A survey and new investigation on $(n, n-k)$-type boundary value problems for higher order impulsive fractional differential equations

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Abstract

A survey for studies on boundary value problems of higher order ordinary differential equations is given firstly. Secondly a simple review for studies on solvability of boundary value problems for impulsive fractional differential equations is presented. Thirdly by using a general method for converting an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation and employing fixed point theorems in Banach space, we establish existence results of solutions for three classes of boundary value problems ($(n, n-k)$ type BVPs) of impulsive higher order fractional differential equations. Some examples are presented to illustrate the efficiency of the results obtained and some mistakes are also corrected at the end of the paper finally. A conclusion section is given at the end of the paper.

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1 Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [41]. Fractional differential equations therefore find numerous applications in different branches of physics, chemistry and biological sciences such as visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles and neuron modelling [43]. The reader may refer to the books and monographs [20, 28, 42] for fractional calculus and developments on fractional differential and fractional integro-differential equations with applications.

On the other hand, the theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such characteristics arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the readers to [32].

The first purpose of this paper is to present a survey for studies on BVPs for higher order ordinary differential equations and BVPs for impulsive fractional differential equations. The second purpose of this paper is to establish a general method for converting an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation.

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The third purpose of this paper is to establish existence results for three classes of $(n, n-k)$ type boundary value problems for higher order fractional differential equations with impulse effects. Our results are new since the methods are different from known ones.

1.1 BVPs for higher order ordinary differential equations

Solvability of boundary value problems for higher order ordinary differential equations were investigated by many authors. These boundary value problems mainly contain $2m$-th order Lidstone BVPs, $(n, n-p)$ type BVPs, anti-periodic BVPs, periodic BVPs and Neumann BVPs.

For examples, in [11, 12, 13, 45, 57], solvability of the following problems were investigated:

$$\begin{cases} y^{(2m)}(t) = f(y(t), \cdots, y^{(2j)}(t), \cdots, y^{(2(m-1))}(t)), 0 \le t \le 1, \\ y^{(2i)}(0) = 0 = y^{(2i)}(1), i \in \mathbb{N}_0^{m-1} \end{cases} \tag{1.1}$$

and

$$\begin{cases} y^{(2m)}(t) = f(y(t), \cdots, y^{(2j)}(t), \cdots, y^{(2(m-1))}(t)), 0 \le t \le 1, \\ y^{(2i)}(0) = 0 = y^{(2i+1)}(1), i \in \mathbb{N}_0^{m-1}. \end{cases} \tag{1.2}$$

Solvability of boundary value problems for higher order ordinary differential equations were investigated by many authors. For examples, in [4, 6, 14, 16, 24, 25, 26, 35, 40, 48, 50], the following $(n, n-k)$ type problems were studied:

$$\begin{cases} (-1)^{n-k} y^{(n)} = f(t, y), t \in (0, 1), \\ y^{(i)}(0) = 0, i \in \mathbb{N}_0^{k-1}, \\ y^{(j)}(1) = 0, j \in \mathbb{N}_0^{n-k-1}. \end{cases} \tag{1.3}$$

In [23, 27], the following more general boundary value problems were studied:

$$\begin{cases} (-1)^{n-k} y^{(n)} = f(t, y), t \in (0, 1), \\ y^{(i)}(0) = 0, i \in \mathbb{N}_0^{k-1}, \\ y^{(j)}(1) = 0, j \in \mathbb{N}_q^{n+q-k-1} \end{cases} \tag{1.4}$$

where $k \in \mathbb{N}_1^{n-1}, q \in \mathbb{N}_0^k$. In [6, 15], authors studied existence of solutions of the following problems:

$$\begin{cases} (-1)^{n-p} y^{(n)} = f(t, y, y', \cdots, y^{(p-1)}), t \in (0, 1), \\ y^{(i)}(0) = 0, i \in \mathbb{N}_0^{p-1}, \\ y^{(j)}(1) = 0, j \in \mathbb{N}_p^{n-1}. \end{cases} \tag{1.5}$$

Solvability of $(n, 1)$ type boundary value problems for higher order ordinary differential equations were investigated by many authors. For examples, the following boundary value problems were

studied in [17, 18, 52]:

$$\begin{cases} (-1)^{n-p}y^{(n)} = f(t, y, y', \cdots, y^{(p-1)}), t \in (0,1), \\ y^{(i)}(0) = 0, i \in \mathbb{N}_0^{n-2}, \\ y^{(p)}(1) = 0. \end{cases} \tag{1.6}$$

In [46], authors studied existence and uniqueness of solutions of anti-periodic boundary value problems for two classes of special second order impulsive differential equations. The methods used are based upon the monotone iterative technique coupled with lower and upper solutions.

In [19], authors investigated existence of positive solutions for Neumann boundary value problem and periodic boundary value problem for second order nonlinear equation $u'' + a(t)g(u(t)) = 0$. Necessary and sufficient conditions for the existence of nontrivial solutions are obtained. The method is based on Mawhin's coincidence degree.

As we know that the general anti-periodic, periodic and Neumann boundary value problems are as follows respectively:

$$y^{(n)} = f(t, y), t \in (0,1), \ \ y^{(i)}(0) = -y^{(i)}(1), i \in \mathbb{N}_0^{n-1}, \tag{1.7}$$

$$y^{(n)} = f(t, y), t \in (0,1), \ \ y^{(i)}(0) = y^{(i)}(1), i \in \mathbb{N}_0^{n-1}, \tag{1.8}$$

and

$$y^{(2n)} = f(t, y), t \in (0,1), \ \ y^{(2i+1)}(0) = y^{(2i+1)}(1) = 0, i \in \mathbb{N}_0^{n-1}. \tag{1.9}$$

Readers may see [1] in which (1.7) was studied with $n = 5$. In [34], the solvability of (1.8) was investigated. Some special cases of (1.9) were studied in [59] and [38].

1.2 BVPs for impulsive fractional differential equations

Impulsive fractional differential equations is an important area of study [5]. Recently, many authors in [2, 3, 7, 8, 21, 29, 47, 55] studied existence of solutions for different kinds of initial value problems or boundary value problems involving impulsive fractional differential equations. The first kind of problems is concerned with impulsive fractional differential equations with multiple starting points $t = t_i (i \in \mathbb{N}_0^m)$. The second kind of problems is concerned with impulsive fractional differential equations with single starting points $t = 0$.

(A) Existence and uniqueness of solutions of boundary value problems of impulsive fractional differential equations with multiple starting points $t = t_i (i \in \mathbb{N}_0^m)$.

Recently, Wang [51] consider the second case in which D^α has multiple start points, i.e., $D^\alpha = D^\alpha_{t_i^+}$. They studied the existence and uniqueness of solutions of the following initial value problem of the impulsive fractional differential equation

$$\begin{cases} {}^cD^\alpha_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u_j, j \in \mathbb{N}_0^{n-1}, \\ \Delta u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}_1^p, j \in \mathbb{N}_0^{n-1}, \end{cases} \tag{1.10}$$

where $\alpha \in (n-1, n)$ with n being a positive integer, ${}^cD^\alpha_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $\mathbb{N}_a^b = \{a, a+1, \cdots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \cdots <$

$t_p < t_{p+1} = 1$, $I_{ji} \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}_1^p, j \in \mathbb{N}_0^{n-1})$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Henderson and Ouahab [22] studied the existence of solutions of the following initial value problems and periodic boundary value problems of impulsive fractional differential equations:

$$\begin{cases} {}^cD^{\alpha}_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u_j, j \in \mathbb{N}_0^1, \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}_1^p, j \in \mathbb{N}_0^1, \end{cases} \tag{1.11}$$

and

$$\begin{cases} {}^cD^{\alpha}_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u^{(j)}(b), j \in \mathbb{N}_0^1, \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}_1^p, j \in \mathbb{N}_0^1, \end{cases} \tag{1.11}$$

where $\alpha \in (1,2]$, $b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$, $f : [0,b] \times \mathbb{R} \to \mathbb{R}$, $I_{ji} : \mathbb{R} \to \mathbb{R}$ are continuous functions. Readers should also refer [53].

In [60], Zhao and Gong studied existence of positive solutions of the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions

$$\begin{cases} {}^cD^{q}_{t_i^+} u(t) = f(t, u(t)), t \in (0,T] \setminus \{t_1, \cdots, t_p\}, \\ \Delta u(t_i)] = I_i(u(t_i)), i \in \mathbb{N}_1^p, \\ \Delta u'(t_i)] = J_i u(t_i)), i \in \mathbb{N}_1^p, \\ \alpha u(0) - \beta u(1) = 0,\ \alpha u'(0) - \beta u'(1) = 0, \end{cases} \tag{1.12}$$

where $q \in (1,2)$, ${}^CD^{q}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order q, $\alpha > \beta > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C([0,+\infty), [0,+\infty))(i \in \mathbb{N}_1^p$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Wang, Ahmad and Zhang [54] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^cD^{\alpha}_{t_i^+} u(t) = f(t, u(t)), t \in (0,T] \setminus \{t_1, \cdots, t_p\}, \\ \Delta u(t_i)] = I_i(u(t_i)), i \in \mathbb{N}_1^p, \\ \Delta u'(t_i)] = I_i^*(u(t_i)), i \in \mathbb{N}_1^p, \\ u'(0) + (-1)^{\theta} u(T) = b u(T),\ u(0) + (-1)^{\theta} u(T) = 0, \end{cases} \tag{1.13}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $\theta = 1, 2$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $I_i, I_i^* \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}_1^p$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [9, 10, 61], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+} x(t) = w(t) f(t, x(t), x'(t)), t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}_1^p, \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}_1^p, \\ ax(0) \pm bx'(0) = g_1(x),\ cx(1) + dx'(1) = g_2(x), \end{cases} \tag{1.14}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $a,b,c,d \geq 0$ with $ac+ad+bc \neq 0$, $0=t_0<t_1<\cdots<t_p<t_{p+1}=1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}_1^p$, $f: [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous, $w:[0,1]\to[0,+\infty)$ is a continuous function, $g_1, g_2 : PC(0,1]\to\mathbb{R}$ are two continuous functions.

In 2015, Zhou, Liu and Zhang [62] studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}x(t) = \lambda x(t) + f(t,x(t),(Kx)(t),(Hx)(t)), t \in (0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}_1^p, \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}_1^p, \\ ax(0)-bx'(0)=x_0,\ cx(1)+dx'(1)=x_1, \end{cases} \tag{1.15}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $a \geq 0, b>0, c \geq 0, d>0$ with $\delta = ac+ad+bc \neq 0$, $\lambda>0$, $x_0, x_1 \in \mathbb{R}$, $0=t_0<t_1<\cdots<t_p<t_{p+1}=1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in N_1^p$, $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is continuous, $(Hx)(t)=\int_0^1 h(t,s)x(s)ds$ and $(Kx)(t)=\int_0^t k(t,s)x(s)ds$.

In [33, 37], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}x(t) = f(t,x(t)), t \in (0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}_1^p, \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}_1^p, \\ ax(0)+bx(1)=g_1(x),\ ax'(0)+bx'(1)=g_2(x), \end{cases} \tag{1.16}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $a,b \in R$ with $a \geq b > 0$, $0=t_0<t_1<\cdots<t_p<t_{p+1}=1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}_1^p$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, $g_1, g_2 : PC(0,1]\to\mathbb{R}$ are two continuous functions.

In [36], Liu and Li investigated the existence and uniqueness of solutions for the following nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}u(t) = f(t,u(t),u'(t),u''(t)), t \in (t_i,t_{i+1}], i \in \mathbb{N}_0^p, \\ u(0)=\lambda_1 u(T)+\xi_1\int_0^T q_1(s,u(s),u'(s),u''(s))ds, \\ u'(0)=\lambda_2 u'(T)+\xi_2\int_0^T q_2(s,u(s),u'(s),u''(s))ds, \\ u''(0)=\lambda_3 u''(T)+\xi_3\int_0^T q_3(s,u(s),u'(s),u''(s))ds, \\ \Delta u(t_i)] = A_i(u(t_i)), i \in \mathbb{N}_1^p, \\ \Delta u'(t_i)] = B_i(u(t_i)), i \in \mathbb{N}_1^p, \\ \Delta u''(t_i)] = C_i(u(t_i)), i \in \mathbb{N}_1^p, \end{cases} \tag{1.17}$$

where $\alpha \in (2,3)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, $\lambda_i, \xi_i \in \mathbb{R}(i=1,2,3)$ are constants, $A_i, B_i, C_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}_1^p$, $f:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ is continuous.

