Multiply this by x^{i-j-1} and use that $x^l m = 0$ for $l \geq i$ to get $\lambda_j x^{i-1}m = 0$. But by assumption, $x^{i-1}m \neq 0$ so that this would imply that $\lambda_j = 0$, contrary to hypothesis. Thus there is a k-linear map $f : M \longrightarrow K$ given by $f(x^j m) = x^{n-i+j}$. Since the x^j are linearly independent, this is k-linear and then it is clearly K-linear. ■

6 Interpretation of the dual of an internal hom

These remarks are especially relevant to the vector spaces, although they are appropriate to the other examples. The fact that $(U \multimap V)^* \cong U \otimes V^*$ can be interpreted as saying that the dual of $U \multimap V$ is a subspace of $V \multimap U$, namely those linear transformations of finite rank. An element of the form $u \otimes v^*$ acts as a linear transformation by the formula $(u \otimes v^*)(v) = \langle v, v^* \rangle u$. This is a transformation of row rank 1. A sum of such elements similarly has finite rank.

This observation generalizes the fact that in the category of finite dimensional vector spaces, we have that $(U \multimap V)^* \cong V \multimap U$ (such a category is called a compact ∗-autonomous category). In fact, Halmos avoids the complications of the definition of tensor product of finite dimensional vector spaces by *defining* $U \otimes V$ as the dual of the space of bilinear forms on $U \oplus V$, which is quite clearly equivalent to the dual of $U \multimap V^* \cong V \multimap U^*$ ([Halmos, 1958, Page 40]). (Incidentally, it might be somewhat pedantic to point out that Halmos's definition makes no sense since $U \oplus V$ is a vector space in its own right and a bilinear form on a vector space is not well defined. It would have been better to use the equivalent form above or to define $\mathrm{Bilin}((U, V), K)$.)

7 Appendix: Some generalities on adjoints.

In the earlier paper, [Barr, 2006], the proof of 2.12 was based on some formal results about adjoints. The argument got greatly simplified and these results were not needed, largely because of the concreteness of the categories involved. Still, it seemed worthwhile to include these formal results.

The following is quite well known, but I have not found an explicit proof of it in the literature. In [Lawvere, 1996, Page 168], Lawvere called this situation an essential localization and gave it as an example of "Unity of opposites".

Proposition 7.1. Suppose $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor that has both a left adjoint L and a right adjoint R. Then L is full and faithful if and only R is.

Proof. Suppose that L is full and faithful. Then we have, for any $B, B' \in \mathcal{B}$,

$$\mathrm{Hom}(B, B') \cong \mathrm{Hom}(LB, LB') \cong \mathrm{Hom}(B, FLB')$$

so that, by the Yoneda lemma, the front adjunction $B' \longrightarrow FLB'$ is an isomorphism. Then

$$\mathrm{Hom}(B, B') \cong \mathrm{Hom}(FLB, B') \cong \mathrm{Hom}(LB, RB') \cong \mathrm{Hom}(B, FRB')$$

which implies that the back adjunction $FRB' \longrightarrow B'$ is also an isomorphism, which is possible only if R is full and faithful. The reverse implication is just the dual. ∎

Proposition 7.2. Suppose

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

is a commutative square and both f and k are isomorphisms. Then the square with f^{-1} and g^{-1} also commutes. ∎

Theorem 7.3. Suppose $\mathcal{C}$ is a category, $I : \mathcal{B} \longrightarrow \mathcal{C}$ the inclusion of a full subcategory with a left adjoint S and $J : \mathcal{D} \longrightarrow \mathcal{C}$ is the inclusion of a full subcategory with a right adjoint T. Let $\alpha : 1 \longrightarrow IS$ and $\beta : SI \xrightarrow{\cong} 1$ be the front and back adjunctions for $S \dashv I$ and $\delta : 1 \xrightarrow{\cong} TJ$ and $\varepsilon : JT \longrightarrow 1$ do the same for $J \dashv T$. Suppose, in addition, that $IS\varepsilon : ISJT \longrightarrow IS$ and $JT\alpha : JT \longrightarrow JTIS$ are isomorphisms. Then $JT \dashv IS$.

