On the numerical solution of a fractional population growth model

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Abstract

In this paper, a fractional model for population growth of species within a closed system is considered. A numerical method which is based on the implementation of the fractional Legendre functions with a pseudospectral approach is applied. The aim is to show the effectiveness of fractional Legendre functions for the numerical simulation of fractional models.

2010 Mathematics Subject Classification. **34A08**. 65M70 92D25
Keywords. Fractional differential equation, population model, numerical solution, pseudospectral method.

1 Introduction

Fractional calculus is regarded as an effective tool for the modelling of many physical phenomena. Its efficiency arises from the relaxing properties of fractional derivative operators, which enable to consider uncertainty and vagueness of physical phenomena. Since fractional approach is successfully applied to several kinds of models, it is inevitable to consider the population dynamics with a fractional approach.

Population dynamics is an essential branch of life sciences that studies size and age composition of populations as dynamical systems. An important model of population dynamics was presented by Volterra.

The Volterra's model for the population of a species within a closed system is defined as:

$$\frac{dp}{dt} = ap - bp^2 - cp\int_0^t p(x)dx, \tag{1.1}$$

$$p(0) = p_0, \tag{1.2}$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the competition between species, $c > 0$ is the toxicity coefficient, p_0 is the initial population and $p(t)$ is the population at time t. The coefficient c indicates the essential behaviour of the population evolution before it falls to zero in the long run. In the case $c = 0$, we have the well-known logistic equation. The last term contains the integral that indicates the total metabolism or total amount of toxins produced. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor u. The presence of the toxic term due to the system being closed always causes the population level to fall to zero in the long run. The relative size of the sensitivity to toxins, c, determines the manner in which the population evolves before its fated decay [1].

The time and population can be suitably scaled by applying the non-dimensional variables

$$t = \frac{tc}{b}, \quad u = \frac{pb}{a}$$

Tbilisi Mathematical Journal 10(1) (2017), pp. 269–278.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 20 August 2016.
Accepted for publication: 31 August 2016.

in order to lead to the non-dimensional problem

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \tag{1.3}$$

$$u(0) = u_0, \tag{1.4}$$

where $u(t)$ is the scaled population of identical individuals at time t, and $\kappa = \frac{c}{ab}$ is a prescribed non-dimensional parameter. The analytical solution of equation (1.3)

$$u(t) = u_0 \exp\left(\frac{1}{\kappa}\int_0^t \left[1 - u(\tau) - \int_0^\tau u(x)dx\right] d\tau\right), \tag{1.5}$$

shows that $u(t) > 0$ for all t when $u_0 > 0$. Problem (1.3)-(1.4) is studied extensively in the literature (see [2, 3] and references therein).

A fractional approach to problem (1.3)-(1.4) allows the establishment of the following fractional differential equation to describe the population growth of species within a closed system with a general sense:

$$\kappa \frac{d^\alpha u}{dt^\alpha} = u - u^2 - u \int_0^t u(x)dx, \tag{1.6}$$

$$u(0) = u_0, \tag{1.7}$$

where $\alpha \in (0, 1]$ is a constant describing the order of fractional derivative and κ is the same parameter of the model (1.3). The most important advantage of using fractional derivative operator is its nonlocal property. It is well-known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This makes the model more realistic. However, equation (1.6) has not been so far extensively studied except for a few papers (e.g., see [4, 5]).

An important contribution to the numerical approximation of fractional population model is the paper by Maleki et al. [5]. They presented a multi-domain Legendre-Gauss pseudospectral method for approximate solutions of the fractional population model. In this method the Volterra population model is replaced with a singular Volterra integro differential equation (SVDIE). Then, by choosing a step size, the replaced problem is converted into a sequence of SVIDEs in subintervals.

