Marvels of fractional calculus

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Abstract

This is an expository article that describes, in brief, one of the preeminent branch of applicable mathematics, roots of which lie in the nucleus of pure mathematics that ruled the research since past six decades. In writing this article though several important research papers were excised yet attempt is made to retain the beauty of fractional calculus. This article, accommodates Stieltjes transform and fractional integral operator on spaces of generalized functions, distributional Laplace-Hankel transform by fractional integral operators, and wavelet transform of fractional integrals for the integral Boehmians.

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1. Minutiae of Fractional Calculus

Many people share the success of this fascinating mathematical marvel, called fractional calculus. The subject has witnessed volcanic growth in the hands of so many professionals of mathematics in the form of research papers and books. According to Phillip J. Davis, in his Chauvenet prize winning paper of 1959, derivatives of arbitrary order are defined in terms of the gamma function. Bertram Ross, having fascinated with the concept of interpolation between integral orders of the derivatives, was led to begin, in 1971, study of what is called fractional calculus. During the First International Conference on Fractional Calculus, held at the University of New Haven in June 1974, see Ross, B. (ed.) [34], Bertram Ross opined the name Fractional Calculus a misnomer, which might be better called generalized integration and differentiation. All the four international conferences that were held in the past, it was realized that fractional calculus is an excellent vehicle to further interest in analysis and applied mathematics. Harold Thayer Davis was found to have stated "The great elegance that can be secured by the use of fractional operators and the power they have in simplifying the solution of complicated functional equations should more than justify a more general recognition and use." Fractional calculus has emerged as one of the most important interdisciplinary subject in Mathematics, Physics, Biology, Engineering among many other. Despite this fact, we dared writing this article with only one hope in mind, that so many new applications can be brought to the knowledge of young mathematicians, to initiate many new comers in this area of research, under one roof. The article will serve the purpose of students and researchers, subject to their need. Other purpose of this article is to popularize the topic in the hope that scientists and mathematicians may find interest in excavating the less explored branches where fractional integral and differential operators may be employed. The present article may encourage the educators to include them in their curricula to provide an opportunity for scholars in this field to meet and socialize.

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We do not intend to keep readers engaged in any of historical notes, but at times such may become unavoidable. Still we tried to be very brief. Though we desired to include some more of our researches with regard to applications of fractional calculus, for example, to integral equations, to certain general class of polynomials, and to Dirichlet averages realizing the paucity of space we restricted our temptation to minimum.

A topic, such as this, with so many different people from a multitude of various countries taking part, gives it a unique and special chance to enhance and attain the height of success. If you are one of the aspirant to search fractional calculus, then we hope you will make good contacts with the remarkable formulae that will live with you as mementos in life. And if you are one of the old member of the school of fractional calculus, then we are sure that you have made it a lasting experience and train the new scholars for them to carry the essence of this beautiful branch of mathematics.

Three international conferences on fractional calculus were held, each of them was individually remarkable. First of its kind was held in June 1974 at the University of New Haven, Connecticut, USA, see Ross [34]. Second was held in August 1984 at Ross Priory, University of Strathclyde, see McBride and Roach [25], and the third was held at the centre of Nihon University, Tokyo, in 1989, see Nishimoto [33]. In the proceedings of each of these international conferences, we notice a section on open questions for further research on fractional calculus and its applications. The first of its kind was edited by Thomas J. Osler, where eleven questions were proposed one of which was by Bertram Ross, University of New Haven, he proposed "A number of inquiries to this writer (Bertram Ross) have stemmed from analogies with ordinary calculus concerning a geometrical interpretation of a derivative of arbitrary order. The consensus of experts in this field is that there is in general no geometrical interpretation of a derivative of arbitrary order. This prompts the question: Can there be a geometrical interpretation for a derivative of a particular order. Related to this, the question raised by André Laurent is "can we do something with differentials of arbitrary order?" Similar question was included in the 3rd International Conference at Nihon University in 1989, in its proceedings, [cf. Nishimoto [33, p. 281]], is an article edited by H.M.Srivastava. The question was raised by Bertram Ross, to which the editor replies "An essentially identical problem (first proposed by Professor Ross at the 1974 conference) has appeared in number of places. In fact, on Page 378 of the Proceedings of 1974 Conference, Professor Ross states '.... The consensus of experts in this field is that there is, in general, no geometrical interpretation of a derivative of arbitrary order....' And then asks for a geometrical interpretation for a derivative of a particular order such as $\frac{1}{2}$. A closely related question would naturally involve differentials of arbitrary order. Till to date no satisfactory answer is given to this question.

Precisely speaking fractional calculus is pragmatic and possesses theoretical importance. We, therefore, wish to enumerate definitions related to fractional differentiation and integration.

Lacroix [*Traité du Calcul differential, et du Calcul Integral, 2nd edn., Courcier, Paris* 1819] developed a **formula for the nth derivative of** $y = x^m$, given by

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!},\ m \geq n \in N, m \in N_0 \tag{1}$$

which in terms of the gamma function is

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \tag{2}$$

By setting $n = \frac{1}{2}$, derivative of order $\frac{1}{2}$ is obtained.

Abel [*Solution de quelques problémes à l'aide d'intégrales définies, Gesammelte Math. Werke, Leipzig, Tubner.* (1881), 11-27 and *Auflösung einer mechanischen Aufgabe, J. für reine und angew. Math.* 1 (1826), 153-157] solved the **integral equation**

$$\int_a^z \frac{f(t)}{(z-t)^\mu} = \varphi(z), \quad z > a, 0 < \mu < 1 \tag{3}$$

in connection with **tautochrone problem** and the solution was given for $\mu = \frac{1}{2}$.