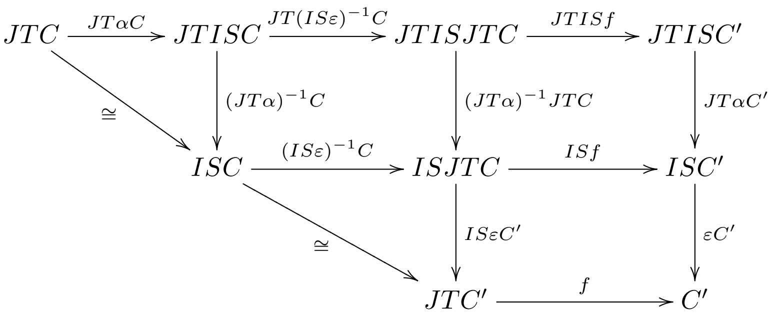
Proof. If $f : JTC \longrightarrow C'$ is given, define $\mu f : C \longrightarrow ISC'$ as the composite

$$C \xrightarrow{\alpha C} ISC \xrightarrow{(IS\varepsilon C)^{-1}} ISJTC \xrightarrow{ISf} ISC'$$

If $g : C \longrightarrow ISC'$ is given, define $\nu g : JTC \longrightarrow C'$ as the composite

$$JTC \xrightarrow{JTg} JTSIC' \xrightarrow{(JT\alpha C')^{-1}} JTC' \xrightarrow{\varepsilon C'} C'$$

We must show that μ and ν are inverse operations. The upper and right hand arrows calculate $\nu\mu f$ and the squares commute by naturality of by applying the preceding proposition to a naturally commuting square.



This shows that $\nu\mu f = f$ and $\mu\nu g = g$ is handled similarly. Thus $\mathrm{Hom}(JTC, C') \cong \mathrm{Hom}(C, ISC')$

Proposition 7.4. $JTI \dashv S$ and $T \dashv ISJ$.

Proof. We have $\mathrm{Hom}(JTIB, C) \cong \mathrm{Hom}(IB, ISC) \cong \mathrm{Hom}(B, SC)$ since I is full and faithful. The second one is proved similarly. ■

Proposition 7.5. $JTI : \mathcal{B} \longrightarrow \mathcal{C}$ and $ISJ : \mathcal{D} \longrightarrow \mathcal{B}$ are full and faithful.

Proof. S has a right adjoint I and a right adjoint JTI. Since I is full and faithful, so is JTI be 7.1. The argument for ISJ is similar. ■

Corollary 7.6. $TI : \mathcal{B} \longrightarrow \mathcal{D}$ is left adjoint to $SJ : \mathcal{D} \longrightarrow \mathcal{C}$ and each is an equivalence.

Proof. $\mathrm{Hom}(TIB, D) \cong \mathrm{Hom}(JTIB, JD) \cong \mathrm{Hom}(B, SJD)$, gives the adjunction. Moreover, $\mathrm{Hom}(TIB, TIB') \cong \mathrm{Hom}(JTIB, JTIB') \cong \mathrm{Hom}(B, B')$ since JTI is full and faithful and a similar argument works for SJ. ■

Applying to the results of Section 2.12, we conclude that:

Theorem 7.7. The functors $TI : \mathcal{V}_w \longrightarrow \mathcal{V}_s$ and $SJ : \mathcal{V}_s \longrightarrow \mathcal{V}_w$ are adjoint equivalences. ■

7.1 Application

This was originally applied to the proof of 2 as follows.

Let $I : \mathcal{V}_s \longrightarrow \mathcal{V}$ and $J : \mathcal{V}_w \longrightarrow \mathcal{V}$ denote the inclusions into $\mathcal{V}$ of the full subcategories consisting of the weak and strong objects, respectively.

Theorem 7.8. The functor $S : \mathcal{V} \longrightarrow \mathcal{V}_w$ for which $SV = \sigma V$ is left adjoint to I. Similarly, the functor $T : \mathcal{V} \longrightarrow \mathcal{V}_s$ for which $TV = \tau V$ is right adjoint to J.

Proof. First we note that for $\sigma I = ISI \cong I$. Then for any $V \in \mathcal{V}$ and $V' \in \mathcal{V}_w$ we have the composite

$$\mathrm{Hom}(V, IV') \longrightarrow \mathrm{Hom}(\sigma V, \sigma IV') \longrightarrow \mathrm{Hom}(V, IV')$$

is the identity and the second arrow is an injection, so that both arrows are isomorphisms. Thus $\mathrm{Hom}(V, IV) \cong \mathrm{Hom}(\sigma V, IV)$. Then we have

$$\mathrm{Hom}(V, IV) \cong \mathrm{Hom}(\sigma V, IV') = \mathrm{Hom}(ISV, IV') \cong \mathrm{Hom}(SV, V')$$

The second assertion is dual. ■

Corollary 7.9. $SITJ \longrightarrow SI$ using the back adjunction and $TJ \longrightarrow TJSI$ using the front adjunction are isomorphisms.

Proof. This is immediate from Corollary 2.10.

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