On the other hand, a useful treatment to obtain approximate solutions of fractional differential equations is replacing the power series term by order α as $\sum_{i=0}^n c_i x^{i\alpha}$. This is implemented in Adomian decomposition method, homotopy perturbation method and He's variational methods. Momani and Xu implemented homotopy and Adomian decomposition methods on the fractional population model in [6] and [7]. However, as it will be illustrated by means of numerical experiments, results are in some cases not sufficiently accurate.

The numerical approximation of Volterra type differential equations has been comprehensively studied in the literature [8, 9, 10, 11]. Many of the methods devised to approximate classical Volterra population models are however not suitable for application to the fractional case.

For instance, Dehghan presented a rational pseudospectral method for the classical Volterra population model in [6]. However, the rational Legendre polynomials are not appropriate for approximation of fractional differential equations.

Orthogonal polynomials and functions are applicable to different differential problems due to their appropriate properties that reduce the solutions of differential equations to algebraic systems. An extension of Legendre polynomials to the fractional case is constructed by Rida and Yousef previously in [12]. However the proposed method was not easy to implement on fractional differential equations due to its complexity. Another method inspired with the idea of fractional power series is the fractional order Legendre functions (FDF) in paper of Kazem et al. [13]. Kazem et al. presented an alternative way of generating orthogonal functions of the form $\varphi_n(x) = \sum_{i=0}^{n} c_i x^{i\alpha}$ to solve fractional differential equations more accurately. They called them as fractional order Legendre functions, presented the operational matrix and applied them on some linear and nonlinear differential equations in combination with the tau method. FDFs will be defined in Section 2 in detail.

In this paper, we aim to show the feasibility and efficiency of fractional Legendre functions for the approximation of the fractional population model (1.6). Here, it is useful to note that for $\alpha = 1$ case, they coincide with shifted Legendre polynomials. Throughout the paper, numerical experiments are carried out and their results support the effectiveness of fractional Legendre functions for solutions of fractional differential equations.

In the present paper, the following definitions of fractional derivative will be employed.

Definition 1.1. A real function $f(x),\ x > 0$ is said to be in space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^n \in C_\mu,\ n \in \mathbb{N}$.

Definition 1.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu,\ \mu \geq -1$, is defined as

$$\begin{aligned} I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,\ \alpha > 0, \\ I^0 f(t) &= f(t). \end{aligned}$$

Definition 1.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = I^{m-\alpha} D^m f(t),$$

for $m - 1 < \alpha \leq m,\ m \in \mathbb{N}, t > 0$ and $f \in C_{-1}^m$.

The rest of the paper is organized as follows: In section 2, a pseudo spectral approximation based on fractional Legendre functions will be presented for equation (1.6). Then, numerical experiments and their results are given to show the effectiveness of the pseudospectral fractional Legendre functions method on approximation of equation (1.6) in Section 3.

2 A pseudospectral approach for the fractional population model

We introduce in this section the fractional order Legendre functions and we present some of their properties. Then, a pseudospectral technique based on these functions and applied to equation (1.6) is investigated. Finally, convergence properties will be presented.

2.1 Fractional order Legendre functions

The shifted Legendre polynomials $L_n(t)$ are constructed by means of the transformation $z = 2t-1$ from standard Legendre polynomials. The shifted Legendre polynomials are orthogonal with respect to the weight function $w_s(t) = 1$ in the interval $(0,1)$ with the orthogonal property

$$\int_0^1 L_n(t)L_m(t)dt = \frac{1}{2n+1}\delta_{nm}.$$

It is well known that the sequece of $L_i(t)$ can be obtained by means of the recursive relationship

$$\begin{aligned} &L_{i+1}(t) = \frac{(2i+1)(2t-1)}{i+1}L_i(t) - \frac{i}{i+1}L_{i-1}(t),\ i = 1,2,..., \\ &L_0(t) = 1, \quad L_1(t) = 2t-1. \end{aligned}$$