Liouville [*Memorie sur quelques questions de géometric et de méchanique, et sur unnouveau genere de calcul pour résoudre ces questions, J. Ecole Roy. Polytéchn. Sect.*, 21 (13) (1832), 1-69] suggested a definition based on the **differentiation of an exponential function** applied to $f(x)$, which, is if given by

$$f(x) = \sum_{i=0}^{\infty} c_i e^{a_i x} \quad , \tag{4}$$

then the definition is

$$D^n f(x) = \sum_{i=0}^{\infty} c_i a_i^n e^{a_i x} \tag{5}$$

for any complex n. A formula that is derived (in the same citation) is

$$D^{-n} f(x) = \frac{1}{(-1)^n \Gamma(n)} \int_0^z f(x+t) t^{n-1} dt \quad , \mathrm{Re}(n) > 0, -\infty < t < \infty \ . \tag{6}$$

Riemann [*Versuch einer allgemeinen auffasung der integration und differentiation, Collected Work of Bernhard Riemann* (*H. Weber, Ed.*) 2nd *edn. Dover, New York*, 1953] in 1847 gave the following definition for fractional integration

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty f(t)(x-t)^{\alpha-1} dt \quad , x > 0. \tag{7}$$

Holmgren, H. J. [*Om differenllcallcylen reed indices affttvad nature sore heist, Kongl. Svenska. Velenskaps-Akad. Hadl.* 5 (1) (1865-66), 1-83] have shown that **fractional differentiation is the inverse of fractional integration.**

Laurent, H. [*Sur le calcul des dérivées a indices quelconques, Nouv. Ann. Math.* 3 (3) (1884), 240-252] used **Cauchy's integral formula**

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)} dz \tag{8}$$

to define

$$ {}_{\alpha}D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_z^{\infty} f(t)(z-t)^{-\mu+1} dt \quad , \mathrm{Re}(\mu) > 0 \ . \tag{9} $$

for integration to an arbitrary order (the fractional integral), which is also **called Riemann version of fractional integral.**

When $\alpha = -\infty$, (9) is **called Liouville version of fractional integral**, that is given by

$$ {}_{-\infty}D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{z} f(t)(z-t)^{\mu-1} dt \quad , \mathrm{Re}(\mu) > 0. \tag{10} $$

Commonly used version is, when we set $\alpha = 0$ in (9), called **Riemann-Liouville fractional integral formula** of order μ, defined by .

$$ {}_{0}D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_{0}^{z} f(t)(z-t)^{\mu-1} dt \quad , \mathrm{Re}(\mu) > 0. \tag{11} $$

For $z \to \infty$, (9) reduces to

$$ {}_{z}W_{\infty}^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_{z}^{\infty} f(t)(t-z)^{\mu-1} dt \quad , \mathrm{Re}(\mu) > 0 \ . \tag{12} $$

This is called **Weyl fractional integral operator** [**Weyl, H. Bemerkungen zum begriff des differential quotienten gebrochener, Ordung. Vier. Natur. Ges. Zurich** 62 (1917), 296-302].

Definition, given by (9), is the integral of an arbitrary order, i.e., the fractional integral formula. **The fractional derivative formula cannot be written directly**. We have this concept: Let we want to obtain the fractional derivative formula $f(z)$ of order ν, i.e., ${}_{\alpha}D_z^{\nu}$, $\mathrm{Re}(\nu) > 0$. Let n be the smallest positive integer greater than ν. Put $\mu = n - \nu$, $0 < \mathrm{Re}(\mu) \leq 1$, the **fractional derivative of order** ν is

$$ \begin{aligned} {}_{\alpha}D_z^{\nu} f(z) &= {}_{\alpha}D_z^{n-\mu} f(z) ={}_{\alpha} D_z^{n}[{}_{\alpha}D_z^{-\mu} f(z)] \\ &= \frac{d^n}{dz^n} \frac{1}{\Gamma(\mu)} \int_{a}^{z} f(t)(z-t)^{\mu-1} dt \quad , z > 0 \end{aligned} $$

In particular, if $\alpha = 0$ and $f(z) = z^{y}$, $\mathrm{Re}(\gamma) > -1$, then

$$ {}_{0}D_z^{\nu} z^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma - \upsilon + 1)} z^{\gamma-\nu} \quad , \tag{13} $$

$\mathrm{Re}(\nu) > 0, \mathrm{Re}(\gamma) > -1, z > 0.$

Generalization of (11) and (12) are known as **Erdélyi-Kober operators**, given by $[\mathrm{Re}(\alpha) > 0]$

$$E_{0,z}^{\alpha,\eta}\{f(z)\} = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z f(t)(x-t)^{\alpha-1} t^{\eta} dt \tag{14}$$

and

$$K_{z,\infty}^{\alpha,\eta}\{f(z)\} = \frac{z^{\eta}}{\Gamma(\alpha)} \int_z^{\infty} f(t)(t-z)^{\alpha} t^{-\alpha-\eta} dt \quad . \tag{15}$$

Generalization of **Erdélyi-Kober operators** are given by [Saxena, R.K. *On fractional integral operators, Math. Z.* 96 (1967), 288-291]. **Saigo** [*A remark on integral operator involving the Gauss hypergeometric functions, Math. Rep. College Gen. Edu.* 11 (1978), 135-143] gave the **generalization of Reimann-Liouville and Weyl fractional integral and derivative operators. Sneddon** [*Mixed Boundary Value Problem in Potential Theory, North-Holland Publ. Co., Amsterdam,* 1966, and *The Use of Operators of Fractional Integration in Applied Mathematics, RWN-Polish Sci. Publishers, Warszawa-Poznan,* 1979] gave the **generalization of the most useful classical fractional integrals, the Erdélyi-Kober operators**.

Fractional integral operators involving other special functions are given by **Lowndes** [*On some generalizations of Riemann-Liouville and Weyl fractional integrals and their applications, Glasgow Math. J.* 22 (2) (1981), 173-180, and *On two new operators of fractional integrations, In : Proc. Fractional Calculus, Glasgow* 1984=*Research Notes in Maths.* 138, pp. 87-98, *Pitman Publ.* 1985]. Applications to fractional differential equations, intgral transforms, Mittag - Leffler functions, of the fractional integral and differential operators, can be stuided in [14, 15].

Sections 2, 3 and 4, and their subsections describe some applications of fractional calculus, resulted out of researches carried out by authors and their co-authors. Included in these sections are use of fractional calculus to distribution spaces (or spaces of generalized functions), to integral transforms, and to Boehmian space. Though at places we have avoided rigorous mathematics, yet the analytical comprehension is saved from damage.