Recently, in [**?**], to extend the problem for impulsive differential equation $u''(t)-\lambda u(t) = f(t,u(t)), u(0)=u(T)=0, \Delta u'(t_i)=I_i(u(t)i)), i=1,2,\cdots,p$ to impulsive fractional differential

equation, the authors studied the existence and the multiplicity of solutions for the Dirichlet's boundary value problem for impulsive fractional order differential equation

$$\begin{cases} {}^cD^{\alpha}_{T^-}({}^CD^{\alpha}_{0^+}x(t)+a(t)x(t)=\lambda f(t,x(t)), t\in[0,T], t\neq t_i, i\in\mathbb{N}_1^m, \\ \Delta {}^CD^{\alpha-1}_{T^-}({}^CD^{\alpha}_{0^+}x(t_i)=\mu I_i(x(t_i^-)), i\in\mathbb{N}_1^m,\ x(0)=x(T)=0, \end{cases} \tag{1.18}$$

where $\alpha\in(1/2,1]$, $\lambda,\mu>0$ are constants, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $I_i:\mathbb{R}\to\mathbb{R}(i\in\mathbb{N}_1^m)$ are continuous functions, ${}^cD^{\alpha}_{0^+}$ (or ${}^cD^{\alpha}_{T^-}$) is the standard left (or right) Liouville-Caputo fractional derivative of order α, $a\in C[0,T]$ and there exist constants $a_1,a_2>0$ such that $a_1\le a(t)\le a_2$ for all $t\in[0,T]$, $\Delta x|_{t=t_i}=\lim\limits_{t\to t_i^+}x(t)-\lim\limits_{t\to t_i^-}x(t)=x(t_i^+)-x(t_i^-)$ and $x(t_i^+),x(t_i^-)$ represent the right and left limits of $x(t)$ at $t=t_i$ respectively, a,b,x_0 a constant with $a+b\neq 0$. One knows that the boundary condition $ax(0)+bx(T)=x_0$ becomes $x(0)-x(T)=\frac{x_0}{a}$ when $a+b=0$, that is so called nonhomogeneous periodic type boundary condition.

For impulsive fractional differential equations whose derivatives have single starting points $t=0$, there has been few papers published. In [49], authors presented a new method to converting the impulsive fractional differential equation (with the Liouville-Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Liouville-Caputo fractional derivatives with single start point. The existence and uniqueness of solutions of the following initial or boundary value problems were discussed in [49]:

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t)=f(t,x(t)), t\in(0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)]=I_i(x(t_i)),\ \Delta x'(t_i)]=J_i(x(t_i)),\ i\in\mathbb{N}_1^p, \\ x(0)=x_0,\ x'(0)=x_1, \end{cases} \tag{1.19}$$

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t)=f(t,x(t)), t\in(0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)]=I_i(x(t_i)),\ \Delta x'(t_i)]=J_i(x(t_i)),\ i\in\mathbb{N}_1^p, \\ x(0)+\varphi(x)=x_0,\ x'(0)=x_1, \end{cases} \tag{1.20}$$

$$\begin{cases} {}^cD^{\beta}_{0^+}x(t)=f(t,x(t)), t\in(0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)]=I_i(x(t_i)),\ i\in\mathbb{N}_1^p, \\ ax(0)+bx(1)=0, \end{cases} \tag{1.21}$$

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t)=f(t,x(t)), t\in(0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)]=I_i(x(t_i)),\ \Delta x'(t_i)]=J_i(x(t_i)),\ i\in\mathbb{N}_1^p, \\ ax(0)-bx'(0)=x_0,\ cx(1)+dx'(1)=x_1, \end{cases} \tag{1.22}$$

and

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t)=f(t,x(t)), t\in(0,1]\setminus\{t_1,\cdots,t_p\}, \\ \Delta x(t_i)]=I_i(x(t_i)),\ \Delta x'(t_i)]=J_i(x(t_i)),\ i\in\mathbb{N}_1^p, \\ x(0)-ax(\xi)=x(1)-bx(\eta)=0, \end{cases} \tag{1.23}$$

where $\alpha\in(1,2],\beta\in(0,1]$, $D^{*}_{0^+}$ is the Liouville-Caputo fractional derivative with order $*$ and single start point $t=0$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $I_i,J_i:\mathbb{R}\to\mathbb{R}$ are continuous functions, $a,b,c,d,x_0,x_1\in\mathbb{R}$ are constants, $\varphi:PC(0,1]\to\mathbb{R}$ is a functional.

In [60], authors studied the existence of positive solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}x(t) = f(t, x(t)), t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}_1^p, \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}_1^p, \\ ax(0) - bx(1) = 0,\ ax'(0) - bx'(1) = 0, \end{cases} \tag{1.24}$$

where $\alpha \in (1,2)$, ${}^{C}D^{\alpha}_{0^+}$ represents the standard Liouville-Caputo fractional derivatives of order α, $a > b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}_1^p$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(B) Existence and uniqueness of solutions of boundary value problems of impulsive fractional differential equations with single starting point $t = 0$.

In [56], authors studied existence of solutions of the following boundary value problem for higher order fractional differential equation

$$\begin{cases} D^{\alpha}_{0^+}u(t) + \lambda f(t, u(t)) = 0, 0 < t < b, \lambda > 0, \alpha \in [n, n+1), \\ \\ u^{(j)}(0) = 0, j \in \mathbb{N}_0^{n-1}, u^{(n-1)}(b) = 0. \end{cases} \tag{1.25}$$

In [58], solutions of the following problem were presented:

$$\begin{cases} D^{\alpha}_{0^+}u(t) + p(t)u(t) = 0, 0 < t < 1, \lambda > 0, \alpha \in [n-1, n), \\ \\ u^{(j)}(0) = 0, j \in \mathbb{N}_0^{n-2}, u(1) = 0. \end{cases} \tag{1.26}$$

In recent paper [31], Liu studied existence of positive solutions for the following boundary value problems (BVP) of fractional impulsive differential equations

$$\begin{cases} D^{\alpha}_{0^+}u(t) = -f(t, u(t)), t \in (0,1), t \neq t_k, k = 1, 2, \cdots, m, \\ \\ u(t_k^+) = (1 - c_k)u(t_k^-), k \in \mathbb{N}_1^m,\ \ u(0) = u(1) = 0, \end{cases} \tag{1.27}$$

where $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$ with the base point 0, m is a positive integer, $c_k \in \left(0, \frac{1}{2}\right)$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function, $u(t_k^+)$ and $u(t_k^-)$ denote the right limit and left limit of u at t_k and $u(t_k^+) = u(t_k)$, i.e., u is right continuous at t_k. By constructing a novel transformation, BVP(1.27) is convert into a continuous system. using a specially constructed cone, the Krein-Rutman theorem, topological degree theory, and bifurcation techniques, some sufficient conditions are obtained for the existence of positive solutions of BVP(1.27). However, we find that Lemma 3.1[31] is un-correct.

1.3 Purposes of this paper

We note that in known papers existence of solutions of boundary value problems for the Liouville-Caputo type fractional differential equations of lower order has been discussed deeply. Solvability of boundary value problems for impulsive higher order fractional differential equations has not been studied. The reasons are as follows: higher order fractional differential equations can not be

converted to fractional differential systems with lower order since $D_{0^+}^{\alpha} D_{0^+}^{\beta} x(t) \neq D_{0^+}^{\alpha+\beta} x(t)$ [44]. it is difficult to convert an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation.

(C) The first purpose of this paper is to establish a general method for converting an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation.

(D) The second purpose of this paper is to establish existence results for the following three classes of $(n, n-k)$ type boundary value problems of higher order fractional differential equations with impulse effects:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0^m, \\ I_{0^+}^{n-\alpha} u(0) = 0,\ D_{0^+}^{\alpha-n+j} u(0) = 0, j \in \mathbb{N}_1^{k-1}, \\ I_{0^+}^{n-\alpha} u(1) = 0,\ D_{0^+}^{\alpha-n+j} u(1) = 0, j \in \mathbb{N}_1^{n-k-1}, \\ \lim\limits_{t \to t_s^+} (t - t_s)^{n-\alpha} u(t) = I_n(t_s, u(t_s)),\ s \in \mathbb{N}_1^m, \\ \Delta D_{0^+}^{\alpha-n+j} u(t_s) = I_j(t_s, u(t_s)), j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m, \end{cases} \tag{1.28}$$

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0^m, \\ I_{0^+}^{n-\alpha} u(0) = 0, \ \ D_{0^+}^{\alpha-n+j} u(0) = 0, j \in \mathbb{N}_1^{k-1}, \\ D_{0^+}^{\alpha-n+j} u(1) = 0, j \in \mathbb{N}_l^{n+l-k-1}, \\ \lim\limits_{t \to t_s^+} (t - t_s)^{n-\alpha} u(t) = I_n(t_s, u(t_s)),\ s \in \mathbb{N}_1^m, \\ \Delta D_{0^+}^{\alpha-n+j} u(t_s) = I_j(t_s, u(t_s)), j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m, \end{cases} \tag{1.29}$$

and

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0^m, \\ D_{0^+}^{\alpha-n+i} u(0) = 0, i \in \mathbb{N}_1^k, \\ D_{0^+}^{\alpha-n+j} u(1) = 0, j \in \mathbb{N}_1^{n-k}, \\ \lim\limits_{t \to t_s^+} (t - t_s)^{n-\alpha} u(t) = I_n(t_s, u(t_s)),\ s \in \mathbb{N}_1^m, \\ \Delta D_{0^+}^{\alpha-n+j} u(t_s) = I_j(t_s, u(t_s)), j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m, \end{cases} \tag{1.30}$$

where

(a) $n-1 < \alpha < n$, n is a positive integer, $D_{0^+}^{\alpha}$ the Riemann-Liouville fractional derivatives of order α with starting point 0 respectively,

(b) $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\mathbb{N}_a^b = \{a, a+1, a+2, \cdots, b\}$ for every pair of integers $a < b$, $l \in \mathbb{N}_1^k$, $k \in \mathbb{N}_1^{n-1}$ is a positive integer,

(c) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$, I, $I_j : \{t_s : s \in \mathbb{N}_1^m\} \times \mathbb{R} \to \mathbb{R}$, f is a Carathéodory function, $I_j (j \in \mathbb{N}_1^n)$ are discrete Carathéodory functions,

A function x with $x : (0,1] \to \mathbb{R}$ is said to be a *solution* of BVP(1.28) (or BVP(1.29), BVP(1.30)) if

$$x|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}], \quad \lim_{t \to t_s^+} (t - t_s)^{n-\alpha} x(t) \text{ exists for all } s \in \mathbb{N}_0^m, D_{0^+}^{\alpha} x \text{ are measurable on } (0,1]$$

and x satisfies all equations in (1.28) or (1.29), (1.30) respectively.

We shall construct a weighted Banach space and apply two standard fixed point theorems to obtain the existence of at least one solution of BVP(1.28), BVP(1.29) and BVP(1.30) respectively. Our results are new and naturally complement the literature on higher order impulsive fractional differential equations. This paper may be the first one concerned with the solvability of boundary value problems for higher order singular fractional differential equations with impulse effects and the Riemann-Liouville fractional derivatives.

The paper is outlined as follows. Section 2 contains some definitions needed. Preliminary results are given in Section 3. Applications (Main results) are given in Section 4. In Section 5 we give examples to illustrate the efficiency of the results obtained. Section 6 is a conclusion section. In Appention section, we given proofs of Theorem 4.1 and Theorem 4.3.