The fractional order Legendre functions (FLF) can be defined by the change of variable $t = x^\alpha$, where and $\alpha > 0$, on shifted Legendre polynomials. Let the FLFs $L_i(x^\alpha)$ be simply denoted by $L_i^\alpha(x)$. The fractional order Legendre functions are particular solutions of normalized eigenfunctions of the singular Sturm-Liouville problem

$$((x - x^{1+\alpha})L_i'^\alpha(x))' + \alpha^2 i(i+1)x^{\alpha-1}L_i^\alpha(x) = 0,\ x \in (0,1).$$

Then, $L_i^\alpha(x)$ can be obtained as follows

$$\begin{aligned} L_{i+1}^\alpha(x) &= \frac{(2i+1)(2x^\alpha-1)}{i+1}L_i^\alpha(x) - \frac{i}{i+1}L_{i-1}^\alpha(x),\ i = 1,2,..., \\ L_0^\alpha(x) &= 1,\ L_1^\alpha(x) = 2x^\alpha - 1. \end{aligned}$$

The analytic form of $L_i^\alpha(x)$ of degree $i\alpha$ is given by

$$L_i^\alpha(x) = \sum_{s=0}^{i} b_{s,i}x^{s\alpha},$$

where

$$b_{s,i} = \frac{(-1)^{i+s}(i+s)!}{(i-s)!(s!)^2}$$

and, moreover, $L_i^\alpha(0) = (-1)^i$ and $L_i^\alpha(1) = 1$.

The FLFs are orthogonal with respect to the weight function $w(x) = x^{\alpha-1}$ in the interval $(0,1]$ with the orthogonal property (See [13])

$$\int_0^1 L_n^\alpha(x)L_m^\alpha(x)w(x)dx = \frac{1}{(2n+1)\alpha}\delta_{nm}.$$

Lemma 2.1. The Caputo fractional derivative of order $\gamma > 0$ of FLFs can be obtained in the form of

$$D^\gamma L_i^\alpha(x) = \sum_{s=0}^{i} b'_{s,i}\frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)}x^{s\alpha-\gamma},$$

where $b'_{s,i} = 0$ when $s\alpha \in \mathbb{N}_0$ and $s\alpha < \gamma$, in other cases $b'_{s,i} = b_{s,i}$.

For the proof we refer to [13].

2.2 The pseudospectral technique

For the pseudospectral technique, let us consider the problem

$$\begin{aligned} Lu &= 0, \\ u(0) &= u_0, \end{aligned}$$

where L is an operator, $u(t)$ is the unknown function and $u(0) = u_0$ denotes the initial condition. To use the pseudospectral method for solving problem (1.1), a set of known functions, $\{\varphi_n(t)\}_{n\geq 0}$, are chosen as basis and $u(t)$ is approximated in terms of these functions:

$$u_N(t) = \sum_{n=0}^{N} a_n \varphi_n(t).$$

Then, the operator L is applied on the approximate function and form the residual function

$$res(t) = Lu_N(t).$$

The next task is to choose the coefficients of the series to minimize the residual function. To do this, N nodal points are selected and the residual function is set equal to zero at these points [8]

$$res(t_i) = 0, \ i = 1, 2, ..., N.$$

These equations with the initial condition, at which the function u is substituted with u_N, form a system of $N + 1$ algebraic equations. By solving this system of equations, we can find the coefficients a_ns and so the approximation function can be determined. Also we refer the interested reader to [14].

It is important to note that appropriate Lobatto-Gauss-Legendre nodes fractional Legendre functions in the case $0 \leq \alpha < 1$ are used to get a better approximation.

2.3 Convergence

Suppose that

$$D^{k\alpha} f(x) \in C(0, 1]$$

for $k = 0, 1, ..., m$ and

$$P_m^\alpha = Span\{L_0^\alpha(x), L_1^\alpha(x), ..., L_{m-1}^\alpha(x)\}.$$

If $f_m = A^T \varphi$ is the best approximation to f from P_m^α then the error bound is presented as follows:

$$\|f(x) - f_m(x)\|_w \leq \frac{M_\alpha}{\Gamma(m\alpha + 1)} \sqrt{\frac{1}{(2m + 1)\alpha}},$$

where $M_\alpha \geq |D^{m\alpha} f(x)|, \ x \in (0, 1]$. For details and the proof see [15] and [13].

