On fractional order Mellin transform and some of its properties

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Abstract

In this work, we introduce fractional Mellin transform of order α, $0 < \alpha \leq 1$ on a function which belongs to the Lizorkin space. Further, some properties and applications of fractional Mellin transform are given.

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1 Introduction

Integral transform appears in many fields of applied mathematics, physics and an engineering. There are several kinds of integral transforms and their applications, see for example [26, 27, 28]. In our study we focus and deal with Mellin transform which will mention as the following section.

1.1 Mellin transform

Mellin transform is closely related to the Laplace and Fourier transforms. In this section, we recall the definition of Mellin transform and the relation with Laplace transform is given. Further we also provide the relation between Mellin transform and Fourier transform with positive image [1, 13].

Definition 1.1. Suppose that f is a complex-valued and locally integrable function which defined over $(0, \infty)$. To avoid further complication, we assume throughout that it is continuous in $(0, \infty)$. The Mellin transform can be defined as

$$\mathcal{M}[f(t); s] = F(s) = \int_0^\infty t^{s-1} f(t) dt. \tag{1.1}$$

Generally, the integral exist only for complex values of $s = a + ib$, such that $a_1 < a < a_2$, where a_1 and a_2 depend on the function f.

In the above definition a_1 and a_2 form the so-called strip of definition $St\,(a_1, a_2)$.
In some statuses , the strip may be extended to a half-plane $(a_1 = -\infty \ \text{ or } \ a_2 = \infty)$ or to the whole complex s-plane $(a_1 = -\infty \ \text{ and } \ a_2 = \infty)$, see [6].

Inversion formula of Mellin transform is given as :

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Definition 1.2. [8] Let $f(x)$ be integrable with fundamental strip $St(\alpha, \beta)$. If a is such that $\alpha < a < \beta$ and $F(s = a + it) = \mathcal{M}\left[f(x); s\right]$ is integrable, then the equality

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}\left[f(x); s\right] x^{-s} ds,$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then the equality holds everywhere on $(0, +\infty)$.

Further information regarding Mellin transform can be found in [3, 4, 5, 7, 9].

Definition 1.3. [11] Fourier transform can be defined on a function f of real variable $x \in (-\infty, \infty)$ as

$$\hat{f}(\beta) = \mathscr{F}[f(x); \beta] = \int_{-\infty}^{\infty} f(x) e^{2\pi i \beta x} dx, \quad x \in \mathbb{R}. \tag{1.2}$$

The inverse Fourier transform can be defined as the following

$$f(x) = \mathscr{F}^{-1}\mathscr{F}[f(x); \beta] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\beta) e^{-2\pi i \beta x} d\beta, \quad x \in (-\infty, \infty). \tag{1.3}$$

For more details see [2, 16, 17, 21, 22].

Mellin transform and relation with Laplace, Fourier transformations

The Mellin transform is closely related to an extended form of Laplace transform. By taking substitute of the variable $t = e^{-x}, dt = -e^{-x}dx$ in(1.1), then the integral (1.1) transforms into

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x}) e^{-sx} dx, \tag{1.4}$$

After replacement of the function

$$g(x) = f(e^{-x})$$

Laplace transform of function $g(x)$ can be written as

$$\mathscr{L}\left[g(x); s\right] = \int_{-\infty}^{\infty} g(x) e^{-sx} dx.$$

Therefore

$$\mathcal{M}[f(t); s] = \mathscr{L}[(e^{-x}); s], \quad s = a + ib.$$

Now, writing $s = a - 2\pi i \beta$ in (1.4) to obtain Fourier's transform

$$\begin{aligned} F(s) = \mathcal{M}[f(t); s] &= \int_{-\infty}^{\infty} f(e^{-x}) e^{-x(a-2\pi i\beta)} dx \\ &= \int_{-\infty}^{\infty} f(e^{-x}) e^{-ax} e^{2\pi i \beta x} dx. \end{aligned} \tag{1.5}$$

Then we get

$$\mathcal{M}[f(t); a - 2\pi i\beta] = \mathscr{F}[f(e^{-x})e^{-ax}; \beta]. \tag{1.6}$$

Hence, for a given value $a \in St(a_1, a_2)$, we can express Mellin transform of a function f as a certain Fourier transform.

