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Wreaths, mixed wreaths and twisted coactions

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Dedicated to Peter Freyd and Bill Lawvere.

Abstract

Distributive laws between two monads in a 2-category $\mathscr{K}$, as defined by Jon Beck in the case $\mathscr{K} = \mathrm{Cat}$, were pointed out by the author to be monads in a 2-category $\mathrm{Mnd}\mathscr{K}$ of monads. Steve Lack and the author defined wreaths to be monads in a 2-category $\mathrm{EM}\mathscr{K}$ of monads with different 2-cells from $\mathrm{Mnd}\mathscr{K}$.

Mixed distributive laws were also considered by Jon Beck, Mike Barr and, later, various others; they are comonads in $\mathrm{Mnd}\mathscr{K}$. Actually, as pointed out by John Power and Hiroshi Watanabe, there are a number of dual possibilities for mixed distributive laws.

It is natural then to consider mixed wreaths as we do in this article; they are comonads in $\mathrm{EM}\mathscr{K}$. There are also mixed opwreaths: comonads in the Kleisli construction completion $\mathrm{Kl}\mathscr{K}$ of $\mathscr{K}$. The main example studied here arises from a twisted coaction of a bimonoid on a monoid. A wreath determines a monad structure on the composite of the two endomorphisms involved; this monad is called the wreath product. For mixed wreaths, corresponding to this wreath product, is a convolution operation analogous to the convolution monoid structure on the set of morphisms from a comonoid to a monoid. In fact, wreath convolution is composition in a Kleisli-like construction. Walter Moreira's Heisenberg product of linear endomorphisms on a Hopf algebra, is an example of such convolution, actually involving merely a mixed distributive law. Monoidality of the Kleisli-like construction is also discussed.

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1 Introduction

While trying to expose the categorical mechanism behind the Heisenberg product of endomorphisms, as defined and studied in [1, 16], we noticed that it has to do with distributive laws in the sense of Beck [2]; also see [18, 21] for the general setting and monad terminology. A distributive law $\xi\colon TS \Rightarrow ST$ of a monad T over a monad S on a category $\mathscr{A}$ gives rise to a monad structure on the composite ST. For a mixed distributive law $\zeta\colon SG \Rightarrow GS$ of a comonad G over a monad S, we do not expect GS to be a monad or comonad, so what takes its place? It is the (internalized) ζ-parametrized convolution of 2-cells $G \Rightarrow S$. To understand this to some extent (externally), consider the case where $T \dashv G$, and ξ and ζ are mates [12] under that adjunction. The adjunction gives an isomorphism

$$[\mathscr{A}, \mathscr{A}](1_{\mathscr{A}}, ST) \cong [\mathscr{A}, \mathscr{A}](G, S)$$

The monoid structure on the left-hand side, arising pointwise from the monad structure on ST determined by ξ, transports to a convolution-like monoid structure on the right-hand side, expressible in terms of ζ.

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Rather than remain at the level of distributive law, since there are articles [7, 17, 6] which study that, we decided to generalize to the wreaths of [14].

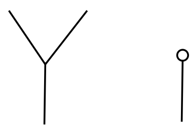
This article begins with a review of wreaths as defined in [14]. We spend some time extending Example 3.2 of [14] to a wreath between monoids rather than groups: the use of fibrations is to bring out the cohomological aspects which permeate the paper.

As for mixed distributive laws [17], there are several possibilities for mixed wreaths. We look at those which are comonads in either the (limit) completion of the ambient 2-category under Eilenberg-Moore construction or the (colimit) completion under the Kleisli construction. The first are called mixed wreaths, the second mixed opwreaths. Mixed Eilenberg-Moore and mixed Kleisli constructions are described and their universal properties presented. Composition in a mixed Kleisli category is convolution parametrized by the mixed wreath.

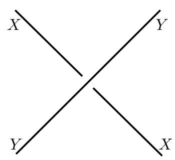
Section 4 provides the construction of a mixed opwreath is a dual of the wreath construction appearing as Example 3.3 in [14] based on Sweedler's crossed product of Hopf algebras. We also generalize to bialgebras (bimonoids). The ingredient is a twisted coaction of a bimonoid on a monoid. Natural connections to cohomological structures are pursued.

Section 5 sets out when a mixed opwreath is opmonoidal. This is about a monoidal structure on the mixed Kleisli construction. The final section gives structure on a twisted coaction so that the associated mixed opwreath becomes opmonoidal.

We will use the string diagrams for monoidal categories as explained in [8]. However, we read the diagrams from top to bottom rather than the reverse. For example, if A is a monoid in any monoidal category $\mathscr{V}$, the multiplication $\mu = \mu_A$ and unit $\eta = \eta_A$ are respectively depicted as follows.

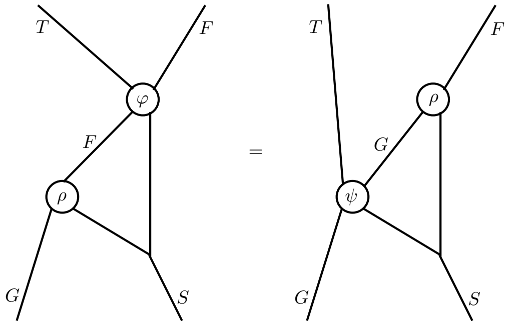


If we are dealing with a braided monoidal category, the braiding $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ will be depicted as a crossing as follows.



2 Review of wreaths

The free completion $\mathrm{EM}(\mathfrak{K})$ of a 2-category $\mathfrak{K}$ (such as Cat) under the Eilenberg-Moore construction was identified in [14]. The objects of $\mathrm{EM}(\mathfrak{K})$ are monads $(\mathscr{A}, T)$ in $\mathfrak{K}$. That is, T is a monoid in the endomorphism category $\mathfrak{K}(\mathscr{A}, \mathscr{A})$, monoidal under composition as tensor product, so we can draw planar diagrams. A morphism $(F, \varphi) \colon (\mathscr{A}, T) \to (\mathscr{B}, S)$ consists of a morphism $F \colon \mathscr{A} \to \mathscr{B}$ and a 2-cell $\varphi \colon SF \Rightarrow FT$ in $\mathfrak{K}$ compatible with the monad structures on T and S. A 2-cell $\rho \colon (F, \varphi) \Rightarrow (G, \psi)$ is a 2-cell $\rho \colon F \Rightarrow GS$ such that the following equation holds.



Also in [14], wreaths were introduced and defined concisely as monads in the 2-category $\mathrm{EM}(\mathfrak{K})$. The wreath product is the monad obtained as the Eilenberg-Moore construction in $\mathrm{EM}(\mathfrak{K})$ on the wreath. As explained in [19], the Eilenberg-Moore construction is a weighted (or "indexed") limit. The completion of a 2-category $\mathfrak{K}$ under that specific limit is $\mathrm{EM}(\mathfrak{K})$; see [14]. So 2-categories admitting the Eilenberg-Moore construction are the algebras for the 2-monad ("doctrine" in Lawvere's terminology [15]) EM on the 2-category of 2-categories; the action 2-functor $\mathrm{EM}(\mathfrak{K}) \to \mathfrak{K}$ takes each monad in $\mathfrak{K}$ to its Eilenberg-Moore object. The action for a free algebra is of course given by the multiplication of the monad. The wreath product

$$\mathrm{wr} = \mathrm{wr}_{\mathfrak{K}} \colon \mathrm{EM}(\mathrm{EM}(\mathfrak{K})) \longrightarrow \mathrm{EM}(\mathfrak{K})$$

gives the multiplication for our 2-monad EM; the unit

$$\mathrm{id} = \mathrm{id}_{\mathfrak{K}} \colon \mathfrak{K} \longrightarrow \mathrm{EM}(\mathfrak{K})$$

simply takes each object $\mathscr{A}$ to the identity monad on $\mathscr{A}$. As always with a completion under limits, EM is a (dual) Kock-Zöberlein monad [13, 20]: the action is adjoint to unit. Therefore, when it exists, the Eilenberg-Moore construction for a 2-category $\mathfrak{K}$ is a right adjoint

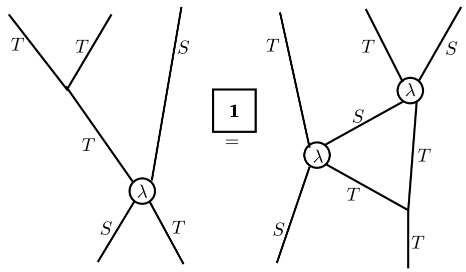
$$\mathrm{em} = \mathrm{em}_{\mathfrak{K}} \colon \mathrm{EM}(\mathfrak{K}) \longrightarrow \mathfrak{K}$$

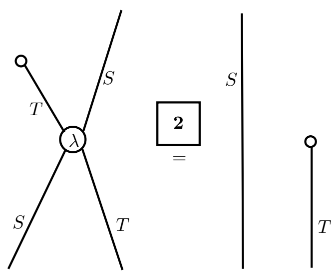
to id; we put $\mathscr{A}^T = \mathrm{em}_{\mathscr{A}} T = \mathrm{em}(\mathscr{A}, T)$.