2. The Distributions: Grappling Experience

Familiarity with the theory of generalized functions (also called ideal functions or distributions) is now an indispensable media of mathematics for the consumer no less than that of the practitioner of mathematical analysis. This is attested to by the stream of applications and extension of the theory, appearing in the current literature of mathematics and physics. Different versions of the theory of distributions (or the generalized functions) were proposed by numerous mathematicians, as a result of which first three decades of this century have shown directions for furthering research in this direction. This span of time may be termed as the period of explosion in functional analysis.

Distributions have different properties. The first (and essential), they are a generalization of the notion of function, and their purpose is to solve problems of differentiation. By a generalization, we mean that the set of distributions is a larger set than the set of functions; every function is a particular distribution, but there are distributions which are not functions. Why does one need a generalization? Gratuitous generalizations are not interesting. Differentiation poses serious problems: there are certainly differentiable functions, but Weierstrass was the first to give an example of a function which is nowhere differentiable. The function $|x|$ is one such function (among others), which is differentiable at most points, but is not differentiable at certain exceptional points. Distributions solve the problems of differentiation in the sense that every distribution is differentiable and even infinitely differentiable, and (moreover) the derivatives are also distributions. If a continuous function is not differentiable, then consider it as a distribution. It always admits a derivative, which is a distribution but not necessarily a function.

Analogous phenomena occurred previously in mathematics. For instance, rational numbers were insufficient to provide the answer to the problem of square roots; integers 2 and 3 do not have rational square roots. But if the rational numbers are generalized to real numbers, then every positive rational numbers has a square root which is real but not necessarily rational, and directly we get the field of real numbers in which every positive number has two square roots of opposite signs. The problem is still not solved, since negative numbers do not have square roots. For this there came the need for yet another generalization, called the complex numbers. It may not be out of place to mention that in the field of complex numbers, every real number has two complex, opposite square roots. In fact every complex number has two complex square roots which generate d'Alemberts Theorem. This is analogous to the relation between functions and distributions. Peano, the mathematician, wrote in 1912 on the difficulties of differentiation: "I am very sure that something must be found. There must exist a notion of generalized functions which are to functions what the real numbers are to rationals". This was a marvelous intuition which arose much before 1944. May be the mathematical knowledge of the time did not make it possible for Peano to find the generalized functions, or even to conceive them.

The electrical engineer Heaviside was the most important precursors of distributions. He wanted to solve ordinary differential equations, particularly those with constant coefficients. He introduced his Symbolic Calculus in 1893-94, much before Peano. There exists a composition product, denoted by $*$, for the which were zero outside of the semi-axis $[0, +\infty]$, introduced by Mercer at the time of integral equations of Volterra – Fredholm. For $t \geq 0$, the composition product is given by

$$(f * g) = \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds, \tag{16}$$

this product is both associative and commutative, and hence defines an algebra. Composition plays a fundamental role in transitory phenomena in electricity. André Weil studied and used composition in his book of 1940 on integration on topological groups. The name "composition" was too vague, which was eventually abandoned and replaced by "convolution". In German, it is called "Faltung". In 1926 Dirac (Paul) introduced a function, which is often called Dirac measure, which is denoted by δ. By this time of the century, Heaviside had already introduced the successive derivative of δ in his symbolic calculus. It is interesting to note that Dirac has used Heaviside's function in some very specific problems, and also suggested an approximation for the same.

In the theory of distributions, there is a topology implying a notion of convergence for which it is true that Dirac's $1/2\varepsilon$ functions converge towards the Dirac distribution δ. He also gave another approximation of the Dirac function, namely, the bell curve

$$\left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^N e^{-|x|^2/2\varepsilon} \tag{17}$$

As $\varepsilon \to 0$, this function converges towards the Dirac function δ. Dirac distribution was actually justified well before the invention of distributions, as the Dirac measure. But like Heaviside, Dirac introduced the derivative of his function, which was not possible owing to the fact that speaking of the derivative as a measure is not justified. He simply took the Dirac function to be the limit of the Gauss function, and then defined its derivative to be the limits of their derivatives. It was not acceptable to define and represent the derivative in this fashion. It is true that the Dirac function has value zero everywhere except at the origin, whereas it should also be zero at the origin since it is an odd function. All other results concerning variable changes, made by Dirac and other physicists on their generalized functions, were justified by the theory of distributions and were reproved with precise computations on these singular functions by Méthée, a Swiss mathematician, who published his work a few years after Schwartz has published distributions in 1954. The theory of harmonic function and holomorphic functions are another necessary prerequisite for studying the theory of distributions.

Bochner [3], in his book on Fourier Integrals, introduced formal functions, which are finite sums of formal derivatives of products of square integrable functions with polynomials. By formal derivatives we mean that these functions are not necessarily differentiable in the usual sense (we have mentioned it in the preceding text). At the first instance, Bochner defined when two formal derivatives define the same generalized function (that is, Bochner has established an equivalent relation). He then defined multiplication and convolution that can be performed on such generalized functions. Bochner failed to clarify exactly when they are defined, except in some simple cases. When the simple conditions are satisfied, the Fourier transform exchanges multiplication and convolution. These generalized functions are exactly what later came to be known as tempered distributions. Schwartz exclaims, "Tempered distributions actually already existed in 1932!".

At one point of time, Bochner was just a few centimeters from introducing distributions. He needed to define a function f on R^n, which would have been a generalized solution to a partial differential equation with constant coefficient of order m:

$$P(D)f = \sum_{p} a_p D^p f = 0 \tag{18}$$

Bochner pointed that the function f is such a solution of the equation (18) that it is a uniform limit on every compact set of sufficient differentiable functions f_n, which are ordinary solution of the equation. He did not establish the relation between his generalized solutions and his generalized functions for the Fourier integral. Jean Leray did some fundamental work on the equation of viscous liquid. He also introduced the notion of a weak solution of a partial differential equation of order 2.

Rest is the story of the advent of the theory of distributions that gives a pleasant sense by reading Theorie des Ditributions [37], which changes the whole concept of functional analysis and renewed the concepts.

There are variety of other approaches to the theory of generalized functions, which are based, in general, on the facts that generalized functions can be represented as sequences of ordinary functions which converge in a certain way and that over a finite interval of a generalized function is a finite order derivative of a continuous function. All these methods can be understood in terms of Schwartz's theory and most of them are, in fact, encompassed by it.