Note that, Fourier transform here is defined in the usual way as

$$\hat{f}(\beta) = \mathscr{F}\left[f(x); \beta\right] = \int_{-\infty}^{\infty} f(x)e^{2\pi i\beta x}dx. \tag{1.7}$$

A direct way to obtain inverse Mellin transformation is by the inverse Fourier transformation, is that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\beta)e^{-2\pi i\beta x}d\beta.$$

Now, using equation (1.5), and above equation we obtain

$$f(e^{-x})e^{-ax} = \int_{-\infty}^{\infty} F(s)e^{-2\pi i\beta x}d\beta \quad with \quad s = a - 2\pi i\beta.$$

By going back to the change of variables $t = e^{-x}$ and $s = a - 2\pi i\beta$, we have

$$f(t)t^a = \int_{-\infty}^{\infty} F(s)t^{2\pi i\beta}d\beta,$$

$$f(t) = \int_{-\infty}^{\infty} F(s)t^{-a+2\pi i\beta}d\beta = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)t^{-s}ds.$$

From above formula we can see that the inverse Mellin transform is obtained.
Otherwise, if we take $\omega = 2\pi\beta$ then equation (1.7) turns to the following equality

$$\hat{f}(\omega) = \mathscr{F}\left[f(x); \omega\right] = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

then

$$f(x) = \mathscr{F}^{-1}\mathscr{F}\left[f(x); \omega\right] = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega.$$

Hence

$$\mathcal{M}[f(t); a - i\omega] = \mathscr{F}[f(e^{-x})e^{-ax}; \omega], \ For\ all\ a \in St(a_1, a_2).$$

2 α-Mellin transform

Using integral transforms to deal with fractional calculus goes back to Riemann and Liouville [18, 23]. In addition, Local fractional integral transforms have possible applications for science and engineering, see [24, 25]. Recently many researchers studied some properties and applications of Mellin transform in fractional sense [12, 14]. In this section we attempt to put the definition of Mellin transform of order α by using the concept of fractional Fourier transform. First of all, we recall α- Fourier transform definition as the following

2.1 Fractional Fourier transform

A new fractional Fourier transformation was introduced in [10, 15, 20] which defined on the Lizorkin function space, concisely.

Definition 2.1. Let $V(\mathbb{R})$ be the set of functions

$$V(\mathbb{R}) = \left\{ v \in \mathcal{S}(\mathbb{R}) : v^{(k)}(0) = 0, \quad k = 0, 1, 2, ... \right\},$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space.

The Lizorkin space of functions $\Phi(\mathbb{R})$ is defined as

$$\Phi(\mathbb{R}) = \{ f \in \mathcal{S}(\mathbb{R}) : \mathscr{F}(f) \in V(\mathbb{R}) \}.$$

Depending on the definition of Lizorkin space, any function $f \in \Phi(\mathbb{R})$ satisfies the orthogonality conditions $\int_{-\infty}^{\infty} x^k f(x) dx = 0, \quad k = 0, 1, 2, ...$
As in [10, 19], the Lizorkin space is invariant with respect to the fractional integration and differentiation operators. In [20] the authors gave a new fractional Fourier definition as the following.

Definition 2.2. [20] Let $f \in \Phi(\mathbb{R})$ then α- Fourier transform can be defined as

$$\hat{f}_\alpha(\omega) = \mathscr{F}_\alpha[f(x); \omega] = \int_{-\infty}^{\infty} e^{i\omega^{\frac{1}{\alpha}} x} f(x) dx, \quad \omega > 0, \tag{2.1}$$

where $0 < \alpha \leq 1$.

It is clear that if $\alpha = 1$, then α-Fourier transform (2.1) reduces to classical Fourier transform (1.2).