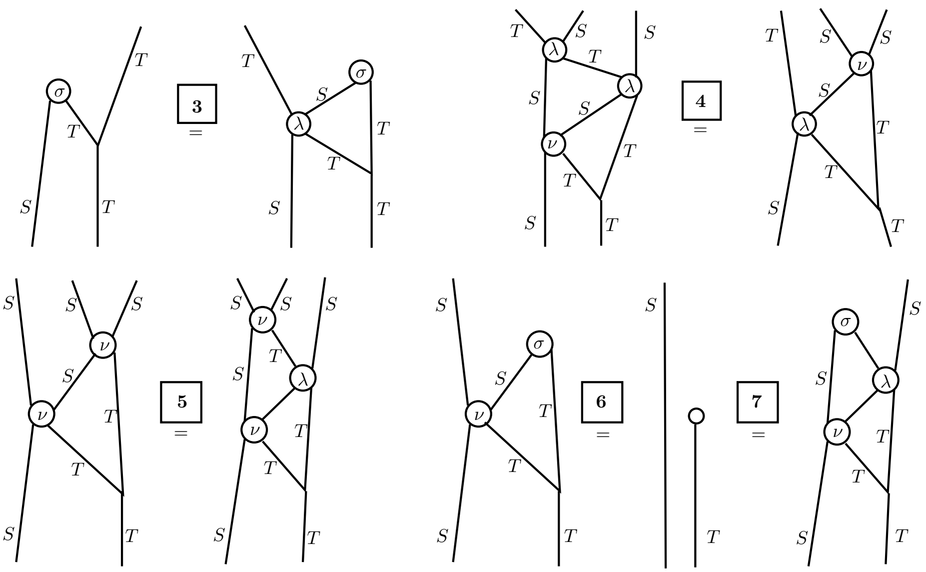
We will now describe wreaths explicitly using string diagrams. The reader wishing to compare the commutative diagrams will find them in [14].

Let $T = (T, \mu, \eta)$ be a monad on an object $\mathscr{A}$ of $\mathscr{K}$.

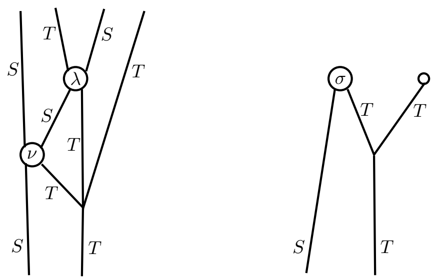
A *wreath around* T consists of an endomorphism S on $\mathscr{A}$, and 2-cells $\nu \colon SS \Longrightarrow ST$, $\sigma \colon 1_{\mathscr{A}} \Longrightarrow ST$ and $\lambda \colon TS \Longrightarrow ST$ satisfying seven axioms.







The *product* of the wreath S around T, or the *wreath product*, is the monad consisting of the endomorphism ST on $\mathscr{A}$ with the multiplication and unit as displayed in the diagram:



where the unlabelled nodes are the ternary and binary multiplications of T.

A *distributive law* [2] of a monad T over a monad S is a special case of a wreath around T consisting of the endofunctor S while the natural transformations ν and σ of the special form

$$\nu = \left(SS \xrightarrow{\mu} S \xrightarrow{S\eta} ST\right) \quad \text{and} \quad \sigma = \left(1_{\mathscr{A}} \xrightarrow{\eta\eta} ST\right) \ , \tag{2.1}$$

and λ remains arbitrary.

Example 2.1. *We now generalise Example 3.2 of [14] from groups to monoids. We call a monoid morphism $p\colon E \to M$ (in the category* Set *of sets) a* normal cloven lax fibration *when it is equipped with a function $j\colon M \to E$ such that $p \circ j = 1_M$, $j(1) = 1$ and, for*

$$A = \{a \in E \colon p(a) = 1\} = p^{-1}(1) \ ,$$

the function $h\colon M \times A \to E$, defined by $h(x,a) = j(x)a$, is invertible. This gives, for each $x \in M$, a pullback square.

$$\begin{tikzcd} A \arrow[r, "h(x,-)"] \arrow[d, "!"'] & E \arrow[d, "p"] \\ 1 \arrow[r, "x"'] & M \end{tikzcd} \tag{2.2}$$

Generally, the kernel of a monoid morphism is a rather strange thing to consider, yet, because we have a fibration, all the fibres $p^{-1}(x)$ of p are isomorphic as sets. Unlike arbitrary fibres, the kernel has the advantage of being a submonoid of E.

We use the pullback (2.2) to obtain a function $\alpha\colon A \times M \to A$ of M on A; indeed, $\alpha(a,x) = a \cdot x \in A$ is characterized by the property

$$j(x)(a \cdot x) = aj(x)\ . \tag{2.3}$$

In other words, this α measures the failure of the kernel to commute with the image of j. Using the pullback uniqueness clause, we see that each $- \cdot x\colon A \to A$ is a monoid morphism.

We also use the pullback (2.2) with x replaced by xy to obtain a function $\rho\colon M \times M \to A$ characterized by the property

$$j(xy)\rho(x,y) = j(x)j(y)\ . \tag{2.4}$$

In other words, ρ measures the failure of j to be a monoid morphism. Indeed, for each $x, y \in M$, we have a 2-cell

$$\begin{tikzcd} A \arrow[rr, "-\cdot x"] \arrow[dr, "-\cdot(xy)"'] & \arrow[d, phantom, "\overset{\rho(x,y)}{\Longleftarrow}"] & A \arrow[dl, "-\cdot y"] \\ & A & \end{tikzcd} \tag{2.5}$$

in the 2-category Mon = MonSet *of monoids; the "naturality" amounts to the equation*

$$(a \cdot (xy))\rho(x,y) = \rho(x,y)((a \cdot x) \cdot y) \tag{2.6}$$

which shows that ρ also measures the failure of α to be an action of the monoid M on A. To prove (2.6), it suffices to prove we have equality after applying $h(xy,-)$, which we do thus:

$$\begin{aligned} j(xy)(a \cdot (xy))\rho(x,y) &= aj(xy)\rho(x,y) \\ &= aj(x)j(y) \\ &= j(x)(a \cdot x)j(y) \\ &= j(x)j(y)((a \cdot x) \cdot y) \\ &= j(xy)\rho(x,y)((a \cdot x) \cdot y)\ . \end{aligned}$$

Let ΣM denote the category with one object 0 and hom $\Sigma M(0,0) = M$; composition is multiplication in M. What we are producing is a normal lax functor

$$P\colon \Sigma M^{\mathrm{op}} \longrightarrow \mathrm{Mon} \tag{2.7}$$

with $P0 = A$ and $Px = - \cdot x \colon A \to A$. The composition constraints are given by (2.5). Clearly $\rho(1,x) = 1 = \rho(x,1)$ so all that remains to prove is the coherence condition (2.8).

$$
\begin{tikzcd}
A \arrow[r, "-\cdot y"] & A \arrow[d, "-\cdot z"] \\
A \arrow[u, "-\cdot x"] \arrow[ur] \arrow[r, "-\cdot(xyz)"'] & A
\arrow[from=1-1, to=2-2, phantom, shift left=3, "\Downarrow\rho(x,y)"]
\arrow[from=1-1, to=2-2, phantom, shift right=3, "\Downarrow\rho(xy,z)"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "-\cdot y"] \arrow[dr] & A \arrow[d, "-\cdot z"] \\
A \arrow[u, "-\cdot x"] \arrow[r, "-\cdot(xyz)"'] & A
\arrow[from=2-1, to=1-2, phantom, shift right=3, "\Downarrow\rho(y,z)"]
\arrow[from=2-1, to=1-2, phantom, shift left=3, "\Downarrow\rho(x,yz)"]
\end{tikzcd}
\tag{2.8}
$$

This amounts to the Schreier factor set or 2-cocycle condition (2.9).

$$\rho(xy,z)(\rho(x,y)\cdot z) = \rho(x,yz)\rho(y,z) \tag{2.9}$$

To prove this, it suffices to check after left multiplication by $j(xyz)$, which we do thus:

$$
\begin{aligned}
j(xyz)\rho(xy,z)(\rho(x,y)\cdot z) &= j(xy)j(z)(\rho(x,y)\cdot z) \\
&= j(xy)\rho(x,y)j(z) \\
&= j(x)j(y)j(z) \\
&= j(x)j(yz)\rho(y,z) \\
&= j(xyz)\rho(x,yz)\rho(y,z)\ .
\end{aligned}
$$