A distribution is a kind of generalized measure, where a derivative assigns another distributions. Thus, continuous function has a derivative but the result may not be a function. The theory of distribution is introduced through two avenues : One of them is the method of functionals and the other is the sequential approach . Both the methods are equally important and advantageous.

Definition : A functional is a map (Φ) from into complex numbers. If $\varphi \in \Phi$, in addition, to be a function of t, depending upon additional variable s, we will write

$$\langle T_t, \varphi(t-s)\rangle \tag{19}$$

to indicate that T acts upon φ as a function of t . Thus (19) is a complex valued function of s , where T is linear and convergent. Distributions are those functionals, defined on class $\mathcal{D}$, which are continuous and linear.

Another way to define distribution is by considering $f(t)$ as a locally integrable function, by which the distribution f is defined through the convergent integral .

$$\langle f, \varphi\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)dt \tag{20}$$

Distributions, that can be generated through (20) from locally integrable functions, are called regular distributions. All other distributions are called singular distributions.

3. Products of Stieltjes transform and fractional integrals on spaces of generalized functions [1]

Results involving products of generalized Stieltjes transform and fractional integral operators on the spaces $F'_{p,\mu}$ of generalized functions are obtained. Techniques employed in the analysis are from those employed elsewhere. $F'_{p,\mu}$ is called the McBride's generalized function space, see [1, 24].

$\mathbb{N}$ (respectively $\mathbb{N}_0$) denote the set of positive (resp. non-negative) integers, $\mathbb{R}^+$ (respectively $\mathbb{R}_0^+$) is the set of positive (resp. non-negative) real numbers, and $\mathbb{R}$ (resp. $\mathbb{C}$) is the field of real (resp. complex) numbers. The sets $\mathbb{C}_+, H_0, \overline{H}_0$ are defined by

$$\begin{aligned} C_+ &= \{s : s : \mathbb{C}\backslash(-\infty, 0)\}, \\ H_0 &= \{s : s \in \mathbb{C}, \mathrm{Re}(s) > 0\}, \\ \overline{H}_0 &= \{s : s \in \mathbb{C}, \mathrm{Re}(s) \geq 0\}. \end{aligned}$$

$L_p(\mathbb{R}^+)$ is the space of measurable function on $\mathbb{R}^+$ such that

$$\|f\|_p = \left[\int_0^\infty |f(t)|^p \, dt\right]^{1/p} \quad , \quad (1 \leq p < \infty)$$

or

$$\|f\|_\infty = \operatorname*{ess \cdot sup}_{0<t<\infty} |f(t)| \quad , \quad (p = \infty)$$

is finite. As usual, let p, q be positive real numbers or ∞ with $1/p + 1/q = 1$. By D_z^k we denote the operator of differentiation, $D_z^k = \frac{d^k}{dz^k}, (k \in \mathbb{N}_0)$.

The generalized Stieltjes transform is defined by

$$G_p[f](s) = \int_0^\infty f(t)(t+s)^{-\rho}dt, \tag{21}$$

for $\rho = 1$, this reduces to classical Stieltjes transform.

Theorem 1 : The Generalized Stieltjes transform $F_\rho(s)$ of $f \in F'_{p,\mu}$ is a holomorphic function of s on C_+ and that

$$\begin{aligned} D_s^k F_\rho(s) &= \left\langle f(x), (-1)^k (\rho)_k (x+s)^{-\rho-k} \right\rangle \\ &= (-1)^k (\rho)_k F_{\rho+k}(s) \ , \ k \in \mathbb{N}_0. \end{aligned}$$

Proof : While being confined to the case $k = 1$, method of induction completes the proof. Let we assume $s, s+h \in C_+$ and $x \in \mathbb{R}^+$, we consider the function

$$\psi_h(x) = \frac{1}{h}\left[(x+s+h)^{-\rho} - (x+s)^{-\rho}\right] - \frac{\partial}{\partial s}(x+s)^{-\rho} \tag{22}$$

$$\lim_{h\to 0} \gamma_k^{p,\mu}(\psi_h) = 0 \ , \ k \in \mathbb{N}_0 \tag{23}$$

We have

$$D_x^k x^{-\mu} \psi_h(x) = \sum_{j=0}^{k} C_j x^{-\mu-j} \eta_j(s,h,x),$$

with certain constants $C_j (j = 0, 1, 2, \ldots, k)$. Cauchy integral formula, then, produces

$$\eta_j = \frac{h}{2\pi i} \int\limits_{C_1} \frac{(x+z)^{-\rho-k+j}}{(z-s-h)(z-s)^2} dz \quad ,$$

where C_1 is the closed path in C_+ which encircles s and $s+h$ once.

Finally, having proved (3) and invoking it, we obtain

$$\lim_{h\to 0} \frac{1}{h}\left[F_\rho(s+h) - F_\rho(s)\right] = \left(f(x), -\rho(s+x)^{-\rho-1}\right)$$

This completes the proof.

3.1 Generalized Stieltjes transform of Fractional Integrals

For the definition of the generalized fractional integrals can be seen in Saigo [35] and Glaeske and Saigo [14].

Theorem 2 : Let $f \in F'_{p,\mu}$, $\mathrm{Re}(\alpha) > 0$ and $1/p - \mathrm{Re}(\beta+\rho) < \mathrm{Re}(\mu) < 1/p - \mathrm{Re}(\beta)$. For $\{0, \beta-\eta\} \subset A_{q;-\mu}, \mathrm{Re}(\rho) + \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta)] > 0,$

$$G_\rho[I^{\alpha,\beta,\eta}f](s) = \langle f(x), \phi_1(x,s)\rangle \tag{24}$$

and for $\{\beta,\eta\} \subset B_{q;-\mu}$ and $\mathrm{Re}(\beta-\eta) < 1$,

$$G_\rho[J^{\alpha,\beta,\eta}f](s) = \langle f(x), \phi_2(x,s)\rangle \tag{25}$$

where

$$\phi_1(x,s) = [J^{\alpha,\beta,\eta}(s+x)^{-\rho}](x)$$

and

$$\phi_2(x,s) = [I^{\alpha,\beta,\eta}(s+x)^{-\rho}](x)\ .$$

Then

$$\begin{aligned}\phi_1(x,s) &= \frac{x^{-\beta-\rho}\Gamma(\beta+\rho)\Gamma(\eta+\rho)}{\Gamma(\rho)\Gamma(\alpha+\beta+\rho+\eta)}\\ &\quad \times {}_2F_1\left[\beta+\rho,\eta+\rho;\alpha+\beta+\rho+\eta;-\frac{s}{x}\right]\end{aligned} \tag{26}$$

and

$$\begin{aligned}\phi_2(x,s) &= \frac{x^{-\beta}\Gamma(\eta-\beta+1)}{s^\rho\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}\\ &\quad \times {}_3F_2\left[1,\rho,\eta-\beta+1;1-\beta,\alpha+\eta+1;-\frac{x}{s}\right]\ .\end{aligned} \tag{27}$$

We deliberately avoid proof here, owing to its complete derivation in [1], which can be referred to.