The inverse α - Fourier transform is defined as

$$f(x) = \mathscr{F}_\alpha^{-1} \left(\hat{f}_\alpha(\omega) \right) = \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} \hat{f}_\alpha(\omega) e^{-i\omega^{\frac{1}{\alpha}} x} \omega^{\frac{1-\alpha}{\alpha}} d\omega. \tag{2.2}$$

2.2 Fractional Mellin transform

In this section we use similar steps in the above section, so we satisfy that for every $0 < \alpha \leq 1$,

$$\mathcal{M}_\alpha(f(t); a - i\omega^{\frac{1}{\alpha}}) = \mathscr{F}_\alpha(f(e^{-x}) e^{-ax}; \omega^{\frac{1}{\alpha}}). \tag{2.3}$$

Thus, for every $a \in St(a_1, a_2)$, and $t = e^{-x}, dx = -t^{-1} dt$

$$\begin{aligned} \mathcal{M}_\alpha(f(t); a - i\omega^{\frac{1}{\alpha}}) = \mathcal{M}_\alpha(f(e^{-x}); a - i\omega^{\frac{1}{\alpha}}) &= \int_{-\infty}^{\infty} f(e^{-x}) e^{-ax} e^{i\omega^{\frac{1}{\alpha}} x} dx, \\ &= \int_{-\infty}^{\infty} f(e^{-x}) e^{-(a - i\omega^{\frac{1}{\alpha}})x} dx, \\ &= \int_0^{\infty} f(t) t^{s_\alpha - 1} dt, \end{aligned}$$

for every $0 < \alpha \leq 1, s_\alpha = a - i\omega^{\frac{1}{\alpha}}$ with $a \in St(a_1, a_2)$.

The inverse of fractional Mellin transform can also be obtained from the inverse fractional Fourier transform by using the formula (2.2), and (2.3) with $s_\alpha = a - i\omega^{\frac{1}{\alpha}}$, then we obtain that

$$\begin{aligned} f(e^{-x})e^{-ax} &= \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \mathscr{F}_\alpha\left[f(e^{-x})e^{-ax}; \omega^{\frac{1}{\alpha}}\right] e^{-i\omega^{\frac{1}{\alpha}}x}\omega^{\frac{1-\alpha}{\alpha}}d\omega, \\ &= \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \mathcal{M}_\alpha\left[f(e^{-x}); a - i\omega^{\frac{1}{\alpha}}\right] e^{-i\omega^{\frac{1}{\alpha}}x}\omega^{\frac{1-\alpha}{\alpha}}d\omega, \end{aligned}$$

and taking the change of variable $t = e^{-x}$, *and* $s_\alpha = a - i\omega^{\frac{1}{\alpha}}$ we get

$$f(t)t^a = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha\left[f(t); s_\alpha\right] t^{i\omega^{\frac{1}{\alpha}}} ds_\alpha,$$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha\left[f(t); s_\alpha\right] t^{-a}t^{i\omega^{\frac{1}{\alpha}}} ds_\alpha, \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha\left[f(t); s_\alpha\right] t^{-s_\alpha} ds_\alpha. \end{aligned}$$

Then we have got from the above discussion, fractional Mellin transform and its inverse can be written as the following definition.

Definition 2.3. For every function $f \in \Phi(\mathbb{R})$, then definition of α- Mellin transform is

$$\mathcal{M}_\alpha\left[f(t); s_\alpha\right] = \int_0^\infty f(t)t^{s_\alpha - 1}dt, \quad where \quad s_\alpha = a - iw^{\frac{1}{\alpha}} \quad with \quad a \in St(a_1, a_2). \tag{2.4}$$

The inverse formula can be defined as

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha\left[f(t); s_\alpha\right] t^{-s_\alpha} ds_\alpha. \tag{2.5}$$

2.3 Some properties of the fractional Mellin transform

The following statements are given by using Definition (2.3) on the function $f \in \Phi(\mathbb{R})$

Proposition 2.1. Let a function $f \in \Phi(\mathbb{R})$, and $p \in \mathbb{R}$ then we have

1. Scaling property
$$\mathcal{M}_\alpha\left(f(pt); s_\alpha\right) = p^{-s_\alpha}\mathcal{M}_\alpha\left(f(t); s_\alpha\right),$$

2. Multiplication by t^p
$$\mathcal{M}_\alpha\left(t^p f(t); s_\alpha\right) = \mathcal{M}_\alpha\left(f(t); s_\alpha + p\right),$$

3. Multiplication by a power $0 \leq \beta \leq 1$
$$\mathcal{M}_\alpha\left[(\log t)^\beta f(t); s_\alpha\right] = (1-\beta)!\frac{d^\beta}{ds^\beta}\mathcal{M}_\alpha\left[f(t); s_\alpha\right].$$