3 Numerical experiments

In the present section, numerical experiments will be presented to show the effectiveness of fractional Legendre functions on problem (1.1) by comparisons with other methods.

There have been many other numerical techniques in the literature for the classical Volterra population model. The present method asserts a pseudospectral approach based on shifted Legendre polynomials for this classical Volterra population model since the fractional Legendre functions coincide with shifted Legendre polynomials in the case $\alpha = 1$.

Table 1 gives the comparison for u_{max} of problem (1.1) of present method with Adomian Decomposition Method (ADM), Composite Spectral Functions (CSF) and Rational Legendre Method (RLM). It can be concluded from Table 1 that the shifted Legendre polynomials are applicable for problem (1.1) with a pseudospectral approach. However, CSF and RLM give better accuracy for u_{max}. Here, it is important to note that determination of u_{max} can be enhanced by some extra techniques like determination of place of u_{max}. Here, different N values $20 - 30$ are used to get u_{max} for different κ values. The maximum value is presented as u_{max}.

TABLE 1. A comparison for absolute errors of u_{max} for $\alpha = 1$.

K	ADM [17]	CSF [18]	RLM [8] ($N = 30$)	Present method
0.02	2.0×10^{-2}	9.7×10^{-7}	1.17×10^{-4}	6.8×10^{-5}
0.04	1.2×10^{-2}	7.8×10^{-7}	3.91×10^{-5}	4.5×10^{-5}
0.1	4.6×10^{-3}	5.9×10^{-7}	1.21×10^{-6}	4.8×10^{-5}
0.2	1.1×10^{-3}	6.8×10^{-7}	4.75×10^{-8}	3.0×10^{-6}

Table 2 shows that the method works very efficiently by comparing the residues for different $\alpha = \gamma$ values when $N = 30$. Again, Table 3 is presented to show the effectiveness of fractional Legendre functions on problem (1.6) for $0 < \alpha < 1$. In Table 3, u_{max} values of successful Multi Domain Pseudospectral Method (MDPM) are presented to compare with the present method when $\alpha = 0.75$. In Table 3 the classical shifted Legendre polynomials are also implemented on problem (1.6) to show the effectiveness of the fractional Legendre functions on fractional models.

TABLE 2. Residue errors $\|res\|_\infty$ for different α, γ values.

K	$\alpha = \gamma = 0.25$	$\alpha = \gamma = 0.75$	$\alpha = \gamma = 1$	RLM ($\alpha = \gamma = 1$)
0.02	2.77×10^{-16}	2.49×10^{-16}	3.88×10^{-16}	1.88×10^{-5}
0.04	2.11×10^{-16}	1.38×10^{-16}	2.22×10^{-13}	1.43×10^{-8}
0.1	1.66×10^{-16}	1.66×10^{-16}	2.22×10^{-16}	1.07×10^{-10}
0.2	1.81×10^{-16}	1.11×10^{-16}	1.66×10^{-16}	3.53×10^{-11}

In Figure 1, other numerical methods ADM and Pade approximants are compared with the present method. However, name of ADM method is skipped due to the equivalence with Pade approximants for $\alpha = 1, \kappa = 0.1, N = 20$ case.

TABLE 3. Absolute errors of u_{max} for different γ values, $\kappa = 0.2$, $\alpha = 0.75$.