3.2 Fractional Integrals of the Generalized Stieltjes Transform

Theorem 3 : Let $f \in F'_{p,\mu}$, $\mathrm{Re}(\alpha) > 0$ and $1/p - \mathrm{Re}(\rho) < -\min[0,\mathrm{Re}(\beta)] < \mathrm{Re}(\mu) < 1/p - \mathrm{Re}(\beta)$.Then

$$I^{\alpha,\beta,\eta}[G_\rho[f]](x) = \langle f(t), \phi_2(t,x)\rangle$$

provided that $\{\beta,\eta\} \subset B_{q;-\mu}$, $\mathrm{Re}(\beta-\eta) < 0$, and

$$J^{\alpha,\beta,\eta}[G_\rho[f]](x) = \langle f(t), \phi_1(t,x)\rangle$$

provided that $\{0,\beta-\eta\} \subset A_{q;-\mu}$ and $\mathrm{Re}(\alpha) + \min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)] > 0$, where $\phi_1(x,s)$ and $\phi_2(x,s)$ are defined in (26) and (27), respectively.

We avoid writing the proof, but reader is referred to [1].

4. Distributional Laplace-Hankel transform by Fractional Integral Operators [17]

Different methods are known to have been applied to study several kinds of generalized functions (Schwartz distributions) of the integral transforms. Described in this section is the Laplace

-Hankel transform, which by invoking convolution method is studied for the generalized functions by employing fractional integral operator.

4.1. Laplace - Hankel transform and Fractional Integral Operators

Laplace-Hankel transformation and its inverse, for the generalized functions, are given by-

$$F(\xi, t) = \int_0^\infty \int_0^\infty f(x,y)e^{-\xi x}(yt)^{1/2} J_\mu(yt)dxdy \tag{28}$$

and

$$f(x,y) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty F(\xi, t)e^{\xi x}(yt)^{1/2} J_\mu(yt)d\xi dt, \tag{29}$$

where $f \in B'_{\mu,a,b}, \mu \geq -1/2, b, t > 0$ and $J_\mu(\cdot)$ is the Bessel function of order μ. If $f \in B'_{\lambda,\omega,z}$, then the complex Laplace-Hankel transform of a generalized function is given by

$$F(\xi, t) = \left\langle f(x,y), e^{-\xi x}(yt)^{1/2} J_\mu(yt)\right\rangle \tag{30}$$

where ω and z are a pair of real numbers.

The integrals, for $\varphi(x) \in L_1(a,b)$

$$(I^\alpha_{a+}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt \ , \ x > a \ , \alpha > 0 \tag{31}$$

and

$$(I^\alpha_{b-}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt \ , \ x < a \ , \alpha > 0 \tag{32}$$

are called Riemann – Liouville fractional operators of order α , details of it can be seen in [36]. These formulae are also called left-sided and right-sided fractional integral operators. Above formulae, (31) and (32) can be extended from the interval $[a,b]$ to half axis, given by

$$(I^\alpha_{0+}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)dt \quad , \quad 0 < x < \infty \tag{33}$$

and to the entire axis, by

$$(I^\alpha_{+}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty < x < \infty \tag{34}$$

and

$$(I^\alpha_{-}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty < x < \infty \ . \tag{35}$$

In terms of the convolution, the relations (34) and (35) can be expressed as

$$(I^\alpha_{\pm}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^\infty t_{\pm}^{\alpha-1}\varphi(x-t)dt$$

i.e.

$$= \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} \varphi(x-t) dt \tag{36}$$

where

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1} , & t > 0 \\ 0 , & t < 0 \end{cases} \tag{37}$$

and

$$t_-^{\alpha-1} = \begin{cases} 0 , & t > 0 \\ \mid t \mid^{\alpha-1} , & t < 0 \end{cases} \tag{38}$$

4.2. Distributional Laplace-Hankel Transformation for Fractional Integral Operator

In the first part of this sub-section, we intend to obtain a formula for the fractional Laplace-Hankel transform and then, in the second part we apply the same to obtain the main result of this investigation, namely, the distributional Laplace-Hankel transform for the Riemann-Liouville fractional integral operator. Let we begin with

$$f \circ g = (2\pi)^{1/2} \int_{-\infty}^{\infty} f(t-u) g(u) du, \tag{39}$$

which is a convolution integral for the classical transform. Thus, the convolution formula for the Hankel transform, with regard to (39) is given by

$$H[h * \varphi](p) = (Hh)(p)(H\varphi)(p) \ . \tag{40}$$

The fractional integral operator, of order α, for the generalized function is

$$(I_{0+}^{\alpha} f)(\xi, t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\xi x - yt)^{\alpha-1} f(x,y) dt \quad , \ \mathrm{Re}(\alpha) > 0 \ \ , \tag{41}$$

which, incidentally, is the Hankel convolution integral, given by

$$(I_{0+}^{\alpha} f)(\xi, t) = \left[f(x,y) * \frac{(yt_+)^{\alpha-1}}{\Gamma(\alpha)} \right] \quad , \ \mathrm{Re}(\alpha) > 0 \ \ , \tag{42}$$

Then we make use of a relation, cf. [12, p.22] , for $f(x) = x^{\mu}$, we consider $f(x,y) = (yt)^{\alpha-1}$,and then employing the Laplace transform with respect to the variable (ξx) in the relation so obtained, we get

$$LH(I_{+}^{\alpha} f)(\xi, t) = \frac{2^{\alpha+\frac{1}{2}}}{\Gamma(\alpha)} \left(-\frac{1}{\xi} \right) \frac{\Gamma\left(\frac{\alpha}{2} + \frac{v}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{v}{2} - \frac{\alpha}{2} + \frac{3}{4}\right)} (\xi x)^{-\alpha} F(\xi, t) \ . \tag{43}$$

Under some substitutions and using the concept of signum function, from the above relation we arrive at

$$F(\xi, t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{f(x,y) e^{-\xi x} (yt)^{1/2} J_{\mu}(yt)}{(\xi x - yt) 1^{-\alpha}} dx dy dt \ . \tag{44}$$

This is true for all sufficiently good functions f, provided $\mathrm{Re}(\alpha) < 0$.