2 Definitions

For the convenience of the readers, we shall state the necessary definitions from fractional calculus theory.

For $\varphi \in L^1(0,1)$, denote $\|\varphi\|_1 = \int_0^1 |\varphi(s)| ds$. Let the Gamma and beta functions $\Gamma(\alpha)(\alpha > 0)$ and $\mathbf{B}(a,b)(a > 0, b > 0)$ be defined by $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$, $\mathbf{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$.

Definition 2.1[44]**.** The left *Riemann-Liouville fractional integral of order* $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is given by $I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, t > 0$ provided that the right-hand side exists.

Definition 2.2[44]**.** The left *Riemann-Liouville fractional derivative of order* $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is given by $D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, t > 0$ where $n - 1 < \alpha < n$, provided that the right-hand side exists.

Remark 2.1. Let $h : (0,1] \to \mathbb{R}$ satisfy $h|_{(t_i, t_{i+1}]} \in C(t_i, t_{i+1}] i \in \mathbb{N}_0^m)$. The left Riemann-Liouville fractional integral of order $\alpha > 0$ of h at point $t \in (t_i, t_{i+1}]$ is given by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (t-s)^{\alpha-1} h(s) ds + \int_{t_i}^{t} (t-s)^{\alpha-1} h(s) ds \right],$$

provided that each term of the right-hand side exists.

Let $h:(0,1]\to\mathbb{R}$ satisfy $h|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}]i\in\mathbb{N}_0^m)$. The left Riemann-Liouville fractional derivative of order $\alpha>0$ of h at point $t\in(t_i,t_{i+1}]$ is given by

$$D_{a^+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\left[\frac{d^n}{dt^n}\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\frac{h(s)}{(t-s)^{\alpha-n+1}}ds+\frac{d^n}{dt^n}\int_{t_i}^{t}\frac{h(s)}{(t-s)^{\alpha-n+1}}ds\right],$$

where $n-1<\alpha<n$, provided that each term of the right-hand side exists.

Definition 2.3. Set $p>-1$ and $q\in(-1,0]$. We say $K:(0,1)\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function if it satisfies the followings:

(i) $t\to K\left(t,\ (t-t_s)^{\alpha-n}x\right)$ is integral on $(t_s,t_{s+1}](s\in\mathbb{N}_0^m)$ for every $x\in\mathbb{R}$,

(ii) $x\to K\left(t,\ (t-t_s)^{\alpha-n}x\right)$ is continuous on $\mathbb{R}$ for all $t\in(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0^m)$;

(iii) for each $r>0$ there exists a constant $A_{r,f}\geq 0$ satisfying

$$|K\left(t,\ (t-t_s)^{\alpha-n}x\right)|\leq A_{r,K}t^p(1-t)^q$$

holds for $t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0^m,\ |x|\leq r$.

Definition 2.4. $G:\{t_s:s\in\mathbb{N}_1^m\}\times\mathbb{R}\to\mathbb{R}$ is called a discrete Carathéodory function if

(i) $x\to G\left(t_s,\ (t_s-t_{s-1})^{\alpha-n}x\right)$ is continuous on $\mathbb{R}$ for each $s\in\mathbb{N}_1^m$,

(ii) for each $r>0$ there exists $A_{r,G,s}\geq 0$ such that

$$|G\left(t_s,\ (t_s-t_{s-1})^{\alpha-n}x\right)|\leq A_{r,G,s}$$

holds for $|x|\leq r,\ s\in\mathbb{N}_1^m$.

Definition 2.5[39]. Let E and Z be Banach spaces. $L:\ D(L)\subset E\to Z$ is called a Fredholm operator of index zero if $\mathrm{Im}L$ is closed in E and $\dim\mathrm{Ker}L=\ \mathrm{co}\dim\mathrm{Im}L<+\infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist the projectors $P:\ E\to E$, and $Q:\ Z\to Z$ such that $\mathrm{Im}\ P=\mathrm{Ker}\ L,\ \mathrm{Ker}\ Q=\mathrm{Im}\ L,\ \ X=\mathrm{Ker}\ L\oplus\mathrm{Ker}\ P,\ \ Y=\mathrm{Im}\ L\oplus\mathrm{Im}\ Q$.

If $L:\ D(L)\subset E\to Z$ is called a Fredholm operator of index zero, the inverse of $L|_{D(L)\cap\mathrm{Ker}\ P}:D(L)\cap\mathrm{Ker}\ P\to\mathrm{Im}\ L$ is denoted by K_p.

Definition 2.6[39]. Let $E_i(i=1,2)$ be a Banach space and $T:E_1\to E_2$ is well defined. If T is continuous and maps bounded subsets of E_1 to relative compact subsets of E_2, we call T a completely continuous operator.

3 Preliminary results

In this section, we introduce a new method for converting an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation.

Theorem 3.1[30]. Suppose that $n-1<\alpha<n$, h is integral on $(0,1)$. Then x satisfying

$$\begin{aligned} &x|_{(t_s,t_{s+1}]},\ D_{0^+}^{\alpha-j}x\in C^0(t_s,t_{s+1}], s\in\mathbb{N}_0^m, j\in\mathbb{N}_1^{n-1},\\ &\lim_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t),\ \lim_{t\to t_s^+}D_{0^+}^{\alpha-j}x(t) \text{ exist for all } s\in\mathbb{N}_0^m, j\in\mathbb{N}_1^{n-1}\end{aligned} \tag{3.1}$$

is a solution of

$$D_{0^+}^{\alpha}x(t)=h(t), a.e., t\in(t_i,t_{i+1}](i\in\mathbb{N}_0^m) \tag{3.2}$$

if and only if there exist constants $c_{vj}\in\mathbb{R}(v\in\mathbb{N}_1^n, j\in\mathbb{N}_0^m)$ such that

$$x(t)=\sum_{j=0}^{i}\sum_{v=1}^{n}\frac{c_{vj}}{\Gamma(\alpha-v+1)}(t-t_j)^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m. \tag{3.3}$$

Define

$$X=\left\{x:(0,1]\to\mathbb{R}:\begin{array}{l} x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]\ (s\in\mathbb{N}_0^m),\\ \lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t) \text{ exists } (s\in\mathbb{N}_0^m)\end{array}\right\}.$$

For $x\in X$, define the norms by $\|x\|=\|x\|_X=\max\left\{\sup\limits_{t\in(t_s,t_{s+1}]}(t-t_s)^{n-\alpha}|x(t)|:s\in\mathbb{N}_0^m\right\}$.

Lemma 3.2. X is a Banach space.
Proof. The proof is standard and omitted. ∎

Lemma 3.3. Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:
(i) $\{t\to(t-t_s)^{n-\alpha}x(t):x\in M\}$ is uniformly bounded,
(ii) $\{t\to(t-t_s)^{n-\alpha}x(t):x\in M\}$ is equicontinuous in any interval $(t_s,t_{s+1}](s\in\mathbb{N}_0^m)$.
Proof. The proof is standard and omitted. ∎

Remark 3.1. Suppose that $x\in C^0(0,t_1]$, $\alpha\in(n-1,n)$, A is a constant. Then $\lim\limits_{t\to0^+}t^{n-\alpha}x(t)=A$ implies $\lim\limits_{t\to0^+}I_{0^+}^{n-\alpha}x(t)=\Gamma(\alpha-n+1)A$. In fact, for each $\varepsilon>0$, by $\lim\limits_{t\to0^+}t^{n-\alpha}x(t)=A$, there exists $\delta\in(0,t_1]$ such that $A-\varepsilon<t^{n-\alpha}x(t)<A+\varepsilon, t\in(0,\delta)$. Note

$$\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}ds=\int_0^1\frac{(1-w)^{n-\alpha-1}}{\Gamma(n-\alpha)}w^{\alpha-n}dw=\Gamma(\alpha-n+1).$$

Then for $t\in(0,\delta)$, we have

$$\begin{aligned}&|I_{0^+}^{n-\alpha}x(t)-\Gamma(\alpha-n+1)A|=\left|\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(s)ds-\Gamma(\alpha-n+1)A\right|\\ &\le\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}\left|s^{n-\alpha}x(s)-A\right|ds<\varepsilon\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}=\Gamma(\alpha-n+1)\varepsilon.\end{aligned}$$

This is $\lim\limits_{t\to0^+}I_{0^+}x(t)=\Gamma(\alpha-n+1)A$. ∎

Remark 3.2. Let $M = (m_{ij})_{n\times n}$ be a matrix with $|M| \neq 0$ and $|m_{ij}| \leq 1 (i, j \in \mathbb{N}_1^n)$. Suppose M_{ij} be the the algebraic cofactors of m_{ij}. Then $|M_{ij}| \leq \Gamma(n)$. In fact, we have

$$|M_{ij}| = \begin{vmatrix} m_{11} & \cdots & m_{1\ j-1} & m_{1\ j+1} & \cdots & m_{1\ n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{i-1\ 1} & \cdots & m_{i-1\ j-1} & m_{i-1\ j+1} & \cdots & m_{i-1\ n} \\ m_{i+1\ 1} & \cdots & m_{i+1\ j-1} & m_{i+1\ j+1} & \cdots & m_{i+1\ n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n1} & \cdots & m_{n\ j-1} & m_{n\ j+1} & \cdots & m_{n\ n} \end{vmatrix}$$

$$= \left| \sum_{i_1, i_2, \cdots, i_n} (-1)^{\sigma(i_1, i_2, \cdots, i_n)} m_{1\ i_1} m_{2\ i_2} \cdots m_{n\ i_n} \right|$$

$$\leq \sum_{i_1, i_2, \cdots, i_n} |(-1)^{\sigma(i_1, i_2, \cdots, i_n)} m_{1\ i_1} m_{2\ i_2} \cdots m_{n\ i_n}| \leq \Gamma(n),$$

where $i_1, i_2, \cdots, i_n$ is an array of the elements of $\mathbb{N}_1^n$ and $\sigma(i_1, i_2, \cdots, i_n)$ is the inverse number of $i_1, i_2, \cdots, i_n$. ∎

To obtain the main results, we need the Leray-Schauder nonlinear alternative.

Lemma 3.4[39]. Let E be a Banach space, U be a closed, bounded and convex subset of E, $\Omega \subset U$ be an open ball aand the zero point $\theta \in \Omega$. Suppose $T : \overline{\Omega} \to E$ is completely contnuous. Then one of the following results holds:

(i) T has atl least fixed point in $\overline{\Omega}$;

(ii) there exist a $x \in \partial\Omega$ and $\lambda \in 01$ such that $x = \mu Tx + (1 - \lambda)\theta$.

Lemma 3.5[39]. Suppose E and Z are Banach spaces. Let $L : D(L) \cap E \to Z$ be a Fredholm operator of index zero and $N : E \to Z$ be $L-$compact on each open nonempty set Ω centered at zero. Assume that the following conditions are satisfied:

(i). $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$;

(ii). $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$;

(iii). $\deg(\wedge^{-1} QN\big|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, where $\wedge^{-1} : Y/\mathrm{Im}L \to \mathrm{Ker}L$ is the inverse of the isomorphism $\wedge : \mathrm{Ker}L \to Y/\mathrm{Im}L$.