We are now in a position to transport the multiplication of E to $M \times A$ across the isomorphism h.

$$
\begin{tikzcd}
M \times A \arrow[rr, "h"] \arrow[dr, "\mathrm{pr}_1"'] & & E \arrow[dl, "p"] \\
& M &
\end{tikzcd}
$$

The resultant multiplication on $M \times A$ is

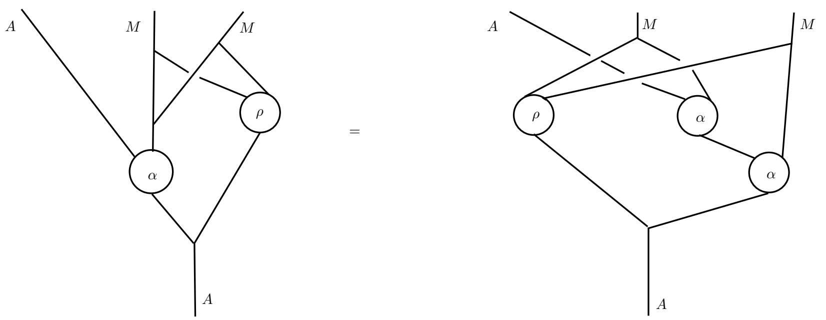
$$(x,a)(y,b) = (xy, \rho(x,y)(a\cdot y)b)\ . \tag{2.10}$$

This gives an equivalence of categories between normal cloven lax fibrations over any monoid M and normal lax functors $P \colon \Sigma M^{\mathrm{op}} \longrightarrow \mathrm{Mon}$. This is essentially classical and is an interpretation theorem for the second cohomology of the monoid M: 2-cocycles equate to certain extensions $E \to M$.

Now we give the wreath. The category is Set. *The remaining data all arise from data in* Set *by applying the strong monoidal functor* $\mathrm{Set} \to [\mathrm{Set}, \mathrm{Set}]$ *which takes K to $K \times -$. The monad T arises from the monoid A. The endofunctor S arises from the set M. The natural transformation ν arises from the function $M \times M \to M \times A, (x,y) \mapsto (xy, \rho(x,y))$. The natural transformation λ arises from the function $A \times M \to M \times A, (a,x) \mapsto (x, \alpha(a,x))$. The natural transformation σ arises from the function $\mathbf{1} \to M \times A$ which picks out* $(1,1)$.

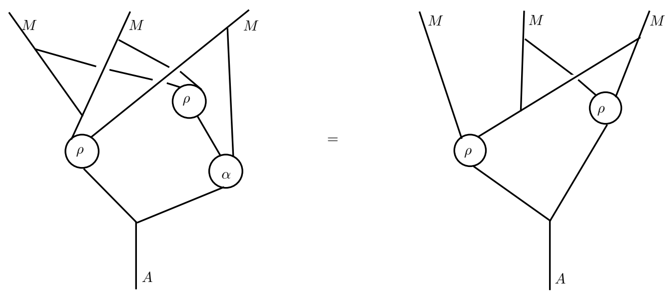
The wreath product of course arises from the monoid $M \times A$ with product (2.10) and so recaptures E up to isomorphism.

Remark 2.2. *1. Here is the string diagram for* (2.6)*.*



(2.11)

2. Here is the string diagram for (2.9)*.*



(2.12)

3. The structure on $p\colon E \to M$ *of* normal cloven (strict) fibration *consists of a function* $j\colon M \to E$ *such that* $p \circ j = 1_M$*,* $j(1) = 1$ *and the square*

$$\begin{tikzcd}
M \times E \arrow[r, "\mu\circ(j\times 1_E)"] \arrow[d, "1_M\times p"'] & E \arrow[d, "p"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

is a pullback. Notice that we have the condition $M \times A \cong E$ *for a lax fibration since we can paste two pullback squares as follows:*

$$\begin{tikzcd}
M \times A \arrow[r, "1_M\times \mathrm{incl}"] \arrow[d, "\mathrm{pr}_1"'] & M \times E \arrow[r, "\mu\circ(j\times 1_E)"] \arrow[d, "1_M\times p"'] & E \arrow[d, "p"] \\
M \arrow[r, "(1_M,1)"'] \arrow[rr, bend right, "1_M"'] & M \times M \arrow[r, "\mu"'] & M\ .
\end{tikzcd}$$

4. *Similarly to Example 2.1, Example 3.3 of [14] can be generalised from Hopf algebras H to bimonoids M in a braided monoidal category. Moreover, there is no need for the convolution invertibility of ρ: however, the one axiom required for A to be a twisted M-module, which is stated in [14] in terms of that inverse of ρ, should be replaced by the naturality condition* (2.11). *We will discuss a dual of this in Section 4.*

3 Mixed wreaths

There are several possibilities for mixed wreaths just as for mixed distributive laws; compare [17].

We will use the notation

$$\mathrm{EM}^{\mathrm{du}}(\mathfrak{K}) = \mathrm{EM}(\mathfrak{K}^{\mathrm{du}})^{\mathrm{du}}$$

for any of the dualities $\mathrm{du} \in \{\mathrm{op}, \mathrm{co}, \mathrm{coop}\}$ (in the notation of [12]). We also put

$$\mathrm{KL}(\mathfrak{K}) = \mathrm{EM}^{\mathrm{op}}(\mathfrak{K})$$

since it is the cocompletion of $\mathfrak{K}$ with respect to the Kleisli construction. This then leads to

$$\mathrm{KL}^{\mathrm{co}}(\mathfrak{K}) = \mathrm{EM}^{\mathrm{coop}}(\mathfrak{K}) \ .$$

Definition 3.1. *Let T be a monad on $\mathscr{A}$ in the 2-category $\mathfrak{K}$. A* mixed wreath around the monad *T is a comonad on $(\mathscr{A}, T)$ in the 2-category* $\mathrm{EM}(\mathfrak{K})$*.*

More explicitly, a mixed wreath structure around T on an endomorphism G of $\mathscr{A}$ consists of 2-cells $\delta\colon G \Rightarrow GGT$, $\varepsilon\colon G \Rightarrow T$ and $\xi\colon TG \Rightarrow GT$ satisfying four axioms which say that $(G,\xi)\colon (\mathscr{A},T) \to (\mathscr{A},T)$ is a morphism, and $\delta\colon (G,\xi) \Rightarrow (G,\xi)(G,\xi)$ and $\varepsilon\colon 1 \Rightarrow (G,\xi)$ are 2-cells, in $\mathrm{EM}(\mathfrak{K})$, and three axioms which say δ is coassociative with counit ε.

Suppose $\mathfrak{K}$ admits the Eilenberg-Moore construction for both monads and comonads. Simply because $\mathrm{em}\colon \mathrm{EM}(\mathfrak{K}) \longrightarrow \mathfrak{K}$ is a 2-functor, each mixed wreath $(G,\xi)\colon (\mathscr{A},T) \to (\mathscr{A},T)$ yields a comonad $G^{\xi} = \mathrm{em}(G,\xi)$ on $\mathscr{A}^T$ in $\mathfrak{K}$. Define

$$\mathrm{mem}(G,\xi,T) = (\mathscr{A}^T)^{G^{\xi}} \ ,$$

the Eilenberg-Moore construction for the comonad G_{ξ}. This gives the object function for a *mixed Eilenberg-Moore construction*

$$\mathrm{mem}\colon \mathrm{EM}^{\mathrm{co}}(\mathrm{EM}(\mathfrak{K})) \xrightarrow{\mathrm{EM}^{\mathrm{co}}(\mathrm{em})} \mathrm{EM}^{\mathrm{co}}(\mathfrak{K}) \xrightarrow{\mathrm{em}^{\mathrm{co}}} \mathfrak{K} \tag{3.13}$$

with an obvious left adjoint.