Proof. 1. Putting $x = pt,\ dx = pdt$ we obtain

$$\begin{aligned}\mathcal{M}_\alpha\left[f(pt);s_\alpha\right] = \int_0^\infty f(pt)t^{s_\alpha-1}dt &= \int_0^\infty f(x)\left(\frac{x}{p}\right)^{s_\alpha-1}\frac{dx}{p},\\ &= \int_0^\infty f(x)x^{s_\alpha-1}p^{-s_\alpha}dx,\\ &= p^{-s_\alpha}\mathcal{M}_\alpha\left(f(t);s_\alpha\right).\end{aligned}$$

2. The result can get by applying Definition (2.3) as follows

$$\begin{aligned}\mathcal{M}_\alpha\left[t^p f(t);s_\alpha\right] = \int_0^\infty f(t)t^p t^{s_\alpha-1}dt &= \int_0^\infty f(t)t^{p+s_\alpha-1}dt\\ &= \int_0^\infty f(t)t^{(p+s_\alpha)-1}dt = \qquad \mathcal{M}_\alpha\left[f(t);s_\alpha+p\right].\end{aligned}$$

3. Overall, let us mention the next formula which help us to get the result

$$(\frac{d}{ds})^\beta u^{s-1} = \frac{1}{(1-\beta)!}u^{s-1}(\log u)^\beta, \quad 0 \le \beta \le 1,$$

which implies that

$$\begin{aligned}\mathcal{M}_\alpha\left[(\log t)^\beta f(t);s_\alpha\right] &= \int_0^\infty (\log t)^\beta f(t)t^{s_\alpha-1}dt,\\ &= (1-\beta)!\left(\frac{d}{ds}\right)^\beta\int_0^\infty t^{s_\alpha-1}f(t)dt,\\ &= (1-\beta)!\frac{d^\beta}{ds^\beta}\mathcal{M}_\alpha\left[f(t);s_\alpha\right].\end{aligned}$$

Q.E.D.

Proposition 2.2. Let f be a function which $f \in \Phi(\mathbb{R})$, then we obtain

(1) $\mathcal{M}_\alpha[f^{'}(x);s_\alpha] = \int_0^\infty f^{'}(x)x^{s_\alpha-1}dx = -(s_\alpha-1)\mathcal{M}_\alpha[f(x);s_\alpha-1]$,

(2) $\mathcal{M}_\alpha[f^{''}(x);s_\alpha] = \int_0^\infty f^{''}(x)x^{s_\alpha-1}dx = (s_\alpha-1)(s_\alpha-2)\mathcal{M}_\alpha[f(x);s_\alpha-2]$,

(3) $\mathcal{M}_\alpha[xf^{'}(x);s_\alpha] = \int_0^\infty xf^{'}(x)x^{s_\alpha-1}dx = -s_\alpha\mathcal{M}_\alpha[f(x);s_\alpha]$,

(4) $\mathcal{M}_\alpha[x^2f^{''}(x);s_\alpha] = \int_0^\infty x^2f^{''}(x)x^{s_\alpha-1}dx = s_\alpha(s_\alpha+1)\mathcal{M}_\alpha[f(x);s_\alpha]$.

In fact the results can be expanded to further derivatives as the following

Theorem 2.3. let $f \in \Phi(\mathbb{R})$, hence we get:

(1) $\mathcal{M}_\alpha[f^{(n)}(x);s_\alpha] = (-1)^n\dfrac{\Gamma(s_\alpha)}{\Gamma(s_\alpha-n)}\mathcal{M}_\alpha[f(x);s_\alpha-n]$,

(2) $\mathcal{M}_\alpha[x^nf^{(n)}(x);s_\alpha] = (-1)^n\dfrac{\Gamma(s_\alpha+n)}{\Gamma(s_\alpha)}\mathcal{M}_\alpha[f(x);s_\alpha]$.