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

4 Applications

In this section, we apply Theorem 3.1 to establish existence results for BVP(1.28), BVP(1.29), BVP(1.30) respectively. We firstly converting these problems into integral equations by using Theorem 3.1. By applying Lemma 3.4, we prove the main results for BVP(1.28) and BVP(1.29). By using Lemma 3.5, we establish existence result for BVP(1.30). Denote $f_x(t) = f(t, x(t)), t \in$

$(t_s, t_{s+1}], s \in \mathbb{N}_0^m$ and $K_x(t_s) = K(t_s, x(t_s)), s \in \mathbb{N}_1^m$. Define the following $(n-k)\times(n-k)$ matrix:

$$M = (m_{ij})_{(n-k)\times(n-k)} =$$

$$\begin{cases} \begin{pmatrix} \frac{1}{\Gamma(n)} & \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-1)} & \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k)} & \frac{1}{\Gamma(n-2k-1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(n-2k-1)} & \frac{1}{\Gamma(n-2k-2)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\ \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \text{ for } k < \frac{n-1}{2}, \\ \begin{pmatrix} \frac{1}{\Gamma(n)} & \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-1)} & \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(2k-n+2)} \end{pmatrix} \text{ for } k \geq \frac{n-1}{2}. \end{cases}$$

Remark 4.1. By direct computation, we know that $|M| = \sum_{j=1}^{n-k} m_{ij}M_{ij} \neq 0$, where M_{ij} is the algebraic cofactors of m_{ij} respectively. Furthermore, we have from Remark 3.2 that $|M_{ij}| \leq \Gamma(n-k)$ and $M^{-1} = \frac{M^*}{|M|}$, where M^* is the adjoint matrix of M, i.e.,

$$M^* = \begin{pmatrix} M_{11} & M_{21} & \cdots & M_{n-k\ 1} \\ M_{12} & M_{22} & \cdots & M_{n-k\ 2} \\ \cdots & \cdots & \cdots & \cdots \\ M_{1\ n-k} & M_{2\ n-k} & \cdots & M_{n-k\ n-k} \end{pmatrix}.$$

Lemma 4.1. Suppose that $h : (0,1) \to \mathbb{R}$ is integral and satisfies $|h(t)| \leq t^p(1-t)^q$ on $(0,1)$, and $I_j \in \mathbb{R}$. Then $x \in X$ is a solution of

$$\begin{cases} D_{0^+}^{\alpha} x(t) = h(t), \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0, \\ I_{0^+}^{n-\alpha} x(0) = 0,\ D_{0^+}^{\alpha-n+j} x(0) = 0,\ j \in \mathbb{N}_1^{k-1}, \\ I_{0^+}^{n-\alpha} x(1) = 0,\ D_{0^+}^{\alpha-n+j} x(1) = 0, j \in \mathbb{N}_1^{n-k-1}, \\ \lim\limits_{t \to t_s^+} (t-t_s)^{n-\alpha} x(t) = I_{ns},\ \ \Delta D_{0^+}^{\alpha-n+j} x(t_s) = I_{n-j\ s}, j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m, \end{cases} \tag{4.1}$$

if and only if

$$x(t) = -\sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{M_{jv}}{|M|} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-(j-1)} \frac{I_{\sigma w}(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)}$$

$$-\sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{M_{jv}}{|M|} \int_0^1 \frac{(1-s)^{\alpha-(j-1)-1}}{\Gamma(\alpha-(j-1))} h(s)ds \tag{4.2}$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n} \frac{I_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

Proof. On sees that

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right| \le t^{\alpha+p+q} \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} \tag{4.3}$$

and

$$\left|\int_0^t \frac{(t-s)^{n-j-1}}{\Gamma(n-j)} h(s)ds\right| \le t^{n-j+p+q} \frac{\mathbf{B}(n-j+q,p+1)}{\Gamma(n-j)}, j \in \mathbb{N}_0^{k-1}. \tag{4.4}$$

Firstly we prove that x satisfies (4.2) if $x \in X$ and x is a solution of (4.1).

Since $x \in X$, there exists $r \ge 0$ such that

$$\|x\| = \max\left\{\sup_{t\in(t_s,t_{s+1}]} (t-t_s)^{n-\alpha}|x(t)| : s \in \mathbb{N}_0^m\right\} = r. \tag{4.5}$$

By Theorem 3.1, there exist constants $c_{vw} \in \mathbb{R}(v \in \mathbb{N}_1^n, w \in \mathbb{N}_0^m)$ such that

$$x(t) = \sum_{w=0}^{s}\sum_{v=1}^{n} \frac{c_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m. \tag{4.6}$$

By Definition 2.1 and Definition 2.2, we have

$$I_{0^+}^{n-\alpha}x(t) = \sum_{w=0}^{s}\sum_{v=1}^{n} \frac{c_{vw}}{\Gamma(n-v+1)}(t-t_w)^{n-v} + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} h(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m,$$

$$D_{0^+}^{\alpha-n+j}x(t) = \sum_{w=0}^{s}\sum_{v=1}^{n-j} \frac{c_{vw}}{\Gamma(n-j-v+1)}(t-t_w)^{n-j-v} + \int_0^t \frac{(t-s)^{n-j-1}}{\Gamma(n-j)} h(s)ds, \tag{4.7}$$

$t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m, j \in \mathbb{N}_1^{n-1}$.

(i) It follows from $I_{0^+}^{n-\alpha}x(0) = 0$, (4.4) and (4.7) that $c_{n0} = 0$.

(ii) From $D_{0^+}^{\alpha-n+j}x(0) = 0$, (4.7), (4.4), $n-k+1+p+q > 0$, we get $c_{n-j\ 0} = 0 (j \in \mathbb{N}_1^{k-1})$.

(iii) From $\lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t) = I_{n\ s}$ and (4.6), we get $c_{n\ s} = I_{n\ s} (s \in \mathbb{N}_1^m)$.

(iv) From $\Delta D_{0^+}^{\alpha-n+j}x(t_s) = I_{n-j\ s}$ and (4.7), we get $c_{n-j\ s} = I_{n-j\ s} (s \in \mathbb{N}_1^m, j \in \mathbb{N}_1^{n-1})$.

(v) From $I_{0^+}^{n-\alpha}u(1) = 0$ and $D_{0^+}^{\alpha-n+j}x(1) = 0$, (4.6) and (4.7), (i)-(iv), we get

$$\sum_{v=1}^{n-k} \frac{c_{v0}}{\Gamma(n-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{n} \frac{I_{vw}}{\Gamma(n-v+1)}(1-t_w)^{n-v} + \int_0^1 \frac{(1-s)^{n-1}}{\Gamma(n)} h(s)ds = 0,$$

$$\sum_{v=1}^{n-j} \frac{c_{v0}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{n-j} \frac{I_{vw}}{\Gamma(n-j-v+1)}(1-t_w)^{n-j-v} + \int_0^1 \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} h(s)ds = 0, j \in \mathbb{N}_1^{n-k-1}.$$

Case 1. $k \leq \frac{n-1}{2}$.
We have

$$\begin{pmatrix} \frac{1}{\Gamma(n)} & \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-1)} & \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k)} & \frac{1}{\Gamma(n-2k-1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(n-2k-1)} & \frac{1}{\Gamma(n-2k-2)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\ \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{n-k\ 0} \end{pmatrix} = - \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=1}^{n} \frac{I_{vw}(1-t_w)^{\alpha-v}}{\Gamma(\alpha-v+1)} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-1} \frac{I_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)} + \int_0^1 \frac{(1-s)^{n-1-1}}{\Gamma(n-1)} h(s)ds \\ \cdots\cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{n-(n-k-1)} \frac{I_{vw}(1-t_w)^{n-(n-k-1)-v}}{\Gamma(n-(n-k-1)-v+1)} + \int_0^1 \frac{(1-s)^{n-(n-k-1)-1}}{\Gamma(n-(n-k-1))} h(s)ds \end{pmatrix}.$$

Hence

$$\begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{n-k\ 0} \end{pmatrix} = -M^{-1} \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=1}^{n} \frac{I_{vw}(1-t_w)^{\alpha-v}}{\Gamma(\alpha-v+1)} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-1} \frac{I_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)} + \int_0^1 \frac{(1-s)^{n-1-1}}{\Gamma(n-1)} h(s)ds \\ \cdots\cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{n-(n-k-1)} \frac{I_{vw}(1-t_w)^{n-(n-k-1)-v}}{\Gamma(n-(n-k-1)-v+1)} + \int_0^1 \frac{(1-s)^{n-(n-k-1)-1}}{\Gamma(n-(n-k-1))} h(s)ds \end{pmatrix}.$$

It follows that

$$c_{i\ 0} = - \sum_{j=1}^{n-k} \frac{M_{ji}}{|M|} \left(\sum_{w=1}^{m} \sum_{\sigma=1}^{n-(j-1)} \frac{I_{\sigma w}(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)} + \int_0^1 \frac{(1-s)^{\alpha-(j-1)-1}}{\Gamma(\alpha-(j-1))} h(s)ds \right), i \in \mathbb{N}_1^{n-k}. \quad (4.8)$$

Case 2. $k > \frac{n-1}{2}$.

We have

$$\begin{pmatrix} \frac{1}{\Gamma(n)} & \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-1)} & \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(2k-n+2)} \end{pmatrix} \begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{n-k\ 0} \end{pmatrix}$$

$$= - \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=1}^{n} \frac{I_{vw}(1-t_w)^{\alpha-v}}{\Gamma(\alpha-v+1)} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-1} \frac{I_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)} + \int_0^1 \frac{(1-s)^{n-1-1}}{\Gamma(n-1)} h(s)ds \\ \cdots\cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{n-(n-k-1)} \frac{I_{vw}(1-t_w)^{n-(n-k-1)-v}}{\Gamma(n-(n-k-1)-v+1)} + \int_0^1 \frac{(1-s)^{n-(n-k-1)-1}}{\Gamma(n-(n-k-1))} h(s)ds \end{pmatrix}.$$

We get (4.8) similarly. Substituting c_{vw} into (4.6), we know that x satisfies (4.2).

Secondly we prove $x \in X$ and x is a solution of (4.1). It is easy to see from (4.2) that $x \in X$ and

$$I_{0^+}^{n-\alpha} x(0) = 0,\ D_{0^+}^{\alpha-n+j} x(0) = 0,\ j \in \mathbb{N}_1^{k-1},$$

$$I_{0^+}^{n-\alpha} x(1) = 0,\ D_{0^+}^{\alpha-n+j} x(1) = 0, j \in \mathbb{N}_1^{n-k-1},$$

$$\lim_{t \to t_s^+} (t - t_s)^{n-\alpha} x(t) = I_{ns},\ s \in \mathbb{N}_1^m,$$

$$\Delta D_{0^+}^{\alpha-n+j} x(t_s) = I_{n-j\ s}, j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m.$$

Now, we prove that x satisfies $D_{0^+}^{\alpha} x(t) = h(t)$. We remember (4.8) and (i)-(iv), then it suffices to prove $D_{0^+}^{\alpha} x(t) = h(t)$ if x satisfies (4.6).

In fact, for $t \in (t_i, t_{i+1}](i \in \mathbb{N}_0^m)$, by Definition 2.2, we have

$$D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t (t-s)^{n-\alpha-1} x(s)ds \right]^{(n)}$$

$$= \frac{1}{\Gamma(n-\alpha)} \left[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} x(s)ds + \int_{t_i}^{t} (t-s)^{n-\alpha-1} x(s)ds \right]^{(n)}$$

$$= h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m.$$

From above discussion, we know that $x \in X$ and x satisfies (4.1). The proof is completed. ∎

Let

$$N=(n_{ij})=\begin{pmatrix}
\frac{1}{\Gamma(n-l)} & \frac{1}{\Gamma(n-l-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k-l)} & \frac{1}{\Gamma(k-l+1)} \\
\frac{1}{\Gamma(n-l-1)} & \frac{1}{\Gamma(n-l-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k-l-1)} & \frac{1}{\Gamma(k-l)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k+l)} & \frac{1}{\Gamma(n-2k+l-1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\
\frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(n-2k+l-1)} & \frac{1}{\Gamma(n-2k+l-2)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{\Gamma(k-l+2)} & \frac{1}{\Gamma(k-l+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\
\frac{1}{\Gamma(k-l+1)} & \frac{1}{\Gamma(k-l)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0
\end{pmatrix}$$

and N_{ij} be the algebraic cofactors of n_{ij} respectively. Furthermore, we have $|N_{ij}| \leq \Gamma(n-k)$ by Remark 3.2 and $N^{-1}=\frac{N^*}{|N|}$, where N^* is the adjoint matrix of N.