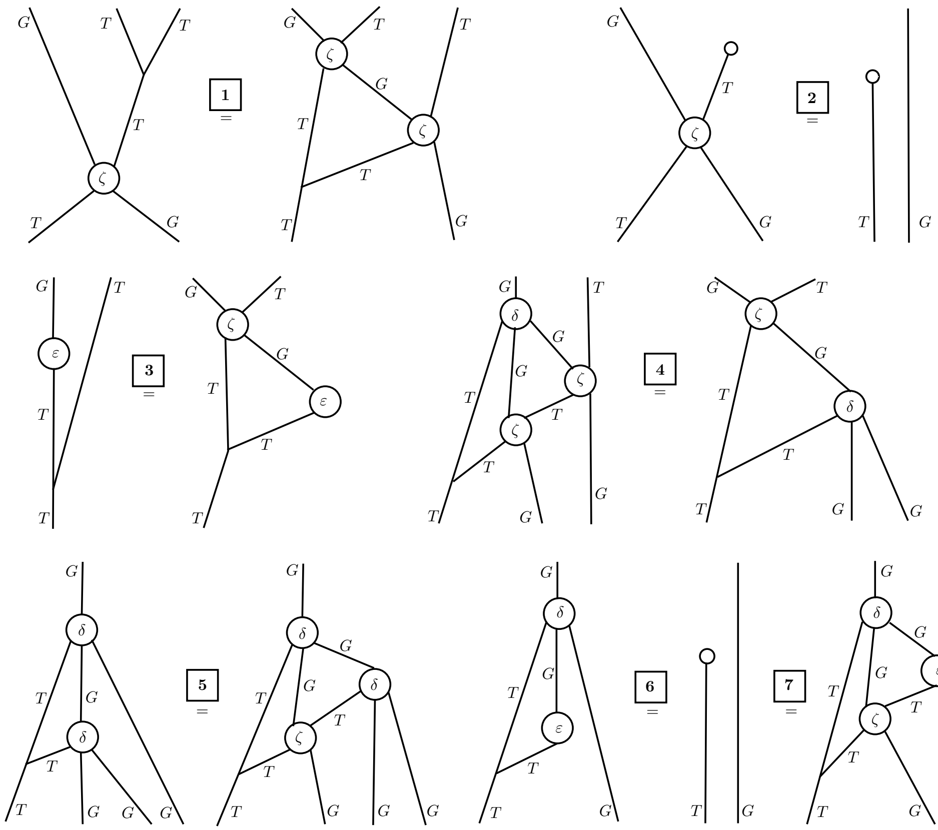
We have the following description of $\mathrm{mem}(G,\xi,T)$ when $\mathfrak{K} = \mathrm{Cat}$. The objects (A,a,c) of the category consist of an Eilenberg-Moore T-algebra $a\colon TA \to A$ and a morphism $c\colon A \to GA$ satisfying the following three conditions.

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tc"'] & A \arrow[dd, "c"] \\ TGA \arrow[d, "\xi_A"'] & \\ GTA \arrow[r, "Ga"'] & GA \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "c"] \arrow[d, "c"'] & GA \arrow[dd, "Gc"] \\ GA \arrow[d, "\delta_A"'] & \\ GGTA \arrow[r, "GGa"'] & GGA \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "c"'] \arrow[ddr, "1"] & \\ GA \arrow[d, "\varepsilon_A"'] & \\ GTA \arrow[r, "a"'] & A \end{tikzcd}$$

To reinforce the limit nature of the $\mathrm{mem}(G, \xi, T)$ construction we next record its characterization as a representing object. This can be taken as the definition when $\mathfrak{K}$ lacks the Eilenberg-Moore construction for monads or comonads in general.

Proposition 3.2. $\mathfrak{K}(\mathscr{X}, \mathrm{mem}(G, \xi, T)) \cong \mathrm{mem}(\mathfrak{K}(\mathscr{X}, G), \mathfrak{K}(\mathscr{X}, \xi), \mathfrak{K}(\mathscr{X}, T))$

Definition 3.3. *Let T be a monoid on $\mathscr{A}$ in the 2-category $\mathfrak{K}$. A* mixed opwreath around the monad *T is a comonad on $(\mathscr{A}, T)$ in the 2-category* $\mathrm{KL}(\mathfrak{K})$. *This consists of an endomorphism G of $\mathscr{A}$ made into a morphism of* $\mathrm{KL}(\mathfrak{K})$ *by a 2-cell $\zeta\colon GT \Rightarrow TG$ and into a comonad by comultiplication $\delta\colon G \Rightarrow TGG$ and counit $\varepsilon\colon G \Rightarrow T$. The seven axioms are shown below in string form.*



At the 2-category level Definition 3.3 really is just an example: a mixed opwreath in $\mathfrak{K}$ is a mixed wreath in $\mathfrak{K}^{\mathrm{op}}$. Indeed, in the presence of right adjoints, we will now point out how a mixed opwreath amounts to a wreath.

Recall from [12] the terminology and concept of mates under adjunction. Here is an exercise on mates using the string calculus.

Proposition 3.4. Suppose T is a monad on $\mathscr{A} \in \mathfrak{K}$. Suppose $G \dashv S$ are adjoint endomorphisms of $\mathscr{A}$. Mixed opwreath structures on G around T correspond under adjoint mateship to wreath structures on S around T.

In the situation of Proposition 3.4, the Eilenberg-Moore construction for the wreath product ST does not easily reinterpret in terms of G and T, rather, as you would expect, the Kleisli construction does. We shall now define this in general.

By applying Proposition 3.2 to $\mathfrak{K}^{\mathrm{op}}$, and by defining the *mixed Kleisli construction* as the composite

$$\mathrm{mkl}\colon \mathrm{KL}^{\mathrm{co}}(\mathrm{KL}(\mathfrak{K})) \xrightarrow{\mathrm{KL}^{\mathrm{co}}(\mathrm{kl})} \mathrm{KL}^{\mathrm{co}}(\mathfrak{K}) \xrightarrow{\mathrm{kl}^{\mathrm{co}}} \mathfrak{K}\ , \tag{3.14}$$

we obtain:

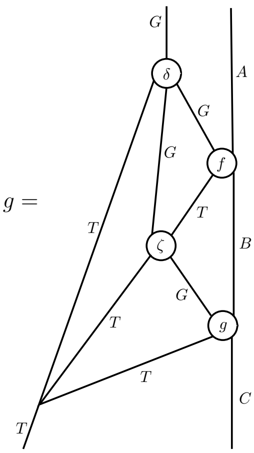
Proposition 3.5. $\mathfrak{K}(\mathrm{mkl}(G,\zeta,T),\mathscr{X}) \cong \mathrm{mem}(\mathfrak{K}(G,\mathscr{X}),\mathfrak{K}(\zeta,\mathscr{X}),\mathfrak{K}(T,\mathscr{X}))$

For $\mathfrak{K} = \mathrm{Cat}$, the category $\mathrm{mkl}(G,\zeta,T)$ has the same objects as $\mathscr{A}$ and has homsets defined by

$$\mathrm{mkl}(G,\zeta,T)(A,B) = \mathscr{A}(GA,TB)\ .$$

Composition is defined by *wreath convolution*: the composite of $f\colon GA \to TB$ and $g\colon GB \to TC$ is $g \circ f = f *_\zeta g$ as in the commutative diagram (3.15).

$$\begin{tikzcd}
GA \arrow[r, "\delta_A"] \arrow[d, "f *_\zeta g"'] & TGGA \arrow[r, "TGf"] & TGTB \arrow[d, "T\zeta_B"] \\
TC & TTTC \arrow[l, "\mu_{3C}"] & TTGB \arrow[l, "TTg"]
\end{tikzcd} \tag{3.15}$$

$f *_\zeta g =$ 

Remark 3.6. *1. For any monoidal category $\mathscr{V}$, the functor $\mathscr{V} \to [\mathscr{V},\mathscr{V}]$, taking X to $X \otimes -$, is strong monoidal. So each monoid $A = (A,m,j)$ in $\mathscr{V}$ is taken to a monad $T = (T,\mu,\eta)$ on $\mathscr{V}$. We will speak of a* mixed opwreath *around A to mean a quadruple (C,d,w,z) consisting of an object C and morphisms $d\colon C \to A \otimes C \otimes C$, $w\colon C \to A$ and $z\colon C \otimes A \to A \otimes C$ satisfying the string diagram conditions of Definition 3.3 with $T,\mu,\eta,G,\delta,\varepsilon,\zeta$ replaced by A,m,j,C,d,w,z, respectively. Moreover, since mixed opwreaths are defined purely in terms of the monoidal structure, each mixed opwreath (C,d,w,z) around A in $\mathscr{V}$ defines gives rise to a mixed opwreath $(G,\delta,\varepsilon,\zeta)$ around the monad $T = A\otimes-$. Furthermore, we write* $\mathrm{mkl}(C,z,A)$ *for the category* $\mathrm{mkl}(G,\zeta,T)$*.*

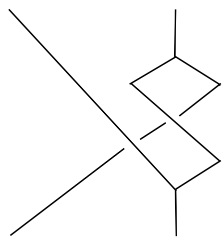
2. *For any monoidal category $\mathscr{V}$, the functor $\mathscr{V}^{\mathrm{rev}} \to [\mathscr{V}, \mathscr{V}]$, taking X to $- \otimes X$, is also strong monoidal. Thus the mixed opwreath around A as in Item 1 is taken to a mixed wreath around the monad $- \otimes A$ on $\mathscr{V}$. We write* $\mathrm{mem}(C, z, A)$ *for the mixed Eilenberg-Moore construction* (3.13) *applied to this mixed wreath.*

Definition 3.7. *The set* $\mathrm{mkl}(C, z, A)(I, I) \cong \mathscr{V}(C, A)$ *of endomorphisms of I in the category* $\mathrm{mkl}(C, z, A)$ *is, of course, a monoid under composition. The multiplication might be called z-*parametrized convolution *on $\mathscr{V}(C, A)$.*

Example 3.8 (The Heisenberg category)**.** *Suppose $A = (A, m, j, c, e)$ is a bimonoid in the braided monoidal category $\mathscr{V}$. Denote the braiding by σ. We obtain a mixed opwreath on the monoid $A = (A, m, j)$ (indeed it is a mixed opdistributive law) in $\mathscr{V}$ by taking the comonoid C to be $A = (A, c, e)$, z to be*

$$
\begin{aligned}
z_{\mathrm{h}} &= \left(A \otimes A \xrightarrow{1_A \otimes \sigma^{-1}_{A,A} c} A \otimes A \otimes A \xrightarrow{\sigma^{-1}_{A,A} \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes m} A \otimes A \right) \\
&= \left(A \otimes A \xrightarrow{1_A \otimes c} A \otimes A \otimes A \xrightarrow{\sigma^{-1}_{A,A\otimes A}} A \otimes A \otimes A \xrightarrow{1_A \otimes m} A \otimes A \right) ,
\end{aligned}
$$

d to be $j \otimes c \colon A \to A \otimes A \otimes A$, and w to be $j \circ e \colon A \to A$.