The *first* approach to define fractional derivatives and integrals of generalized functions is due to the definition of fractional integral operator as the *convolution*

$$\frac{1}{\Gamma(\alpha)}(yt_{\pm})^{\alpha-1} * f \tag{45}$$

of the function $\frac{1}{\Gamma(\alpha)}(yt)^{\alpha-1}$, f being the generalized function. This method is suitable for cases dealing with half axis.

The *second* approach, commonly used, is based on use of *adjoint operators*. The generalized function for fractional integration is defined by

$$(I^{\alpha}_{a+}f, \varphi) = (f, I^{\alpha}_{b-}\varphi) \tag{46}$$

which make sense if I^{α}_{b-} continuously maps the space of test functions X into itself and, when f and $I^{\alpha}_{a+}f$ are considered as generalized functions on different test function spaces X and Y, respectively such that $f \in$X$'$ (the dual of the test function space X), $I^{\alpha}_{a+}(f) \in$Y$'$(the dual of test function space Y). This asserts that I^{α}_{b-}, indeed, maps continuously Y into X.

The Laplace - Hankel transform of fractional integral operator defines generalized functions in the dual space $B'_{\mu,a,b}$ as

$$F(\xi, t) = \left\langle f(x, y), e^{-\xi x}(yt)^{1/2} J_{\mu}(yt)(\xi x - yt)^{\alpha-1} \right\rangle \ , \tag{47}$$

where $\mu \geq -1/2, b, t > 0$.

Since the function f is the complex Laplace-Hankel transform of a generalized function , thus f belongs $C^{\infty}_{0}(R_1)$ in (47).

Using the notion of convolution [12, p. 154], by virtue of (34), the fractional integral formula for the generalized function $f \in C^{\infty}_{0}(R_1)$ is

$$(I^{\alpha}_{0+}f, \varphi) = (I^{\alpha}_{+}f, \varphi) = \left(\frac{1}{\Gamma(\alpha)}(yt^{\alpha-1}_{+}) * f, \varphi \right) \ , \tag{48}$$

which holds true when f is supported on the half axis $(x > 0)$.

This implies, indeed, that $I^{\alpha}_{0+}f$ (the Riemann - Liouville fractional integral operator) is also supported on half axis. Thus, (32) is true for all values of α.

5. Wavelet Transform of Fractional Integrals for Integrable Boehmians [19]

In what follow is an excellent combination of three most powerful entity of applicable analysis, which is not found to have appeared before. This deals with the wavelet transform of fractional integral operator (the Riemann-Liouville operators) on Boehmian spaces [36] *. By virtue of the existing relation between the wavelet transform and the Fourier transform, we obtained integrable Boehmians defined on the Boehmian space for the wavelet transform of fractional integrals.*

5.1 Brief Description

The concept of wavelets or ondelettes started to appear in the literature only in early in 1980s. French geophysicist, **Morlet** [31, 32], introduced the operation of dilation, while keeping the translation operations developed an algorithm for the recovery of the signals under investigation from this wavelet transform. Then, a French theoretical physicists, **Alex Grossman**, who quickly recognized the importance of the Morlet wavelet transform, which is something similar to coherent states formalism in quantum mechanics, developed an exact inversion formula for the wavelet transform. Then, by the joint venture of mathematical physics group in Marseille, led by Grossman, in collaboration with Daubechies, Paul and others, extended Morlet's discrete version of wavelet transform to the continuous version by relating it to the theory of coherent states in quantum physics. This was the notion of the integral (or continuous) wavelet transform.

In order to eliminate the above said weakness of the Fourier analysis, Dennis Gabor [13], a Hungarian British physicist and engineer, first introduced the **Windowed Fourier transform (or the short-time Fourier transform,** or more appropriately the **Gabor transform**) by using a Gaussian distribution function as the window function. The idea of using a window function lies in order to localize the Fourier transform and then shift the window to another position, and so on.

The remarkable feature of the Gabor transform is the local aspect of the Fourier analysis with the time resolution is equal to the size of the window. In fact, it deals with discrete set of coefficients which allows efficient numerical computation of those coefficients. However, the Gabor wavelets suffers from some serious algorithmic handicaps and shortcomings which have, successfully, been solved by Henrique Malvar [22, 23]. Malvar wavelets are much more effective and superior to other wavelets, including Gabor wavelets and Morlet-Grossman wavelets.

The development of the wavelet transform and mathematical analysis of the wavelet transform had really not begun until a year later in 1985, when Meyer learnt about the work of Morlet and the Marseille group, recongnized immediately the deep connection of Morlet's algorithm to the notion of resolution of identity in harmonic analysis due to Calderón in 1964. He then applied the Littlewood-Paley theory to study wavelet decomposition. In this regard, Yves Meyer may be considered as the founder of this mathematical subject, **which we call wavelet analysis**.

Since wavelet analysis is built on Fourier analysis, Meyer's book [26] devotes a brief discussion on distributions, the Poisson summation formula, Shannon's sampling theorem, and the Littlewood-Paley theory. He also explains the construction of wavelets and the application of wavelet series representations to the analysis of the most important function spaces, such as Hölder, Hardy, Block and Besov, and also the notion of holomorphic wavelets.