Lemma 4.2. Suppose that $h:(0,1)\to\mathbb{R}$ is integral and satisfies $|h(t)|\leq t^p(1-t)^q$ on $(0,1)$, and $I_j\in\mathbb{R}$. Then $x\in X$ is a solution of

$$\begin{cases}
D^{\alpha}_{0^+}x(t)=h(t), \quad t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0, \\
I^{n-\alpha}_{0^+}x(0)=0,\ D^{\alpha-n+j}_{0^+}x(0)=0, j\in\mathbb{N}_1^{k-1}, \\
D^{\alpha-n+j}_{0^+}x(1)=0, j\in\mathbb{N}_l^{n+l-k-1}, \\
\lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t)=I_{ns},\ s\in\mathbb{N}_1^m, \\
\Delta D^{\alpha-n+j}_{0^+}x(t_s)=I_{n-j\ s}, j\in\mathbb{N}_1^{n-1}, s\in\mathbb{N}_1^m,
\end{cases} \tag{4.9}$$

if and only if

$$\begin{aligned}
x(t)=&-\sum_{v=1}^{n-k}\tfrac{t^{\alpha-v}}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\tfrac{N_{ji}}{|N|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\tfrac{I_{vw}}{\Gamma(n-j-v+1)}(1-t_w)^{n-j-v} \\
&\sum_{v=1}^{n-k}\tfrac{t^{\alpha-v}}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\tfrac{N_{ji}}{|N|}\int_0^1\tfrac{(1-s)^{n-j-1}}{\Gamma(n-j)}h(s)ds \\
&+\sum_{w=1}^{s}\sum_{v=1}^{n}\tfrac{I_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, t\in(t_s,t_{s+1}], s\in\mathbb{N}_0^m.
\end{aligned} \tag{4.10}$$

Proof. Suppose that $x\in X$ is a solution of (4.9). Then by Theorem 3.1, we have (4.6). Furthermore, we have same results (i)-(iv) in the proof of Lemma 4.1 and

$$\sum_{v=1}^{n-j}\tfrac{c_{v0}}{\Gamma(n-j-v+1)}+\sum_{w=1}^{m}\sum_{v=1}^{n-j}\tfrac{I_{vw}}{\Gamma(n-j-v+1)}(1-t_w)^{n-j-v}+\int_0^1\tfrac{(1-s)^{n-j-1}}{\Gamma(n-j)}h(s)ds=0, j\in\mathbb{N}_l^{n+l-k-1}.$$

Note $l \in \mathbb{N}_1^k$. It follows that

$$\begin{pmatrix} \frac{1}{\Gamma(n-l)} & \frac{1}{\Gamma(n-l-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k-l)} & \frac{1}{\Gamma(k-l+1)} \\ \frac{1}{\Gamma(n-l-1)} & \frac{1}{\Gamma(n-l-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k-l-1)} & \frac{1}{\Gamma(k-l)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k+l)} & \frac{1}{\Gamma(n-2k+l-1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(n-2k+l-1)} & \frac{1}{\Gamma(n-2k+l-2)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k-l+2)} & \frac{1}{\Gamma(k-l+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\ \frac{1}{\Gamma(k-l+1)} & \frac{1}{\Gamma(k-l)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{k-l+1\ 0} \\ c_{k-l+2\ 0} \\ \cdots \\ c_{n-k-1\ 0} \\ c_{n-k\ 0} \end{pmatrix} = - \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=1}^{n-1} \frac{I_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)} + \int_0^1 \frac{(1-s)^{n-1-1}}{\Gamma(n-1)} h(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-2} \frac{I_{vw}(1-t_w)^{n-2-v}}{\Gamma(n-2-v+1)} + \int_0^1 \frac{(1-s)^{n-2-1}}{\Gamma(n-2)} h(s)ds \\ \cdots\cdots\cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{n-(n+l-k-1)} \frac{I_{vw}(1-t_w)^{n-(n+l-k-1)-v}}{\Gamma(n-(n+l-k-1)-v+1)} \\ + \int_0^1 \frac{(1-s)^{n-(n+l-k-1)-1}}{\Gamma(n-(n+l-k-1))} h(s)ds \end{pmatrix}. \tag{4.11}$$

Then we get

$$c_{i\ 0} = - \sum_{j=1}^{n-k} \frac{N_{ji}}{|N|} \left(\sum_{w=1}^{m} \sum_{v=1}^{n-j} \frac{I_{vw}}{\Gamma(n-j-v+1)} (1-t_w)^{n-j-v} + \int_0^1 \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} h(s)ds \right), i \in \mathbb{N}_1^{n-k}.$$

Substituting all of c_{vw} into (4.6), we get (4.10). The remainder of the proof is similar to that of Lemma 4.1 and omitted. ∎

Now, we define the following operators on X by

$$\begin{aligned} (T_1 x)(t) = & - \sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{M_{jv}}{|M|} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-(j-1)} \frac{I_{\sigma x}(t_w)(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)} \\ & - \sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{M_{jv}}{|M|} \int_0^1 \frac{(1-s)^{\alpha-(j-1)-1}}{\Gamma(\alpha-(j-1))} f_x(s)ds \\ & + \sum_{w=1}^{s} \sum_{v=1}^{n} \frac{I_{vx}(t_w)}{\Gamma(\alpha-v+1)} (t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_x(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m. \end{aligned} \tag{4.12}$$

and

$$(T_2x)(t) = -\sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{N_{ji}}{|N|} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-j} \frac{I_{\sigma x}(t_w)}{\Gamma(n-j-\sigma+1)}(1-t_w)^{n-j-\sigma}$$

$$\sum_{v=1}^{n-k} \frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{N_{ji}}{|N|} \int_0^1 \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} f_x(s)ds \tag{4.13}$$

$$+ \sum_{w=1}^{s} \sum_{v=1}^{n} \frac{I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_x(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

Lemma 4.3. Suppose that (a)-(c) hold and $l \in \mathbb{N}_1^k$. Then Both operators $T_1 : X \to X$ and $T_2 : X \to X$ are well defined and completely continuous.

Proof. The proof is standard and is omitted, one may see [56]. ∎

Choose $Z = L^1(0,1) \times \mathbb{R}^{nm}$ with the norm $||(x, a_{ij} : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)|| = \max\{||x||_1, |a_{ij}| : i \in \mathbb{N}_1^n, j \in \mathbb{N}_1^m\}$. Choose $E = X$ and

$$D(L_1) = \{x \in X : {}^cD_{0^+}^{\alpha} x \in L^1(0,1), D_{0^+}^{\alpha-n+i}u(0) = 0, i \in \mathbb{N}_1^k, D_{0^+}^{\alpha-n+j}u(1) = 0, j \in \mathbb{N}_1^{n-k}\}.$$

Define the linear operator $L_1 : X \bigcap D(L_1) \to Z$ and the nonlinear operator $N_1 : X \to Z$ by

$$(L_1x)(t) = \begin{pmatrix} {}^cD_{0^+}^{\alpha} x(t) \\ \Delta D_{0^+}^{\alpha-(n-1)} x(t_s) : s \in \mathbb{N}_1^m \\ \cdots \\ \Delta D_{0^+}^{\alpha-1} x(t_s) : s \in \mathbb{N}_1^m, \\ \lim\limits_{t \to t_s^+} (t-t_s)^{n-\alpha} x(t) : s \in \mathbb{N}_1^m \end{pmatrix}, \ (N_1x)(t) = \begin{pmatrix} f_x(t) \\ I_{1x}(t_s) : s \in \mathbb{N}_1^m \\ \cdots \\ I_{n-1\ x}(t_s) : s \in \mathbb{N}_1^m, \\ I_{nx}(t_s) : s \in \mathbb{N}_1^m \end{pmatrix}. \tag{4.14}$$

Lemma 4.4. Suppose that (a)-(c) hold. Then

(i) L_1 is a Fredholm operator of index zero.

(ii) $N_1 : \overline{\Omega} \to Z$ is called L_1-compact for bounded set $\overline{\Omega} \sqsubseteq X$.

(iii) x is a solution of BVP(1.30) if and only if $L_1x = N_1x$.

Proof. Let

$$K = (k_{ij})_{(n-k-1)\times(n-k-1)} = \begin{pmatrix} \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \cdots & \frac{1}{\Gamma(n-2k+1)} & \frac{1}{\Gamma(n-2k)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k)} & \frac{1}{\Gamma(n-2k-1)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \end{pmatrix}.$$

Then $|K| \neq 0$. Let K_{ij} be the the algebraic cofactors of k_{ij}. Then $|K_{ij}| \leq \Gamma(n-k)$.

Firstly we prove that $L_1 : D(L_1) \subset E \to Z$ is a Fredholm operator of index zero.

Claim 1.

$$\mathrm{Ker}L_1 = \{c_{n0}t^{\alpha-n} : c_{n\ 0} \in \mathbb{R}\}. \tag{4.15}$$

In fact, $x \in \mathrm{Ker}L_1$ if and only if

$$\begin{pmatrix} {}^cD^{\alpha}_{0^+}x(t) \\ \Delta D^{\alpha-(n-1)}_{0^+}x(t_s) : s \in \mathbb{N}^m_1 \\ \cdots \\ \Delta D^{\alpha-1}_{0^+}x(t_s) : s \in \mathbb{N}^m_1, \\ \lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t) : s \in \mathbb{N}^m_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 : s \in \mathbb{N}^m_1 \\ 0 : s \in \mathbb{N}^m_1 \\ \cdots \\ 0 : s \in \mathbb{N}^m_1 \end{pmatrix}.$$

Use Lemma Theorem 3.1, we have $x \in D(L_1)$ and

$$x(t) = \sum_{w=0}^{i}\sum_{v=1}^{n}\frac{c_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}, t \in (t_i, t_{i+1}], i \in \mathbb{N}^m_0,$$

$$D^{\alpha-n+j}_{0^+}x(t) = \sum_{w=0}^{i}\sum_{v=1}^{n-j}\frac{c_{vw}}{\Gamma(n-j-v+1)}(t-t_w)^{n-j-v}, t \in (t_i, t_{i+1}], i \in \mathbb{N}^m_0, j \in \mathbb{N}^{n-1}_1.$$

By $D^{\alpha-n+i}_{0^+}x(0) = 0, i \in \mathbb{N}^k_1$, we get $c_{i\ 0} = 0, i \in \mathbb{N}^{n-1}_{n-k}$. By $D^{\alpha-n+j}_{0^+}x(1) = 0, j \in \mathbb{N}^{n-k}_1$, we have

$$\begin{pmatrix} \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \cdots & \frac{1}{\Gamma(n-2k+1)} & \frac{1}{\Gamma(n-2k)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k)} & \frac{1}{\Gamma(n-2k-1)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\ \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{k\ 0} \\ c_{k+1\ 0} \\ \cdots \\ c_{n-(k+2)\ 0} \\ c_{n-(k+1)\ 0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}.$$

Then $c_{i\ 0} = 0, i \in \mathbb{N}^{n-k-1}_1$. Hence $x(t) = c_{n\ 0}t^{\alpha-n}$. On the other hand, we have $c_{n\ 0}t^{\alpha-n} \in \mathrm{Ker}L_1$. Then (4.15) holds.