We put $\mathrm{Hb}(A) = \mathrm{mkl}(A, z_{\mathrm{h}}, A)$ *and call it the* Heisenberg category *of the bimonoid A in $\mathscr{V}$. Here is the reason.*

Proposition 3.9. Suppose $\mathscr{V} = \mathrm{Vect}$ is the symmetric monoidal category of vector spaces over a fixed field and H is a Hopf algebra, then the z_{h}-parametrized convolution of linear endomorphisms of H is the Heisenberg product as defined in [16, 1].

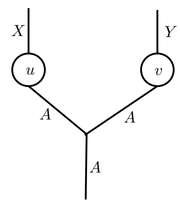
4 Twisted coactions

The construction of a mixed opwreath explained in this section is a dual of the wreath construction appearing as Example 3.3 in [14] based on Sweedler's crossed product of Hopf algebras.

We begin by pointing out that, given a monoid A in any monoidal category $\mathscr{V}$, the representable functor $\mathscr{V}(-, A) \colon \mathscr{V}^{\mathrm{op}} \to \mathrm{Set}$ becomes monoidal when equipped with the natural family of functions

$$
\mathscr{V}(X, A) \times \mathscr{V}(Y, A) \longrightarrow \mathscr{V}(X \otimes Y, A) , \tag{4.16}
$$

defined by $(u, v) \mapsto u \bullet v$ as depicted below, and $\eta_A \in \mathscr{V}(I, A)$. The reason is that the Yoneda embedding $\mathscr{V} \to [\mathscr{V}^{\mathrm{op}}, \mathrm{Set}]$ is monoidal, where monoids in the codomain are precisely monoidal functors.



Here are the properties of the dot product:

$\eta_A \bullet u = u = u \bullet \eta_A$ *(unitality)*

$(u \bullet v) \bullet w = u \bullet (v \bullet w)$ (associativity)

$(u \bullet v) \circ (f \otimes g) = (u \circ f) \bullet (v \circ g)$ *(naturality)*

We also recall that, if $\mathscr{V}$ is (lax) braided then the tensor product $A \otimes B$ of monoids A and B is again a monoid: the (lax) braiding gives a distributive law of A over B used in defining $\mu_{A\otimes B}$ in terms of the multiplications μ_A and μ_B.

Definition 4.1. *Let A be a monoid and B be a bimonoid in a (lax) braided monoidal category $\mathscr{V}$. A* twisted (right) coaction *of B on A consists of a monoid morphism $\gamma\colon A \to A \otimes B$ and a morphism $\tau\colon I \to A \otimes B^{\otimes 2}$ such that*

$(1_A \otimes \varepsilon_B) \circ \gamma = 1_A$ (counitality)

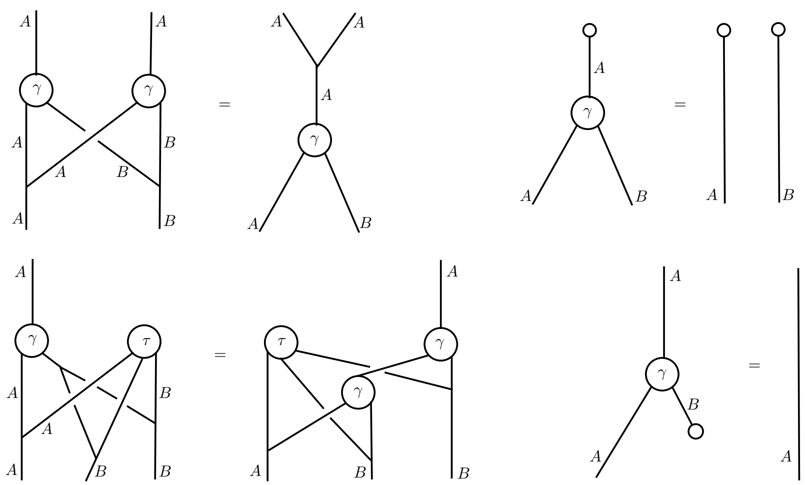
$\tau \bullet ((\gamma \otimes 1_B) \circ \gamma) = ((1_A \otimes \delta_B) \circ \gamma) \bullet \tau$ *(τ-coassociativity)*

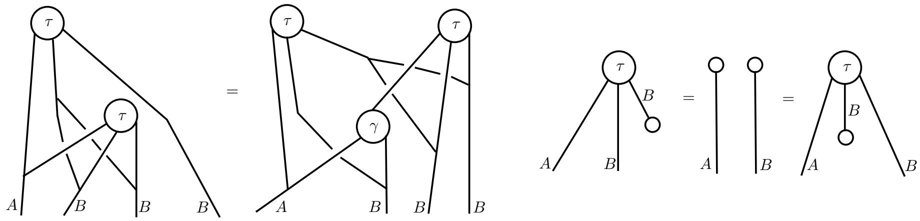
$((1_A \otimes \delta_B \otimes 1_B) \circ \tau) \bullet (\tau \otimes \eta_B) = ((1_{A\otimes B} \otimes \delta_B) \circ \tau) \bullet ((\gamma \otimes 1_{B\otimes B}) \circ \tau))$ (2-cocyclicity)

$(1_{A\otimes B} \otimes \varepsilon_B) \circ \tau = \eta_A \otimes \eta_B = (1_A \otimes \varepsilon_B \otimes 1_B) \circ \tau$ *(normality)* .

Note that we do not require τ to be $\bullet$-invertible.

Here in string form are the conditions on a twisted coaction.



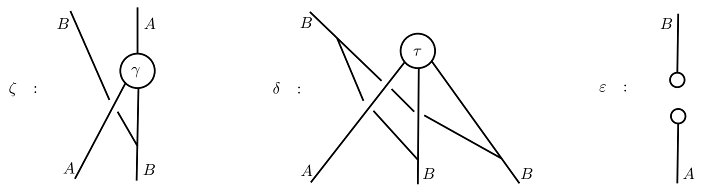


Proposition 4.2. Given a twisted coaction of a monoid A on a bimonoid B in a braided monoidal category $\mathscr{V}$, using the notation of Definition 4.1, a mixed opwreath around A is defined by the comonoid B equipped with the morphisms

$$\zeta = (\eta_A \otimes 1_B) \bullet \gamma \colon B \otimes A \longrightarrow A \otimes B$$
$$\delta = (\eta_A \otimes \delta_B) \bullet \tau \colon B \longrightarrow A \otimes B \otimes B$$
$$\varepsilon = \eta_A \circ \varepsilon_B \colon B \longrightarrow A$$

as required by Definition 3.3.

Here are the string diagrams for these ζ, δ and ε.



In proving Proposition 4.2, a lemma will be useful.

Lemma 4.3. The following equations hold:

(i) $\delta_B \bullet \delta_B = \delta_B \circ \mu_B$

(ii) $(1 \otimes \delta_B) \bullet \tau = (1_A \otimes \eta_B \otimes \eta_B) \bullet \delta$

(iii) $(\eta_A \otimes \delta_B) \bullet ((1 \otimes \delta_B) \circ \gamma) = (1 \otimes \delta_B) \circ \zeta$

Proof. Item (i) is a restatement of the bimonoid axiom for B asserting that μ_B preserves comultiplication. Item (ii) is immediate on drawing the string diagrams. Item (iii) is immediate from the string diagrams and using the bimonoid axiom. Q.E.D.