Proposition 2.4. Let $f \in \Phi(\mathbb{R})$, then we have

$$\mathcal{M}_\alpha[f^{(\frac{1}{2})}(t); s_\alpha] = \int_0^\infty t^{s-1} f^{(\frac{1}{2})}(t)dt.$$

by using fractional integration by parts and fractional derivative of power function, we get

$$\mathcal{M}_\alpha[f^{(\frac{1}{2})}(t); s_\alpha] = \int_0^\infty t^{s_\alpha - 1} f^{(\frac{1}{2})}(t)dt = \int_0^\infty f(t) D^{\frac{1}{2}} t^{s_\alpha - 1} dt,$$
$$= \frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha - \frac{1}{2})} \mathcal{M}_\alpha \left[f(t); s_\alpha - \frac{1}{2} \right].$$

$$\mathcal{M}_\alpha[f^{(\frac{3}{2})}(t); s_\alpha] = \int_0^\infty t^{s_\alpha - 1} f^{(\frac{3}{2})}(t)dt,$$

by using fractional integration by parts and fractional derivative of power function, then

$$\mathcal{M}_\alpha[f^{(\frac{3}{2})}(t); s_\alpha] = \int_0^\infty t^{s_\alpha - 1} f^{(\frac{3}{2})}(t)dt = \int_0^\infty f(t) D^{\frac{3}{2}} t^{s_\alpha - 1} dt,$$
$$= \frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha - \frac{3}{2})} \mathcal{M}_\alpha \left[f(t); s_\alpha - \frac{3}{2} \right].$$

Continuing by the induction, hence the results above can be extended as the following

Theorem 2.5. Let f be a fractional derivative function for all $n - 1 < \beta < n,$ *where* $n \in \mathbb{N}$ and f is α -Mellin transformable function defined on Lizorkin space, then

$$\mathcal{M}_\alpha[D^\beta f(t); s_\alpha] = \frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha - \beta)} \mathcal{M}_\alpha[f(t); s_\alpha - \beta], \quad 0 < \alpha \leq 1.$$

By using the same technique in above, α- Mellin transform of fractional integral can be given as the following

$$\mathcal{M}_\alpha[I^\beta f(t); s_\alpha] = \frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha + \beta)} \mathcal{M}_\alpha[f(t); s_\alpha + \beta], \; n - 1 < \beta < n \;\text{ and }\; 0 < \alpha \leq 1.$$

3 Applications of fractional Mellin transform

In following example, we are going to apply fractional Mellin transform to solve the differential equation.

Example 3.1. Let us consider the differential equation:

$$x^2 f^{'}(x) + x f(x) = \delta(x + c),$$

where $\delta(x + c)$ is Dirac delta function.

By applying fractional Mellin transform of both sides of above equation and using the property of Dirac delta function

$$\int_0^\infty \delta(x+c)f(x)dx = f(c),$$

we obtain

$$(s_\alpha+1)\mathcal{M}_\alpha[f(x);s_\alpha+1]+\mathcal{M}_\alpha[f(x);s_\alpha+1]=c^{s_\alpha-1}.$$

By solving the difference equation we get

$$\mathcal{M}_\alpha[f(x);s_\alpha+1]=\frac{c^{s_\alpha-1}}{(s_\alpha+1)}.$$

The solution can be obtained by using the inverse of fractional Mellin transform

$$f(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{(s_\alpha+2)c^{s_\alpha}}x^{-(s_\alpha+1)}ds_\alpha.$$

We can also apply fractional Mellin transform of series to get summation for infinite series as the following example.

Example 3.2. Let us consider the series $g(t)=\sum_{p=0}^{\infty} f(pt)$, where $t\in(0,\infty)$.

By taking fractional Mellin transform on both sides and using Proposition 2.1 we can get

$$\mathcal{M}_\alpha[g(t);s_\alpha]=\sum_{p=0}^{\infty}\mathcal{M}_\alpha[f(pt);s_\alpha]=\sum_{p=0}^{\infty}\frac{1}{p^{s_\alpha}}\mathcal{M}_\alpha[f(t);s_\alpha]=\zeta(s_\alpha)\mathcal{M}_\alpha[f(t);s_\alpha],$$

where $\zeta(s_\alpha)$ is the Riemann Zeta function. Therefore, by applying the inverse of fractional Mellin transform that gives

$$g(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s_\alpha)\mathcal{M}_\alpha[f(t);s_\alpha]t^{-s_\alpha}ds_\alpha.$$

4 Conclusion

In this paper fractional Mellin transform is presented depended on the definition of fractional Fourier transform with positive variable image, and its inverse for each $0<\alpha\leq 1$. When $\alpha=1$ the classical definition of Mellin transform will be obtained. Furthermore, some properties and applications of the usual Mellin transform remain valid for fractional Mellin transform.

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