Claim 2.

$$\mathrm{Im}L_1 = \left\{ (u, a_{ij}) \;\middle|\; \begin{array}{l} -\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{k}\frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)} \\ -\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds + \int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds = 0 \end{array} \right\}. \tag{4.16}$$

In fact, $(u, a_{ij} : i \in \mathbb{N}^{n-1}_0, j \in \mathbb{N}^m_1) \in \mathrm{Im}L_1$ if and only if there exists $x \in D(L_1)$ such that

$$\begin{pmatrix} {}^cD^{\alpha}_{0^+}x(t) \\ \Delta D^{\alpha-(n-1)}_{0^+}x(t_s) : s \in \mathbb{N}^m_1 \\ \cdots \\ \Delta D^{\alpha-1}_{0^+}x(t_s) : s \in \mathbb{N}^m_1, \\ \lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t) : s \in \mathbb{N}^m_1 \end{pmatrix} = \begin{pmatrix} u(t) \\ a_{1s} : s \in \mathbb{N}^m_1 \\ a_{2s} : s \in \mathbb{N}^m_1 \\ \cdots \\ a_{n-1s} : s \in \mathbb{N}^m_1 \\ a_{n\ s} : s \in \mathbb{N}^m_1 \end{pmatrix}.$$

By Theorem 3.1, we know that there exist constants $c_{vw} \in \mathbb{R}(v \in \mathbb{N}_1^m, w \in \mathbb{N}_0^m)$ such that $x \in D(L_1) \bigcap X$ and

$$x(t) = \sum_{w=0}^{s} \sum_{v=1}^{n} \frac{c_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m. \tag{4.17}$$

By Definition 2, we have for $j \in \mathbb{N}_0^{n-1}$ that

$$\begin{aligned} D_{0^+}^{\alpha-n+j} x(t) &= \sum_{w=0}^{s} \sum_{v=1}^{n-j} \frac{c_{vw}}{\Gamma(n-j-v+1)}(t-t_w)^{n-j-v} + \int_0^t \frac{(t-s)^{n-j-1}}{\Gamma(n-j)} u(s)ds, \\ & t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m, j \in \mathbb{N}_1^{n-1}. \end{aligned} \tag{4.18}$$

(i) From $\Delta D_{0^+}^{\alpha-n+j} x(t_s) = a_{js}$ and (4.18), we get $c_{js} = a_{js}(j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m)$.

(ii) From $D_{0^+}^{\alpha-n+i} x(0) = 0, i \in \mathbb{N}_1^k$, we get $c_{i\ 0} = 0, i \in \mathbb{N}_{n-k}^{n-1}$.

(iii) By $D_{0^+}^{\alpha-n+j} x(1) = 0, j \in \mathbb{N}_1^{n-k}$, we have from (4.18), (i), (ii) that

$$\begin{pmatrix} \frac{1}{\Gamma(n-1)} & \cdots & \frac{1}{\Gamma(n-k)} & \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(k+2)} & \frac{1}{\Gamma(k+1)} \\ \frac{1}{\Gamma(n-2)} & \cdots & \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(n-k)} & \cdots & \frac{1}{\Gamma(n-2k+1)} & \frac{1}{\Gamma(n-2k)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(n-k-1)} & \cdots & \frac{1}{\Gamma(n-2k)} & \frac{1}{\Gamma(n-2k-1)} & \cdots & \frac{1}{\Gamma(1)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Gamma(k+1)} & \cdots & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} & \cdots & 0 & 0 \\ \frac{1}{\Gamma(k)} & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{1\ 0} \\ c_{2\ 0} \\ \cdots \\ c_{k\ 0} \\ c_{k+1\ 0} \\ \cdots \\ c_{n-(k+2)\ 0} \\ c_{n-(k+1)\ 0} \end{pmatrix}$$

$$= - \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=1}^{n-1} \frac{a_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)} + \int_0^1 \frac{(1-s)^{n-1-1}}{\Gamma(n-1)} u(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-2} \frac{a_{vw}(1-t_w)^{n-2-v}}{\Gamma(n-2-v+1)} + \int_0^1 \frac{(1-s)^{n-2-1}}{\Gamma(n-2)} u(s)ds \\ \cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{n-k} \frac{a_{vw}(1-t_w)^{n-k-v}}{\Gamma(n-k-v+1)} + \int_0^1 \frac{(1-s)^{n-k-1}}{\Gamma(n-k)} u(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{n-(k+1)} \frac{a_{vw}(1-t_w)^{n-(k+1)-v}}{\Gamma(n-(k+1)-v+1)} + \int_0^1 \frac{(1-s)^{n-(k+1)-1}}{\Gamma(n-(k+1))} u(s)ds \\ \cdots \\ \sum_{w=1}^{m} \sum_{v=1}^{k+1} \frac{a_{vw}(1-t_w)^{k+1-v}}{\Gamma(k+1-v+1)} + \int_0^1 \frac{(1-s)^{k+1-1}}{\Gamma(k+1)} u(s)ds \\ \sum_{w=1}^{m} \sum_{v=1}^{k} \frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)} + \int_0^1 \frac{(1-s)^{k-1}}{\Gamma(k)} u(s)ds \end{pmatrix}.$$

Then $c_{i\ 0}=0, i\in\mathbb{N}_1^{n-k-1}$. It follows that

$$\begin{pmatrix} c_{1\ 0}\\ c_{2\ 0}\\ \cdots\\ c_{k\ 0}\\ c_{k+1\ 0}\\ \cdots\\ c_{n-(k+2)\ 0}\end{pmatrix}=-N^{-1}\begin{pmatrix} \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-1}\frac{a_{vw}(1-t_w)^{n-1-v}}{\Gamma(n-1-v+1)}+\int_0^1\frac{(1-s)^{n-1-1}}{\Gamma(n-1)}u(s)ds\\ \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-2}\frac{a_{vw}(1-t_w)^{n-2-v}}{\Gamma(n-2-v+1)}+\int_0^1\frac{(1-s)^{n-2-1}}{\Gamma(n-2)}u(s)ds\\ \cdots\\ \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-k}\frac{a_{vw}(1-t_w)^{n-k-v}}{\Gamma(n-k-v+1)}+\int_0^1\frac{(1-s)^{n-k-1}}{\Gamma(n-k)}u(s)ds\\ \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-(k+1)}\frac{a_{vw}(1-t_w)^{n-(k+1)-v}}{\Gamma(n-(k+1)-v+1)}+\int_0^1\frac{(1-s)^{n-(k+1)-1}}{\Gamma(n-(k+1))}u(s)ds\\ \cdots\\ \sum\limits_{w=1}^{m}\sum\limits_{v=1}^{k+1}\frac{a_{vw}(1-t_w)^{k+1-v}}{\Gamma(k+1-v+1)}+\int_0^1\frac{(1-s)^{k+1-1}}{\Gamma(k+1)}u(s)ds\end{pmatrix}$$

and

$$\sum_{i=1}^{k}\frac{c_{i\ 0}}{\Gamma(k-i+1)}+\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds=0. \tag{4.19}$$

So

$$c_{i\ 0}=-\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{|K|}\left(\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}+\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds\right), i\in\mathbb{N}_1^{n-k-1}. \tag{4.20}$$

Substituting (4.20) into (4.19), we have

$$-\sum_{i=1}^{k}\frac{\sum\limits_{j=1}^{n-k-1}\frac{K_{j\ i}}{|K|}\left(\sum\limits_{w=1}^{m}\sum\limits_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}+\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds\right)}{\Gamma(k-i+1)}+\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)}$$

$$+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds=0.$$

That is

$$\begin{aligned}&-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}+\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)}\\ &-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds=0.\end{aligned} \tag{4.21}$$

On the other hand, if $(u,a_{ij}: i\in\mathbb{N}_0^{n-1}, j\in\mathbb{N}_1^m)$ satisfies (4.21), we can prove that there exists $x\in D(L_1)\bigcap X$ such that $L_1x=(u,a_{ij}: i\in\mathbb{N}_0^{n-1}, j\in\mathbb{N}_1^m)$. Hence (4.16) is valid.

Claim 3. For $(u,a_{ij}: i\in\mathbb{N}_0^{n-1}, j\in\mathbb{N}_1^m)\in \mathrm{Im}L_1$, there exists $x\in D(L_1)\bigcap X$ such that

$L_1x = (u, a_{ij} : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)$ with

$$x(t) = c_{n\ 0}t^{\alpha-n} - \sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-i+1)|K|}\left(\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds\right)t^{\alpha-i}$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{a_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m. \tag{4.22}$$

This claim follows from Claim 2.

It follows from Claim 1 and Claim 2 that $\dim \mathrm{Ker}L_1 = 1$ and $\mathrm{Im}L_1$ is closed in Z. Furthermore, define projectors $P : X \to \mathrm{Ker}L_1$ and $Q : Z \to \mathrm{Im}L_1$ by

$$Px(t) = \frac{I_{0+}^{n-\alpha}x(0)}{\Gamma(\alpha-n+1)}t^{\alpha-n},\ x \in X,$$

$$Q(u, a_{ij} : i \in \mathbb{N}_1^n, j \in \mathbb{N}_1^m) = \left(\overline{Q}_{u,a_{ij}}, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m\right),$$

where

$$\overline{Q}_{u,a_{ij}} = \left[\int_0^1\left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right)ds\right]^{-1} \times$$

$$\left[-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}(1-t_w)^{k-v}}{\Gamma(k-v+1)}\right.$$

$$\left.+\int_0^1\left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right)u(s)ds\right].$$

It is easy to see that $P : X \to \mathrm{Ker}L_1$ and $Q : Z \to Z/\mathrm{Im}L_1$ are well defined and

$$\mathrm{Im}\ P = \mathrm{Ker}\ L_1,\ \mathrm{Ker}\ Q = \mathrm{Im}\ L_1,\ \ X = \mathrm{Ker}\ L_1 \oplus \mathrm{Ker}\ P,\ \ Z = \mathrm{Im}\ L_1 \oplus \mathrm{Im}\ Q.$$

So $\dim \mathrm{Ker}L_1 = \ \mathrm{co\ dim}\ \mathrm{Im}L_1 = 1 < +\infty$. Then $L_1 :\ D(L) \subset X \to Z$ is a Fredholm operator of index zero.

The inverse of $L_1|_{D(L_1)\cap \mathrm{Ker}\ P} :\ D(L_1) \cap \mathrm{Ker}\ P \to \mathrm{Im}\ L_1$ is denoted by $K_P : \mathrm{Im}L_1 \to D(L_1) \cap \mathrm{Ker}\ P$ with

$$K_p(u, a_{ij} : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)(t)$$

$$= -\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-v+1)|K|}\left(\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{a_{vw}(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}u(s)ds\right)t^v$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{a_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

Secondly for each nonempty open bounded subset Ω of E satisfying $D(L_1)\cap\overline{\Omega} \neq \varnothing$, we prove that $N_1 :\ \overline{\Omega} \to Z$ is called L_1-compact. It suffices to prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is bounded and relatively compact.

One sees that

$$QN_1x(t) = Q\begin{pmatrix} f_x(t) \\ I_{1x}(t_s) : s \in \mathbb{N}_1^m \\ I_{2x}(t_s) : s \in \mathbb{N}_1^m \\ \cdots \\ I_{n\ x}(t_s) : s \in \mathbb{N}_1^m \end{pmatrix} = \left(\overline{Q}_{f_x(t),I_{ix}(t_j)}, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m\right),$$

where

$$\overline{Q}_{f_x(t),I_{ix}(t_j)} = \left[\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) ds\right]^{-1} \times$$

$$\left[-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_{vx}(t_w)(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_{vx}(t_w)(1-t_w)^{k-v}}{\Gamma(k-v+1)}\right.$$

$$\left. + \int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) f_x(s)ds\right].$$

By direct computation, we have

$$K_p(I-Q)N_1x(t) = K_PN_1x(t) - K_PQN_1x(t) = K_p\begin{pmatrix} f_x(t) - \overline{Q}_{f_x,I_{ix}(t_j)} \\ I_{1x}(t_s) : s \in \mathbb{N}_1^m \\ I_{2x}(t_s) : s \in \mathbb{N}_1^m \\ \cdots \\ I_{n\ x}(t_s) : s \in \mathbb{N}_1^m \end{pmatrix}$$

$$= -\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(\alpha-v+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{(1-t_w)^{n-j-v}t^v}{\Gamma(n-j-v+1)}I_{vx}(t_w) + \sum_{w=1}^{s}\sum_{v=1}^{n}\frac{(t-t_w)^{\alpha-v}}{\Gamma(\alpha-v+1)}I_{vx}(t_w)$$

$$-\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}t^v}{\Gamma(\alpha-v+1)|K|}\int_0^1 \frac{(1-s)^{n-j-1}}{\Gamma(n-j)}[f_x(s) - \overline{Q}_{f_x,I_{ix}(t_j)}]ds$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[f_x(s) - \overline{Q}_{f_x,I_{ix}(t_j)}]ds, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

By Lemma 3.4, we can prove that $QN_1(\overline{\Omega})$ is bounded and $K_p(I-Q)N_1(\overline{\Omega})$ is bounded and relatively compact. Hence $N_1 : \overline{\Omega} \to Z$ is called L_1-compact for bounded set $\overline{\Omega} \sqsubseteq X$.