Method	$N = 15$	$N = 20$	$N = 25$
MDPM [5]	0.636284	0.636283	0.636282
$\gamma = 1$	0.636109	0.636154	0.636177
$\gamma = 0.75$	0.636230	0.636243	0.636038

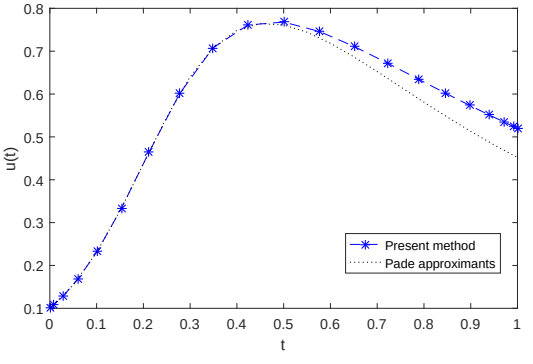

FIGURE 1. A comparison when $N = 20$, $\alpha = 1$, $K = 0.1$.

In Figure 2, solutions of the proposed method when $\gamma = 1$ are presented for different α values and $N = 20$. The solutions are valid when we compare with the findings in [5].

To show the effectiveness of the fractional Legendre functions, solutions of the method for different $\alpha = \gamma$ values is illustrated in Figure 3 for $N = 20$. It gives a better convergence than Figure 2.

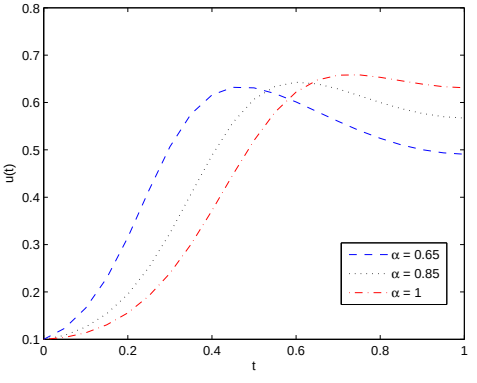

FIGURE 2. Resulting graph of $u(t)$ for different α values, $N = 20$.

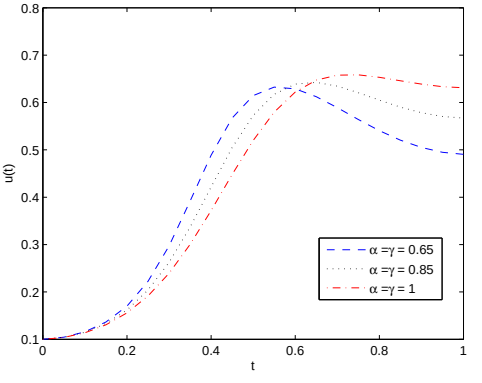

FIGURE 3. Resulting graph of $u(t)$ for different $\alpha = \gamma$ values, $N = 20$.

Lastly, Figure 4 shows the effect of different κ values on the solution of problem (1.1) when $\alpha = \gamma = 1, N = 20$. The results are valid when we analyze the similar figures in [5].

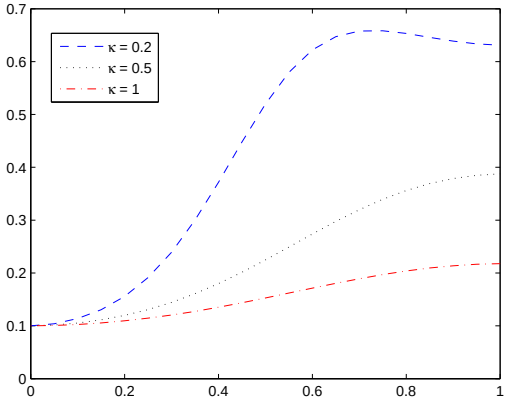

FIGURE 4. Resulting graph of $u(t)$ for different κ values, $N = 20$.

This section shows that the pseudospectral fractional Legendre functions method is feasible and efficient for approximate solution of fractional population model when a pseudospectral approach is considered.

4 Conclusions

In the present paper, an efficient pseudospectral method is presented via fractional Legendre functions for the fractional Volterra population model. Obtained numerical results support that the

presented method is valid and sufficiently accurate for the fractional population growth model.

Acknowledgments

This research is supported by Istanbul Medeniyet University, Turkey under Grant F-BAG-2016-808.

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