Here now are some clues on proving Proposition 4.2. There are seven conditions satisfied by the twisted coaction. There are seven axioms to verify for the mixed opwreath. For condition 1, we can express the fact that γ preserves multiplication in the form $\gamma \circ \mu_A = \gamma \bullet \gamma$, then dot both sides on the left with $\eta_A \otimes 1_B$. Condition 2 follows by dotting on both sides by $\eta_A \otimes 1_B$ the equation expessing the fact that γ preserves unit. Condition 3 follows from counitality of γ and the bimonoid condition that μ_B preserves counit. Condition 4 is obtained by dotting both sides of the τ-coassociativity equation on the left by $\eta_A \otimes \delta_B$ and employing Lemma 4.3. Condition 5 follows by dotting both sides of the cocyclicity condition by $\eta_A \otimes \mu_{B3}$, where μ_{B3} is the ternary multiplication $\mu_B \circ (\mu_B \otimes 1_B) = \mu_B \circ (1_B \otimes \mu_B)$, and employing Lemma 4.3. Unsurprisingly by now, conditions 6 and 7 follow from the two equations of normality and that μ_B preserves counit.

Remark 4.4. *If the 2-cocycle τ has the form $\eta_A \otimes \tau'$ for some $\tau' \colon I \to B \otimes B$ then the mixed opwreath of Proposition 4.2 is a mixed opdistributive law.*

There is a 2-categorical viewpoint on twisted coactions. Recall (see [10] or Chapter 15 of [22], for example) that the category $\mathrm{Mon}\mathscr{V}$ of monoids in the braided monoidal category $\mathscr{V}$ is a monoidal 2-category. If $f, g \colon M \to N$ are monoid morphisms then a 2-cell $\xi \colon f \Rightarrow g \colon M \to N$ is a morphism $\xi \colon I \to N$ in $\mathscr{V}$ satisfying the *naturality condition* $\xi \bullet f = g \bullet \xi$. The vertical composite of ξ with a 2-cell $\zeta \colon g \Rightarrow h \colon M \to N$ is $\zeta \bullet \xi$. The horizontal composite of ξ with a 2-cell $\xi' \colon f' \Rightarrow g' \colon N \to L$ is $\xi' \bullet (f' \circ \xi) = (g' \circ \xi) \bullet \xi'$. The tensor product in $\mathrm{Mon}\mathscr{V}$ is the tensor product of monoids that we have been dealing with already (it uses the braiding of $\mathscr{V}$ yet is not itself a braided tensor product unless $\mathscr{V}$ is symmetric).

Now we can think of our bimonoid B as a comonoid in the 2-category $\mathrm{Mon}\mathscr{V}$.

Proposition 4.5. A twisted right coaction of the bimonoid B on the monoid A in $\mathscr{V}$ is precisely a normal lax right coaction of the comonoid B on the object A in the 2-category $\mathrm{Mon}\mathscr{V}$.

Proof. We need to see what is involved in a normal lax right coaction. Indeed we have a morphism $\gamma \colon A \to A \otimes B$ in $\mathrm{Mon}\mathscr{V}$, as required. We have a 2-cell

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\overset{\tau}{\Rrightarrow}"] & A \otimes B \arrow[d, "1 \otimes \delta_B"] \\ A \otimes B \arrow[r, "\gamma \otimes 1"'] & A \otimes B \otimes B \end{tikzcd} \tag{4.17}$$

in $\mathrm{Mon}\mathscr{V}$; the 2-cell condition is precisely τ-coassociativity. A lax coaction also involves a 2-cell $\tau_0 \colon 1_A \Rightarrow (1_A \otimes \varepsilon) \circ \gamma$ however the normality condition is that this should be an identity; this precisely amounts to counitality. The axioms on τ for a lax coaction are precisely cocyclicity and normality. Q.E.D.

We now remind the reader of the role that variants of the (algebraist's) simplicial category $\mathbf{\Delta}$ play as host to generic monoids, comonoids, actions and coactions (see [15]). We write $\mathbf{\Delta}_{\perp,\top}$ for the strict monoidal category whose objects are the strictly positive finite ordinals, whose morphisms are order and first-and-last-element preserving functions; the tensor product is $m \oplus n = m + n - 1$ thought of, for the purposes of the value at morphisms, as identifying the last element of m with the first element of n. Similarly, $\mathbf{\Delta}_\top$ denotes the category whose objects are the strictly positive finite ordinals, whose morphisms are order and last-element preserving functions. There is a strict right action

$$\oplus \colon \mathbf{\Delta}_\top \otimes \mathbf{\Delta}_{\perp,\top} \longrightarrow \mathbf{\Delta}_\top \tag{4.18}$$

of $\mathbf{\Delta}_{\perp,\top}$ on $\mathbf{\Delta}_\top$ defined by $m \oplus n = m + n - 1$ as before except that on morphisms, the left morphism in the operation need not preserve the first element, so the result may not either.

Here is a picture of some generating morphisms of $\mathbf{\Delta}_\top$.

$$\mathbf{\Delta}_\top \ : \qquad \begin{tikzcd} 1 \arrow[r, shift left, "\partial_0"] & 2 \arrow[l, shift left, "\sigma_0"] \arrow[r, shift left=3, "\partial_0"] \arrow[r, shift right, "\partial_1"] & 3 \arrow[l, shift right=1, "\sigma_0"'] \arrow[l, shift left=3, "\sigma_1"] \arrow[r, shift left=5, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=5, "\partial_2"] & \cdots \arrow[l, shift left=3, "\sigma_0"] \arrow[l, shift right=2, "\sigma_1"] \arrow[l, shift right=7, "\sigma_2"] \end{tikzcd} \tag{4.19}$$

The corresponding picture for $\mathbf{\Delta}_{\perp,\top}$ is obtained by deleting all the morphisms labelled ∂_0. There is a canonical inclusion $\mathbf{\Delta}_{\perp,\top} \rightarrowtail \mathbf{\Delta}_{\top}$ which respects the right actions by $\mathbf{\Delta}_{\perp,\top}$. The corresponding picture for $\mathbf{\Delta}$ is obtained by adjoining the object 0 and morphisms $\partial_n \colon n \to (n+1)$. There is a canonical inclusion $\mathbf{\Delta}_{\top} \rightarrowtail \mathbf{\Delta}$. Moreover, $\mathbf{\Delta}^{\mathrm{op}} \cong \mathbf{\Delta}_{\perp,\top}$.

A comonoid $B = (B, \delta_B, \varepsilon_B)$ in a monoidal category $\mathscr{W}$ defines a strong monoidal functor $\bar{B} \colon \mathbf{\Delta}_{\perp,\top} \longrightarrow \mathscr{W}$ whose value at n is $B^{\otimes(n-1)}$, whose value at $\sigma_r \colon (n+1) \to n$ is

$$\sigma_r = 1_{B^{\otimes r}} \otimes \varepsilon_B \otimes 1_{B^{\otimes(n-r-1)}} \colon B^{\otimes n} \to B^{\otimes(n-1)} \ , \tag{4.20}$$

and whose value at $\partial_r \colon n \to (n+1)$ is

$$\partial_r = 1_{B^{\otimes r}} \otimes \delta_B \otimes 1_{B^{\otimes(n-r-2)}} \colon B^{\otimes(n-1)} \to B^{\otimes n} \ . \tag{4.21}$$

In fact this gives an equivalence of categories implying that, up to isomorphism, the comonoid B can be recaptured from the strong monoidal functor.

Suppose $\mathscr{W}$ acts on a category $\mathscr{A}$ via a functor $\star \colon \mathscr{A} \times \mathscr{W} \to \mathscr{A}$. The comonoid B in $\mathscr{W}$ defines a comonad $- \star B$ on $\mathscr{A}$. We define a *right action* of B on an object $A \in \mathscr{A}$ to be the structure $\gamma \colon A \to A \star B$ of an Eilenberg-Moore $(- \star B)$-coalgebra on A. There is a functor $\bar{A} \colon \mathbf{\Delta}_{\top} \longrightarrow \mathscr{A}$ whose value at the object n is $A \star B^{\otimes(n-1)}$, whose value at $\partial_0 \colon n \to (n+1)$ is

$$\gamma \otimes 1_{B^{\otimes(n-1)}} \colon A \star B^{\otimes(n-1)} \longrightarrow A \star B^{\otimes n} \ ,$$

and whose value at the other morphisms ∂_{r+1} in (4.19) is $\partial_{r+1} = 1_A \star \partial_r$ where ∂_r comes from $\bar{B}$. Then $\bar{A}$ and $\bar{B}$ comprise an action morphism.

$$\begin{tikzcd}
\mathbf{\Delta}_{\top} \times \mathbf{\Delta}_{\perp,\top} \arrow[r, "\oplus"] \arrow[d, "\bar{A}\times\bar{B}"'] & \mathbf{\Delta}_{\top} \arrow[d, "\bar{A}"] \\
\mathscr{A} \times \mathscr{W} \arrow[r, "\star"'] & \mathscr{A}
\end{tikzcd} \tag{4.22}$$

Again, this is part of an equivalence of categories between action morphisms and pairs (A, B).