Thirdly, it is easy to see that x is a solution of BVP(1,30) if and only if $L_1x = N_1x$. The proof is completed. ∎

(H1) there exist nonnegative numbers $b_f,\ a_f,\ B_I, A_I, \sigma \geq 0$ such that

$$\left|f\left(t,\ \frac{x}{(t-t_s)^{n-\alpha}}\right)\right| \leq t^p(1-t)^q[b_f + a_f|x|^\sigma], x \in \mathbb{R}, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m,$$

$$\left|I_j\left(t_s,\ \frac{x}{(t_s-t_{s-1})^{n-\alpha}}\right)\right| \leq B_I|x| + A_I|x|^\sigma, x \in \mathbb{R}, s \in \mathbb{N}_1^m, j \in \mathbb{N}_1^n.$$

Denote

$$M_0 = \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||M||} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-(j-1)} \frac{(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)} B_I + \sum_{v=1}^{n} \frac{mB_I}{\Gamma(\alpha-v+1)}$$

$$+ \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||M||} \frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))} b_f + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} b_f,$$

$$M_1 = \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||M||} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-(j-1)} \frac{(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)} A_I + \sum_{v=1}^{n} \frac{mA_I}{\Gamma(\alpha-v+1)}$$

$$+ \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||M||} \frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))} a_f + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} a_f.$$

Theorem 4.1. Suppose that (a)–(c) (defined in Section 1), (H1) holds. Then, the system (1.28) has at least one solution in X if one of the following items hold:

(i) $\sigma < 1$; **(ii)** $\sigma = 1$ with $M_1 < 1$; **(iii)** $\sigma > 1$ with $M_1 M_0^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. The details may be seen in Appendix section. ∎

(H2) there exist constants $M_f, M_I \geq 0$ such that

$$\left| f\left(t, \frac{x}{(t-t_s)^{n-\alpha}}\right) \right| \leq M_f, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m,$$

$$\left| I_j\left(t_s, \frac{x}{(t_s-t_{s-1})^{n-\alpha}}\right) \right| \leq M_I, j \in \mathbb{N}_1^n, s \in \mathbb{N}_1^m.$$

Corollary 4.1. Suppose that (a)-(c) and (H2) hold. Then BVP(1.28) has at least one solution.

Proof. Choose $p = q = 0$, $b_f = 0, a_f = M_f, B_I = 0, A_I = M_I$, $\sigma = 0$. One sees by (H2) that (H1) holds. By Theorem 4.1 (i), we get its proof. ∎

Denote

$$N_0 = \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||N||} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-j} \frac{(1-t_w)^{\alpha-j-\sigma}}{\Gamma(\alpha-j-\sigma+1)} B_I + \sum_{v=1}^{n} \frac{mB_I}{\Gamma(\alpha-v+1)}$$

$$+ \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||N||} \frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))} b_f + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} b_f,$$

$$N_1 = \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||N||} \sum_{w=1}^{m} \sum_{\sigma=1}^{n-j} \frac{(1-t_w)^{\alpha-j-\sigma}}{\Gamma(\alpha-j-\sigma+1)} A_I + \sum_{v=1}^{n} \frac{mA_I}{\Gamma(\alpha-v+1)}$$

$$+ \sum_{v=1}^{n-k} \frac{1}{\Gamma(\alpha-v+1)} \sum_{j=1}^{n-k} \frac{\Gamma(n-k)}{||N||} \frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))} a_f + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} a_f.$$

Theorem 4.2. Suppose that (a)–(c) (defined in Section 1), (H1) holds. Then, the system (1.29) has at least one solution in X if one of the following items hold:

(i) $\sigma < 1$; **(ii)** $\sigma = 1$ with $N_1 < 1$; **(iii)** $\sigma > 1$ with $N_1 N_0^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. It is similar to the proof of Theorem 4.1 and omitted. ∎

Corollary 4.2. Suppose that (a)-(c) and (H2) hold. Then BVP(1.29) has at least one solution.

Proof. It is similar to the proof of Corollary 4.1 and omitted. ∎

(H3) there exist nonnegative non-decreasing functions $\prod_f, \prod_I : [0,\infty) \to [0,\infty)$ such that

$$\left| f\left(t, \tfrac{x}{(t-t_s)^{n-\alpha}}\right)\right| \leq \prod_f(|x|)t^p(1-t)^q, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m,$$

$$\left| I_j\left(t_s, \tfrac{x}{(t_s-t_{s-1})^{n-\alpha}}\right)\right| \leq \prod_I(|x|), s \in \mathbb{N}_1^m, j \in \mathbb{N}_1^n.$$

(H4) there exists a constant $M > 0$ such that $x \in X \bigcap D(L_1)$ with $|x(t)| > M$ for all $t \in (0,1]$ implies

$$-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \sum_{w=1}^{m}\sum_{v=1}^{n-j} \tfrac{I_{vx}(t_w)(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k} \tfrac{I_{vx}(t_w)(1-t_w)^{k-v}}{\Gamma(k-v+1)}$$

$$-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \int_0^1 \tfrac{(1-s)^{n-j-1}}{\Gamma(n-j)} f_x(s)ds + \int_0^1 \tfrac{(1-s)^{k-1}}{\Gamma(k)} f_x(s)ds \neq 0.$$

(H5) there exists a constant $M_0 > 0$ such that

$$c\left[-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \sum_{w=1}^{m}\sum_{v=1}^{n-j} \tfrac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k} \tfrac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)}\right.$$

$$\left.-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \int_0^1 \tfrac{(1-s)^{n-j-1}}{\Gamma(n-j)} f(s, cs^{\alpha-n})ds + \int_0^1 \tfrac{(1-s)^{k-1}}{\Gamma(k)} f(s, cs^{\alpha-n})ds\right] > 0$$

holds for all $|c| > M_0$ or

$$c\left[-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \sum_{w=1}^{m}\sum_{v=1}^{n-j} \tfrac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k} \tfrac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)}\right.$$

$$\left.-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \tfrac{K_{j\ i}}{\Gamma(k-i+1)|K|} \int_0^1 \tfrac{(1-s)^{n-j-1}}{\Gamma(n-j)} f(s, cs^{\alpha-n})ds + \int_0^1 \tfrac{(1-s)^{k-1}}{\Gamma(k)} f(s, cs^{\alpha-n})ds\right] < 0.$$

holds for all $|c| > M_0$.

Theorem 4.3. Suppose that (a)-(c), (H3)-(H5) hold. Then BVP(1.30) has at least one solution if

$$\lim_{r\to\infty} \tfrac{r}{M_1 + M_2 \prod_I(r) + M_3 \prod_f(r)} > 1, \tag{4.23}$$

where $M_1 = \frac{M}{\Gamma(\alpha-n+1)}$ and

$$M_2 = \frac{1}{\Gamma(\alpha-n+1)} \sum_{i=1}^{n-k-1} \sum_{j=1}^{n-k-1} \frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||} \sum_{w=1}^{m} \sum_{v=1}^{n-j} \frac{(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}$$

$$+\frac{1}{\Gamma(\alpha-n+1)} \sum_{v=1}^{n} \frac{m}{\Gamma(\alpha-v+1)} + \sum_{i=1}^{n-k-1} \sum_{j=1}^{n-k-1} \frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||} \sum_{w=1}^{m} \sum_{v=1}^{n-j} \frac{(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{v=1}^{n} \frac{m}{\Gamma(\alpha-v+1)},$$

$$M_3 = \frac{1}{\Gamma(\alpha-n+1)} \sum_{i=1}^{n-k-1} \sum_{j=1}^{n-k-1} \frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||} \frac{\mathbf{B}(n+q-j,p+1)}{\Gamma(n-j)} + \frac{1}{\Gamma(\alpha-n+1)} \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}$$

$$+ \sum_{i=1}^{n-k-1} \sum_{j=1}^{n-k-1} \frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||} \frac{\mathbf{B}(n+q-j,p+1)}{\Gamma(n-j)} + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}.$$

Proof: The details can be seen in Apendix section. ∎

5 Some examples

In this section, we firstly point out a mistake occurred in [31]. Then to illustrate the usefulness of our main result, we present some examples.

Remark 5.1. In [31], existence of positive solutions of (1.27) was studied. Suppose $\alpha \in (1, 2)$. Lemma 3.1[31] claimed that if $u \in PC[0, 1]$ is fixed point of the operator $A : PC[0, 1] \to PC[0, 1]$ defined by

$$(Au)(t) = \int_0^1 G(t, s)f(s, u(s))ds + t^{\alpha-1} \sum_{t<t_k<1} \frac{c_k}{1-c_k} u(t_k),\ u \in PC[0, 1],$$

where

$$G(t, s = \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1},\ 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1},\ \ 0 \le t \le s \le 1, \end{cases}$$

then u is a solution of BVP(1.27). However we find that this lemma is in-correct.

Proof. In fact, if u is a fixed point of A, we have

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds + t^{\alpha-1} \sum_{t<t_k<1} \frac{c_k}{1-c_k} u(t_k),\ u \in PC[0, 1],$$

For $t \in (t_1, t_2)$, by Definition 2.2, we have by direct computation that

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(2-\alpha)}\left[\int_0^t (t-s)^{1-\alpha}\left(\int_0^1 G(s,v)f(v,u(v))dv + s^{\alpha-1}\sum_{s<t_k<1}\frac{c_k}{1-c_k}u(t_k)\right)ds\right]''$$

$$= \frac{\left[\int_0^t (t-s)^{1-\alpha}\int_0^1 G(s,v)f(v,u(v))dvds\right]'' + \left[\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}\sum_{s<t_k<1}\frac{c_k}{1-c_k}u(t_k)ds\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\int_0^t (t-s)^{1-\alpha}\left(\int_0^s G(s,v)f(v,u(v))dv + \int_s^1 G(s,v)f(v,u(v))dv\right)ds\right]''}{\Gamma(2-\alpha)}$$

$$+ \frac{\left[\int_0^{t_1}(t-s)^{1-\alpha}s^{\alpha-1}\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k)ds + \int_{t_1}^t (t-s)^{1-\alpha}s^{\alpha-1}\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)ds\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\int_0^t (t-s)^{1-\alpha}\int_0^s\left([s(1-v)]^{\alpha-1} - (s-v)^{\alpha-1}\right)f(v,u(v))dvds + \int_0^t (t-s)^{1-\alpha}\int_s^1 [s(1-v)]^{\alpha-1}f(v,u(v))dvds\right]''}{\Gamma(2-\alpha)}$$

$$+ \frac{\left[\int_0^{t_1}(t-s)^{1-\alpha}s^{\alpha-1}ds\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k) + \int_{t_1}^t (t-s)^{1-\alpha}s^{\alpha-1}ds\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}.$$

By interchange integral order for the first term and $w = \frac{s}{t}$ for the second term, we get

$$D^{\alpha}_{0+}u(t) = \frac{\left[\int_0^t\int_s^t (t-s)^{1-\alpha}\left([s(1-v)]^{\alpha-1} - (s-v)^{\alpha-1}\right)ds f(v,u(v))dv + \int_0^t\int_0^s (t-s)^{1-\alpha}s^{\alpha-1}ds(1-v)^{\alpha-1}f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}$$

$$+ \frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k) + w\int_{\frac{t_1}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\left(\int_0^t\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}ds(1-v)^{\alpha-1}f(v,u(v))dv - \int_0^t\int_s^t (t-s)^{1-\alpha}(s-v)^{\alpha-1}ds\right)f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}$$

$$+ \frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k) + t\int_{\frac{t_1}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\left(t\mathbf{B}(2-\alpha,\alpha)\int_0^t (1-v)^{\alpha-1}f(v,u(v))dv - \int_0^t\int_s^t (t-s)^{1-\alpha}(s-v)^{\alpha-1}ds\right)f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}$$

$$+ \frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\frac{c_1}{1-c_1}u(t_1) + t\mathbf{B}(2-\alpha,\alpha)\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)} \neq -f(t,u(t)).$$

Then Lemma 3.1[31] is in-correct. ∎

Remark 5.2. Impulse conditions $u(t_k^+) = (1-c_k)u(t_k^-)$ in (1.27) is unsuitable for this kind of problem.