The next great achievement of wavelet analysis was due to Daubechies et al. [10] which suggests a new construction of painless non -orthogonal wavelet expansion. During 1985-86, further work of Lemarié and Meyer [16] on the first construction of a smooth orthonormal basis on $\mathbb{R}$ and $\mathbb{R}^N$, marked the beginning of their famous contributions to the wavelet theory. The collaborations of Meyer and Mallat, culminated with the remarkable discovery by Mallat of new formalism [20, 21], came to be known as **multiresolution analysis**.

Inspired by the work of Meyer, Daubechies [9], made a remarkable contribution to wavelet theory by constructing families of **compactly supported orthonormal wavelets** with some degree of smoothness. But after a great success, she reconginzed that it is difficult to construct wavelets that are symmetric, orthogonal and compactly supported. Chui and Wang [5, 6] introduced **compactly spline wavelets,** and **semi-orthogonal wavelet** analysis**.** As a natural extension of wavelet analysis, Coifman et al. [7, 8] discovered wavelet packets which can be used to design efficient schemes for the representation and compression of acoustic signals and images.

The Gabor transform (i.e., the windowed Fourier transform) of f with respect to g [[11], p.688] is

$$\mathcal{G}[f](\upsilon,t)=\widetilde{f}_g(\upsilon,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\tau)\overline{g}(\tau-t)e^{i\tau t}d\tau$$

$$=\frac{1}{\sqrt{2\pi}}(f,\overline{g}_{\upsilon,t}) \qquad , \tag{49}$$

where $f,g\in L^2(\mathbb{R})$ with the inner product (f,g). For a fixed t ,

$$\mathcal{G}\left|f\right|(\omega,t)=\widetilde{f}_g(\upsilon,t)=\mathrm{F}\{f_t(\tau)\}=\widehat{f}_t(\upsilon)\ ,$$

where F is the Fourier transform and $\mathcal{G}$ is the Gabor transform, respectively.

Definition 3 [36, p.33] : Let $\varphi(x)\in L_1(a,b)$. Then the integrals

$$(I^{\alpha}_{a+}\varphi)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\varphi(t)dt\ ,\ x>a\ , \tag{50}$$

$$(I^{\alpha}_{b-}\varphi)(x):=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}\varphi(t)dt\ ,\ x<b\ , \tag{51}$$

where $\alpha>0$, are Riemann-Liouville fractional integrals of order α . They are also known as *left-sided* and *right-sided fractional integrals*, respectively. Indeed, these integrals are extensions from the case of a finite interval $[a,b]$ to the case of a half-axis, given by

$$(I^{\alpha}_{0+}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\varphi(t)dt \quad , \quad 0<x<\infty \tag{52}$$

while for the whole axis, it is given, respectively, by [36, p. 94]

$$(I^{\alpha}_{+}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x(x-t)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty<x<\infty \tag{53}$$

and

$$(I^{\alpha}_{-}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty<x<\infty\ . \tag{54}$$

The Fourier transform of the fractional integrals $I^{\alpha}_{\pm}\varphi$ are [36, p. 147]

$$\mathrm{F}(I^{\alpha}_{\pm}\varphi)=(\mp ix)^{-\alpha}\widehat{\varphi}(x) \quad ,\ \varphi\in L_1(a,b) \quad . \tag{55}$$

Study of *regular operators* of Mikusiński by Boehme [4] resulted into the theory of Boehmians, the generalization of Schwartz distribution theory. These regular operators form a subalgebra of Mikusiński operators such that they include only such functions whose support is bounded from the left, and at the same time do not have any restriction on the support. The general construction of Boehmians gives rise to various function spaces, which are known as **Boehmian spaces** [cf. Mikusiński and Mikusinski [27] and Mikusiński [28, 29]]. It is observed that these spaces contain all Schwartz distributions, Roumieu ultradistributions and tempered distributions.

The name *Boehmian* is used for all objects by an algebraic construction, which is similar to the construction of the field of quotients. Suppose G is an additive commutative semigroup, S be a subset of group G such that $S \subseteq G$ is a sub semigroup, for which we define a mapping * from $G \times S$ to G such that following conditions are satisfied (these condition are for the mapping *) :

(i) if $\delta, \eta \in S$, then $(\delta * \eta) \in S$ and $\delta * \eta = \eta * \delta$

(ii) if $\alpha \in G, \delta, \eta \in S$, then $(\alpha * \delta) * \eta = \alpha * (\ \delta * \eta)$

(iii) if $\alpha, \beta \in G, \delta \in S$, then $(\alpha + \beta) * \delta = (\alpha * \ \delta) + (\beta * \delta)$.

The delta sequence, denoted by Δ , is defined as members of class delta which are the sequences of subset S, and satisfies the conditions

(i) if $\alpha, \beta \in G, (\delta_n) \in \Delta$ and $(\alpha * \delta_n) = (\beta * \delta_n)$, $\forall\ n$, then $\alpha = \beta$ in G.

(ii) if $(\delta_n), (\varphi_n) \in \Delta$, then $(\delta_n * \varphi_n) \in \Delta$.

Then the *quotient of sequences* is defined as the element of certain class A of pair of sequences defined by .

$$A = \{(f_n), (\varphi_n) : (f_n) \subseteq G^N, (\varphi_n) \in \Delta\}.$$

This is denoted f_n/φ_n by such that .

$$f_m * \varphi_n = f_n * \varphi_m, \quad \forall\ m, n \in N.$$

Further, the quotients of sequences f_n/φ_n and g_n/ψ_n are called *equivalent* if .

$$f_n * \psi_n = g_n * \varphi_n, \qquad \forall\ n \in N.$$

The equivalence relation defined on A and the equivalence classes of quotient of sequence are called *Boehmians* .