This is all standard material, albeit maybe not explicitly in the above dual version.

Now suppose $\mathscr{A}$ is a 2-category and the action $\star \colon \mathscr{A} \times \mathscr{W} \to \mathscr{A}$ corresponds to a functor $\mathscr{W} \to [\mathscr{A}, \mathscr{A}]$ into the 2-functor 2-category. Suppose $A \in \mathscr{A}$ has merely a morphism $\gamma \colon A \to A \star B$. We can define $\bar{A}$ on objects and generating morphisms as before but it is not quite a functor.

Proposition 4.6. A normal lax $(- \star B)$-coalgebra structure on $\gamma \colon A \to A \star B$ amounts to a normal lax functor structure on $\bar{A} \colon \mathbf{\Delta}_{\top} \to \mathscr{A}$ which has its constraints $\bar{A}(\zeta) \circ \bar{A}(\xi) \to \bar{A}(\zeta \circ \xi)$ identities unless neither ξ nor ζ is in $\mathbf{\Delta}_{\perp,\top}$. In particular, $\bar{A}$ restricts along the inclusion $\mathbf{\Delta}_{\perp,\top} \rightarrowtail \mathbf{\Delta}_{\top}$ to a strict functor, that is, a simplicial object of $\mathscr{A}$.

Proposition 4.7. In the situation of Proposition 4.6, suppose A is pointed by a morphism $\eta_A \colon I \to A$ in $\mathscr{A}$, the comonoid B is pointed by a comonoid morphism $\eta_B \colon I \to B$ in $\mathscr{W}$, and $\gamma \colon A \to A \star B$ respects the pointings, then each lax functor $\bar{A} \colon \mathbf{\Delta}_{\top} \to \mathscr{A}$ extends along the inclusion $\mathbf{\Delta}_{\top} \rightarrowtail \mathbf{\Delta}$ to a lax functor $\hat{A} \colon \mathbf{\Delta} \to \mathscr{A}$ by defining $\hat{A}(0) = I$, $\partial_0 = \eta_A \colon I \to A$, and

$$\partial_n \colon = 1_{A \otimes B^{\otimes(n-1)}} \otimes \eta_B \colon A \otimes B^{\otimes(n-1)} \to A \otimes B^{\otimes n} \ .$$

In particular, for braided monoidal $\mathscr{V}$ and $\mathscr{W} = \mathscr{A} = \mathrm{Mon}\mathscr{V}$ (with the action on itself by its own tensor product), each twisted coaction of a bimonoid B on a monoid A determines a slightly lax (augmented) cosimplicial monoid $\hat{A}$ in $\mathscr{V}$ with $\hat{A}(n) = A \otimes B^{\otimes(n-1)}$. Our terminology that τ is a normalized 2-cocycle is justified by the formulas

$$(\partial_1\tau) \bullet (\partial_3\tau) = (\partial_2\tau) \bullet (\partial_0\tau) \quad , \quad \sigma_1\tau = 1 = \sigma_0\tau \ .$$

5 Monoidality

The basis of this section is the pioneering work of Day [3, 4].

Suppose $T = (T, \mu, \eta)$ is a monoidal monad on the monoidal category $\mathscr{A}$. Then the Kleisli category $\mathscr{A}_T$ is canonically monoidal: on objects, which are the same as for $\mathscr{A}$, the tensor product is that of $\mathscr{A}$; on homs it is equal to

$$\mathscr{A}_T(X,Y) \times \mathscr{A}_T(X',Y') = \mathscr{A}(X,TY) \times \mathscr{A}(X',TY') \xrightarrow{\otimes} \mathscr{A}(X \otimes X', TY \otimes TY')$$
$$\xrightarrow{\mathscr{A}(1_{X\otimes X'},\varphi_{Y,Y'})} \mathscr{A}(X \otimes X', T(Y \otimes Y')) = \mathscr{A}_T(X \otimes X', Y \otimes Y') \ .$$

The canonical functor $\mathscr{A} \to \mathscr{A}_T$ is strict monoidal.

Definition 5.1. *A mixed opwreath $(G, \zeta, \delta, \varepsilon)$ around the monoidal monad T on $\mathscr{A}$ (see Definition 3.3) is* opmonoidal *when the lifted comonad $\bar{G} = (\bar{G}, \bar{\delta}, \bar{\varepsilon})$ on $\mathscr{A}_T$ is equipped with opmonoidal structure.*

By the dual of the fact that Kleisli categories of monoidal monads are canonically monoidal, for an opmonoidal opwreath, the category $\mathrm{mkl}(G, \zeta, T)$ is canonically monoidal since it is the Kleisli category for the opmonoidal comonoid $\bar{G}$ on $\mathscr{A}_T$.

Let us spell out the data and axioms involved in Definition 5.1, and the monoidal structure on $\mathrm{mkl}(G, \zeta, T)$.

The data are morphisms

$$\psi_{X,X'} \colon G(X \otimes X') \longrightarrow T(GX \otimes GX') \tag{5.23}$$

indexed by pairs of objects $X, X' \in \mathscr{A}$. There are six axioms.

$$\begin{tikzcd}
G(X \otimes X') \arrow[r, "\psi_{X,X'}"] \arrow[d, "G(f\otimes f')"'] & T(GX \otimes GX') \arrow[r, "T(Gf\otimes Gf')"] & T(GTY \otimes GTY') \arrow[d, "T(\zeta_Y\otimes\zeta_{Y'})"] \\
G(TY \otimes TY') \arrow[d, "G\varphi_{Y,Y'}"'] & & T(TGY \otimes TGY') \arrow[d, "T\varphi_{GY,GY'}"] \\
GT(Y \otimes Y') \arrow[d, "\zeta_{Y\otimes Y'}"'] & & TT(GY \otimes GY') \arrow[d, "\mu_{GY\otimes GY'}"] \\
TG(Y \otimes Y') \arrow[r, "T\psi_{Y,Y'}"'] & TT(GY \otimes GY') \arrow[r, "\mu_{GY\otimes GY'}"'] & T(GY \otimes GY')
\end{tikzcd} \tag{5.24}$$

$$\begin{tikzcd}
G(X\otimes X'\otimes X'') \arrow[r, "\psi_{X\otimes X',X''}"] \arrow[d, "\psi_{X,X'\otimes X''}"'] & T(G(X\otimes X')\otimes GX'') \arrow[d, "T(\psi_{X,X'}\otimes\eta_{GX''})"] \\
T(GX\otimes G(X'\otimes X'')) \arrow[d, "T(\eta_{GX}\otimes\psi_{X',X''})"'] & T(T(GX\otimes GX')\otimes TGX'') \arrow[d, "T\varphi_{GX\otimes GX',GX''}"] \\
T(TGX\otimes T(GX'\otimes GX'')) \arrow[d, "T\varphi_{GX,GX'\otimes GX''}"'] & TT(GX\otimes GX'\otimes GX'') \arrow[d, "\mu_{GX\otimes GX'\otimes GX''}"] \\
TT(GX\otimes GX'\otimes GX'') \arrow[r, "\mu_{GX\otimes GX'\otimes GX''}"'] & T(GX\otimes GX'\otimes GX'')
\end{tikzcd} \tag{5.25}$$

$$\begin{tikzcd}
& T(TGX\otimes TI) \arrow[dr, "T\varphi_{GX,I}"] & \\
T(GX\otimes GI) \arrow[ur, "T(\eta_{GX}\otimes\varepsilon_I)"] & & TTGX \arrow[d, "\mu_{GX}"] \\
GX \arrow[u, "\psi_{X,I}"] \arrow[rr, "\eta_{GX}"'] & & TGX
\end{tikzcd} \tag{5.26}$$

$$\begin{tikzcd}
& T(TI\otimes TGX) \arrow[dr, "T\varphi_{I,GX}"] & \\
T(GI\otimes GX) \arrow[ur, "T(\varepsilon_I\otimes\eta_{GX})"] & & TTGX \arrow[d, "\mu_{GX}"] \\
GX \arrow[u, "\psi_{I,X}"] \arrow[rr, "\eta_{GX}"'] & & TGX
\end{tikzcd} \tag{5.27}$$

$$\begin{tikzcd}
& TGG(X\otimes X') \arrow[dr, "TG\psi_{X,X'}"] & \\
G(X\otimes X') \arrow[ur, "\delta_{X\otimes X'}"] \arrow[d, "\psi_{X,X'}"'] & & TGT(GX\otimes GX') \arrow[d, "T\zeta_{GX\otimes GX'}"] \\
T(GX\otimes GX') \arrow[d, "T(\delta_X\otimes\delta'_X)"'] & & TTG(GX\otimes GX') \arrow[d, "TT\psi_{GX,GX'}"] \\
T(TGGX\otimes TGGX') \arrow[d, "T\varphi_{GGX,GGX'}"'] & & TTT(GGX\otimes GGX') \arrow[d, "T\mu_{GGX\otimes GGX'}"] \\
TT(GGX\otimes GGX') \arrow[dr, "\mu_{GGX\otimes GGX'}"'] & & TT(GGX\otimes GGX') \arrow[dl, "\mu_{GGX\otimes GGX'}"] \\
& T(GGX\otimes GGX') &
\end{tikzcd} \tag{5.28}$$

$$\begin{tikzcd}
& T(TX\otimes TX') \arrow[dr, "T\varphi_{X,X'}"] & \\
T(GX\otimes GX') \arrow[ur, "T(\varepsilon_X\otimes\varepsilon_{X'})"] & & TT(X\otimes X') \arrow[d, "\mu_{X\otimes X'}"] \\
G(X\otimes X') \arrow[u, "\psi_{X,X'}"] \arrow[rr, "\varepsilon_{X\otimes X'}"'] & & T(X\otimes X')
\end{tikzcd} \tag{5.29}$$

Diagram (5.24) expresses the naturality of ψ. Diagrams (5.25), (5.26), (5.27) express the opmonoidality of $\bar{G}$ when equipped with ψ. Diagrams (5.28), (5.29) express the opmonoidality of $\bar{\delta}$, $\bar{\varepsilon}$, respectively. One of the other conditions is that the nullary piece ψ_0 of opmonoidal structure on $\bar{G}$ must be ε_I. This means that the nullary conditions for $\bar{\delta}$, $\bar{\varepsilon}$ to be opmonoidal are automatically satisfied; the less trivial of these is the former, which amounts to the Diagram (5.30), yet that follows using 3 and 6 for a mixed opwreath.

$$\begin{tikzcd}
TGTI \arrow[r, "T\zeta_I"] & TTGI \arrow[r, "TT\varepsilon_I"] & TTTI \arrow[d, "\mu_{TI}"] \\
TGGI \arrow[u, "TG\varepsilon_I"] & & TTI \arrow[d, "\mu_I"] \\
GI \arrow[u, "\delta_I"] \arrow[rr, "\varepsilon_I"'] & & TI
\end{tikzcd} \tag{5.30}$$

Now we come to the monoidal structure on $\mathrm{mkl}(G,\zeta,T)$. The tensor product of two objects X, X' is the tensor product $X \otimes X'$ of the objects as objects of $\mathscr{A}$. The tensor product of morphisms $f\colon X \to Y$ and $f'\colon X' \to Y'$ in $\mathrm{mkl}(G,\zeta,T)$, which are morphisms $f\colon GX \to TY$ and $f'\colon GX' \to TY'$ in $\mathscr{A}$, is the composite

$$G(X \otimes X') \xrightarrow{\psi_{X,X'}} T(GX \otimes GX') \xrightarrow{T(f\otimes f')} T(TY \otimes TY') \\ \xrightarrow{T\varphi_{Y,Y'}} TT(Y \otimes Y') \xrightarrow{\mu_{Y\otimes Y'}} T(Y \otimes Y').$$

The unit of this monoidal structure is the unit I of $\mathscr{A}$. The Eckmann-Hilton argument [5] yields:

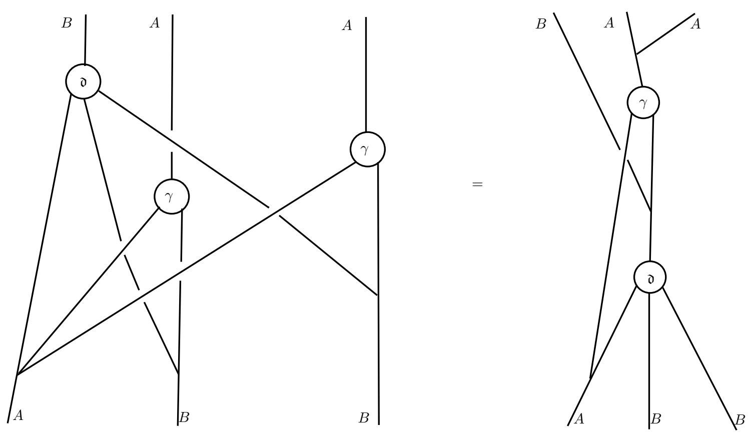
Corollary 5.2. If $(G,\zeta,\delta,\varepsilon)$ is an opmonoidal mixed opwreath around the monoidal monad T on $\mathscr{A}$ then the monoid $\mathrm{mkl}(G,\zeta,T)(I,I)$ of endomorphisms of I is commutative.

6 Monoidal twisted coactions

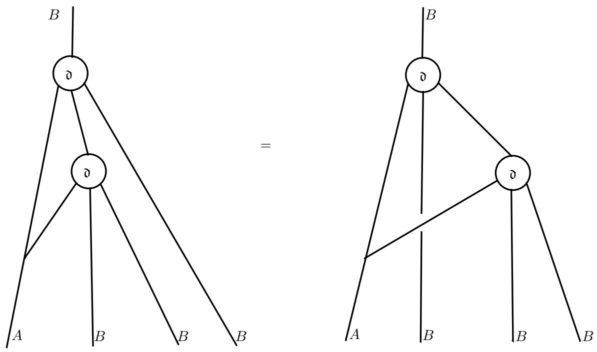
In this section we will show what structure on a twisted coaction leads to opmonoidality of the generated mixed opwreath of Proposition 4.2.

We work in a braided monoidal category $\mathscr{V}$.

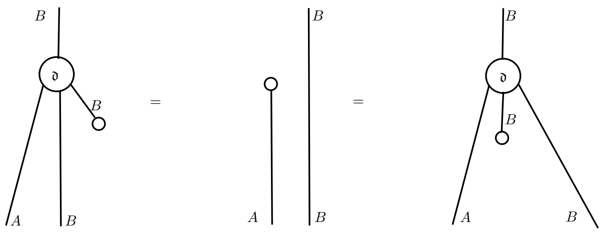
Definition 6.1. *A twisted coaction (γ,τ) of a bimonoid B on a monoid A (see Definition 4.1) is* monoidal *when it is equipped with a morphism $\mathfrak{d}\colon B \to A\otimes B\otimes B$ which satisfies the five conditions* (6.31) *to* (6.34).



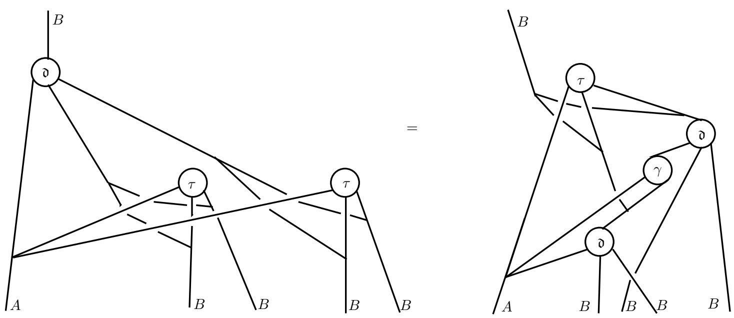
(6.31)



(6.32)



(6.33)



(6.34)

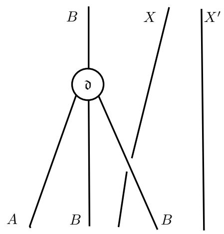
Proposition 6.2. Let $\mathscr{V}$ be a braided monoidal category. Given a monoidal twisted coaction (γ, τ) of a bimonoid B on a commutative monoid A (see Definition 4.1), the mixed opwreath described in Proposition 4.2, equipped with the morphisms

$$\psi_{X,X'} = (1_A \otimes 1_B \otimes c^{-1}_{X,B} \otimes 1_{X'}) \circ (\mathfrak{d} \otimes 1_X \otimes 1_{X'}) \ ,$$

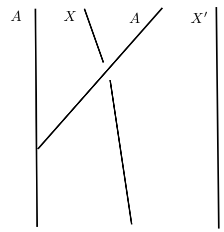
is opmonoidal.

Proof. Since A is commutative, the arising monad $T = A \otimes -$ is monoidal with $\varphi_0 = \eta \otimes 1_I$ and $\varphi_{X,X'} = \mu \circ (1_A \otimes c_{X,A} \otimes 1_{X'})$. Here are the string diagrams for ψ and φ.

$\psi_{X,X'} \quad =$



$\varphi_{X,X'} \quad =$



(6.35)

With these data, and that of Proposition 4.2, draw the string diagrams for Diagrams (5.24) to (5.29). The remarkable fact is that the variables f and f' can be moved out of the top of Diagrams (5.24), while X and X' can be disconnected from all the diagrams. Diagram (5.29) follows from Diagram (6.33). Then Diagrams (6.31) to (6.34) are what remains. So indeed we obtain an opmonoidal mixed opwreath. Q.E.D.

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