The *space of all Boehmians*, denoted by B, has the properties addition, multiplication and differentiation. The Boehmian space B_{L_1} will be called the space of *locally integrable Boehmians* if the group G be the set of all locally integrable function on $\mathbb{R}$ and possibly two such functions are identified with respect to Lebesgue measure (these functions are equal almost everywhere) and the topology of this space is taken to be the semi-norm topology generated by

$$p_n(f) = \int_{-n}^{n} |f|\, d\lambda \quad , n = 1, 2, \ldots .$$

where λ is the usual Lebesgue measure on $\mathbb{R}$ and $D(\mathbb{R})$. In other words, if $f \in L_1$ and (δ_n) is the delta sequence, then $\|(f * \delta_n) - f\| \to 0$, as $n \to \infty$. A pair of sequences (f_n, φ_n) is called a quotient of sequences, and is denoted by f_n/φ_n if $f_n \in L_1(n = 1, 2, \ldots)$ where (φ_n) is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m, \forall\ m, n \in N$, whereas, two quotients of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n * \psi_n = g_n * \varphi_n, \forall\ n \in N$. The equivalence class of quotient of sequences will be called an *integrable Boehmain*, the space of all integrable Boehmian will be denoted by B_{L_1} .Convergence of Boehmians is defined in [Mikusiński [28]]. The terminologies regarding Boehmians and Boehmian spaces can be referred to in [Mikusiński and Mikusiński [27], Mikusiński [28, 29]]. Authors of this paper also investigate the Gabor transform for integrable Boehmian [2], and applications in Fourier and Laplace transform and distribution spaces to fractional calculus in [18].

5.2 Main Results

Using the relation between the Gabor and the Fourier transform,

$$\mathcal{G}\,|f|\,(\omega,t)=\widetilde{f}_g(\upsilon,t)=\mathrm{F}\{f_t(\tau)\}=\widehat{f}_t(\upsilon)\ ,$$

where F is the Fourier transform and $\mathcal{G}$ is the Gabor transform, respectively. The fractional integrals for the Gabor transform, can be written in the form .

$$\mathrm{F}(I^{\alpha}_{\pm}f_t(\tau))=(\mp i\upsilon)^{-\alpha}\widehat{f}_t(\upsilon)\quad ,\ f\in L_1(a,b)\ . \tag{56}$$

In other words, (56) can be written as

$$\mathcal{G}(I^{\alpha}_{\pm}f)=(\mp i\upsilon)^{-\alpha}\widehat{f}_t(\upsilon) \tag{57}$$

i.e.

$$\begin{aligned}\mathcal{G}(I^{\alpha}_{\pm}f_n) &= (\mp i\upsilon)^{-\alpha}(\widehat{f}_t(\upsilon))_n\\ &= (\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\upsilon)\ .\end{aligned} \tag{58}$$

Theorem 3 [19] : If $[f_n/\delta_n]\in B_{L_1}$, then the sequence

$$\mathcal{G}(I^{\alpha}_{\pm}f_n)=(\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\upsilon) \tag{59}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof : If (δ_n) is a delta sequence, then $(\widehat{\delta}_t)_n$ converges uniformly on each compact set to the constant function unity. Therefore, $(\widehat{\delta}_k)>0$ on K (the compact set) and, thus, the left hand side of (38) gives

$$\begin{aligned}\mathcal{G}(I^{\alpha}_{\pm}f_n) &= \frac{(I^{\alpha}_{\pm}\widehat{f}_n)(\widehat{\delta}_k)}{(\widehat{\delta}_k)}=\frac{(I^{\alpha}_{\pm}f_n*\delta_k)^{\wedge}}{(\widehat{\delta}_k)}=\frac{(I^{\alpha}_{\pm}\widehat{f}_k)(\widehat{\delta}_n)}{(\widehat{\delta}_k)}\quad \text{on } K\\ &= \frac{(\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\widehat{\delta}_n)}{(\widehat{\delta}_k)}\qquad ,\ \text{[cf. Eqn. (58)]}\end{aligned}$$

This shows that the Gabor transform of fractional integrals for an integrable Boehmian $F=[f_n/\delta_n]$ can be expressed as the limit of the sequence $\mathcal{G}(I^{\alpha}_{\pm}f_n)$, which, in fact, is the space of all continuous functions on $\mathbb{R}$. This proves the theorem completely.

Property 1 [19] : Let $[f_n/\delta_n]\in B_{L_1}$. Then $\Delta-\lim_{n\to\infty}F_n=F$, $\mathcal{G}(I^{\alpha}_{\pm}F_n)\to\mathcal{G}(I^{\alpha}_{\pm}F)$uniformly on each compact set .

Proof : We have $\delta-\lim_{n\to\infty}F_n-F\Rightarrow\ \mathcal{G}(F_n)\to\mathcal{G}(F)$, uniformly on each compact set. The sequence can be expressed as $F_n*\delta_k, F*\delta_k\in L_1$, $\forall\ n,k\in N$ which has a norm

$$||(F_n-F)*\delta_k||\to 0,\text{as}\quad n\to\infty,\forall k\in N\ \ .$$

where K is well defined. Since $\mathcal{G}\{\delta_k\}$ is a continuous function, we have $\mathcal{G}\{\delta_k\} > 0$ on K for $k \in N$. It is, therefore, enough to prove that

$$\mathcal{G}\{F_n\} \cdot \mathcal{G}\{\delta_k\} \to \mathcal{G}\{F\} \cdot \mathcal{G}\{\delta_k\} \quad ,$$

uniformly on K. We have,

$$\mathcal{G}\{F_n\} \cdot \mathcal{G}\{\delta_k\} - \mathcal{G}\{F\} \cdot \mathcal{G}\{\delta_k\} = \mathcal{G}\{(F_n - F) * \delta_k\} \quad ,$$

such that $||(F_n - F) * \delta_k|| \to 0$, as $n \to \infty$.

This justifies the existence and validity of the property.

Conclusions: Usually all integral transforms support the fractional derivative and integral of an arbitrary order. This fact has given rise to many fruitful results in the past. This article extends, among other results, the amicable combination of Stieltjes integral transform and the generalized fractional integral on certain spaces of generalized functions. Generalized Stieltjes transform of fractional integrals and fractional integrals of generalized Stieltjes transform are studied. Further, having prescribed the simple definition of Laplace-Hankel transformation for the generalized functions, and using the Riemann-Liouville fractional integral operator, in Section 4.1 results relating to distributional Laplace-Hankel transformation for fractional integral operator have been established. Section 5, believed to be ventured for the first time, focuses on the application of the Riemann Liouville type fractional integral operator to the Gabor transform and the integrable Boehmians. The fractional integral formula for the Gabor transform is given by using the relation between the Gabor and the Fourier transforms. The formula and the property established in this paper are suitable for certain Boehmian space for an integrable Boehmian. The compact set and the continuity of the function used, approves the existence of the results given in this paper.

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