Proof. By Theorem 3.1, we have from $D^{\alpha}_{0+}u(t) = -f(t,u(t)), t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m$ that

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds + \sum_{v=0}^{s}\frac{c_{1v}(t-t_v)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{v=0}^{s}\frac{c_{2v}(t-t_v)^{\alpha-2}}{\Gamma(\alpha-1)}, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

According to assumptions $u(0) = 0$ and u is right continuous at $t_k (k \in \mathbb{N}_1^m$, we have $c_{2s} = 0, s \in \mathbb{N}_0^m$. Then

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds + \sum_{v=0}^{s} \frac{c_{1v}(t-t_v)^{\alpha-1}}{\Gamma(\alpha)}, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m.$$

However, we find that $u(t_s^+) - u(t_s^-) \equiv 0$. So impulse conditions $u(t_k^+) = (1 - c_k)u(t_k^-)$ is unsuitable. ∎

Example 5.1. Consider the following impulsive boundary value problem

$$\begin{cases} D_{0^+}^{\frac{8}{5}} u(t) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}} \left[b_0 + a_0[(t - i/2)^{2/5} u(t)]^{\sigma}\right], \quad t \in \left(\frac{i}{2}, \frac{i}{2} + \frac{1}{2}\right], \ i = 0, 1, \\ I_{0^+}^{\frac{2}{5}} u(0) = 0, \ I_{0^+}^{\frac{2}{5}} u(1) = 0, \\ \lim\limits_{t \to s^+} (t - 1/2)^{\frac{2}{5}} u(t) = I, \quad \Delta D_{0^+}^{\frac{3}{5}} u(1/2) = J, \end{cases} \tag{5.1}$$

where b_0, a_0, I, J are constants.

BVP(5.1) is a revised form of BVP(1.27). The boundary conditions and impulse functions are changed. Corresponding to BVP(1.28), we have $0 = t_0 < t_1 = 1/2 < t_2 = 1$, $\alpha = \frac{8}{5}$ with $n = 2$ and $k = 1$, $f(t, x) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}} \left[b_0 + a_0[(t - i/2)^{2/5} x]^{\sigma}\right]$, $t \in \left(\frac{i}{2}, \frac{i}{2} + \frac{1}{2}\right]$, $i = 0, 1$, $I_2(1/2, x) = I$, and $I_1(1/2, x) = J$, f is a Caratheódory function with $p = q = -\frac{1}{5}$. One finds that $M = (1)$ with $||M|| = 1$. It is easy to see that (H1) holds with $b_f = |b_0|, a_f = |a_0|$ and $B_I = \max\{|I|, |J|\}, A_I = 0$. By direction computation, using Mathlab 7.0, we have

$$M_0 = \left[\frac{1}{\Gamma(8/5)}\left(\frac{(1/2)^{3/5}}{\Gamma(8/5)} + \frac{(1/2)^{-2/5}}{\Gamma(3/5)}\right) + \frac{1}{\Gamma(8/5)} + \frac{1}{\Gamma(3/5)}\right] \max\{|I|, |J|\}$$

$$+ \left(\frac{1}{\Gamma(8/5)} \frac{\mathbf{B}(2/5, 4/5)}{\Gamma(8/5)} + \frac{\mathbf{B}(7/5, 4/5)}{\Gamma(8/5)}\right) |b_0| < 3.7 \max\{|I|, |J|\} + 4.6|b_0|,$$

$$M_1 = \left(\frac{1}{\Gamma(8/5)} \frac{\mathbf{B}(2/5, 4/5)}{\Gamma(8/5)} + \frac{\mathbf{B}(7/5, 4/5)}{\Gamma(8/5)}\right) |a_0| < 4.6|a_0|.$$

By Theorem 4.1, BVP(5.1) has at least one solution if one of the following items hold:

(i) $\sigma < 1$; **(ii)** $\sigma = 1$ with $4.6|a_0| < 1$; **(iii)** $\sigma > 1$ with $4.6|a_0|(3.7 \max\{|I|, |J|\} + 4.6|b_0|)^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^{\sigma}}$. ∎

Example 5.2. Consider the following impulsive boundary value problem

$$\begin{cases} D_{0^+}^{\frac{38}{5}}u(t) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0 + a_0[(t-i/2)^{2/5}u(t)]^{\sigma}\right], \quad t \in \left(\frac{i}{2}, \frac{i}{2}+\frac{1}{2}\right], \ i = 0, 1, \\ I_{0^+}^{\frac{2}{5}}u(0) = 0, \ D_{0^+}^{-\frac{2}{5}+j}u(0) = 0, j \in \mathbb{N}_1^3, \\ D_{0^+}^{-\frac{2}{5}+j}u(1) = 0, j \in \mathbb{N}_1^4, \\ \lim\limits_{t\to s^+}(t-1/2)^{\frac{2}{5}}u(t) = I_8, \ \ \Delta u^{(i)}(1/2) = I_i, \ i \in \mathbb{N}_1^7, \end{cases} \tag{5.2}$$

where $b_0,\ a_0, d_0, I, I_i (i \in \mathbb{N}_1^7)$ are constants.

Corresponding to BVP(1.29), we have $\alpha = \frac{38}{5}$ with $n = 8$, $m = 1$ with $0 = t_0 < t_1 = 1/2 < t_2 = 1$, $k = 4, l = 1$, and $f(t,x) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0 + a_0[(t-i/2)^{2/5}x]^{\sigma}\right]$, $I_i(i,,x) = I_i(i \in \mathbb{N}_1^8)$, f is a Caratheódory function with $p = q = -\frac{1}{5}$. One finds that

$$N = (n_{ij}) = \begin{pmatrix} \frac{1}{\Gamma(7)} & \frac{1}{\Gamma(6)} & \frac{1}{\Gamma(5)} & \frac{1}{\Gamma(4)} \\ \frac{1}{\Gamma(6)} & \frac{1}{\Gamma(5)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(3)} \\ \frac{1}{\Gamma(5)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(3)} & \frac{1}{\Gamma(2)} \\ \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(3)} & \frac{1}{\Gamma(2)} & \frac{1}{\Gamma(1)} \end{pmatrix}$$

with $||N|| = \frac{1}{1036800}$. It is easy to see that (H1) holds with $b_f = |b_0|, a_f = |a_0|$ and $B_I = \max\{|I_i| : i \in \mathbb{N}_1^8\}, A_I = 0$. By direction computation, using Mathlab 7.0, we have

$$N_0 = \left(\sum_{v=1}^{4}\frac{1036800\Gamma(4)}{\Gamma(43/5-v)}\sum_{j=1}^{4}\sum_{\sigma=1}^{8-j}\frac{(1/2)^{38/5-j-\sigma}}{\Gamma(43/5-j-\sigma)} + \sum_{v=1}^{8}\frac{1}{\Gamma(43/5-v)}\right)\max\{|I_i| : i \in \mathbb{N}_1^8\}$$

$$+\left(\sum_{v=1}^{4}\frac{1036800\Gamma(4)}{\Gamma(43/5-v)}\sum_{j=1}^{4}\frac{\mathbf{B}(37/5-j,4/5)}{\Gamma(43/5-j)} + \frac{\mathbf{B}(37/5,4/5)}{\Gamma(38/5)}\right)|b_0| \le 4469800\max\{|I_i| : i \in \mathbb{N}_1^8\} + 23664|b_0|,$$

$$N_1 = \left(\sum_{v=1}^{4}\frac{1036800\Gamma(4)}{\Gamma(43/5-v)}\sum_{j=1}^{4}\frac{\mathbf{B}(37/5-j,4/5)}{\Gamma(43/5-j)} + \frac{\mathbf{B}(37/5,4/5)}{\Gamma(38/5)}\right)|a_0| < 23664|a_0|.$$

By Theorem 4.2, BVP(5.2) has at least one solution in X if one of the following items hold:

(i) $\sigma < 1$; **(ii)** $\sigma = 1$ with $23664|a_0| < 1$; **(iii)** $\sigma > 1$ with $23664|a_0|[4469800\max\{|I_i| : i \in \mathbb{N}_1^8\} + 23664|b_0|]^{\sigma-1} \le \frac{(\sigma-1)^{\sigma-1}}{\sigma^{\sigma}}$. ∎

Example 5.3. Consider the following impulsive boundary value problem

$$\begin{cases} D_{0+}^{\frac{8}{5}}u(t) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0 + a_0[(t-i/2)^{2/5}u(t)]^{\frac{1}{3}}\right], \quad t \in \left(\frac{i}{2}, \frac{i}{2}+\frac{1}{2}\right], \ i = 0, 1, \\ D_{0+}^{\frac{3}{5}}u(0) = 0, \ D_{0+}^{\frac{3}{5}}u(1) = 0, \\ \lim\limits_{t\to s^+}(t-1/2)^{\frac{2}{5}}u(t) = I_2[u(1/2)]^{1/3}, \ \Delta D_{0^+}^{3/5+i}u(1/2) = I_1[u(1/2)]^{1/3}, \end{cases} \tag{5.3}$$

where $b_0, a_0, I_1, I_2 \in \mathbb{R}$, $I_1 \geq 0, I_2 > 0, a_0 > 0$ are constants.

Corresponding to BVP(1.30), we have $\alpha = \frac{8}{5}$ with $n = 2$, $0 = t_0 < t_1 = \frac{1}{2} < t_2 = 1$, $k = 1$, and $f(t,x) = t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0 + a_0[(t-i/2)^{2/5}x]^{\frac{1}{3}}\right]$, $I_j(1/2, x) = I_j x^{\frac{1}{3}}$, $j \in \mathbb{N}_1^2$, f is a Carathódoey function with $p = q = -\frac{1}{5}$. It is easy to see that (H3) holds with $\prod_f(x) = |b_0| + a_0 x^{1/3}$, $\prod_I(x) = \max\{|I_1|, I_2\}x^{\frac{1}{3}}$.

One finds

$$\overline{M} = b_0 \int_0^{1/2} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}ds + b_0 \int_{1/2}^{1} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}ds$$

$$+I_1 x(1/2)^{\frac{1}{3}} + a_0 \int_0^{1/2} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}s^{2/15}[x(s)]^{\frac{1}{3}}ds$$

$$+a_0 \int_{1/2}^{1} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}(s-1/2)^{2/15}[x(s)]^{\frac{1}{3}}ds.$$

Choose

$$M = \sqrt[3]{\frac{\left|b_0 \int_0^{1/2} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}ds + b_0 \int_{1/2}^{1} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}ds\right|}{I_1 + a_0 \int_0^{1/2} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}s^{2/15}ds + a_0 \int_{1/2}^{1} s^{-\frac{1}{5}}(1-s)^{-\frac{1}{5}}(s-1/2)^{2/15}ds}}.$$

If $|x(t)| > M$ on $(0,1]$, by $\lim\limits_{t\to(1/2)^+}(t-1/2)^{2/5}x(t) = I_2[x(t)]^{1/3}$ with $I_2 > 0$, we know that $x(t)$ has same sign on $(0,1]$. Then either $x(t) > M$ for all $t \in (0,1]$ or $x(t) < -M$ for all $t \in (0,1]$. It is easy to see that $\overline{M} \neq 0$. Hence (H4) holds.

Similarly we know (H5) holds. By direct computation, we have $M_1 = \frac{M}{\Gamma(3/5)}$ and

$$M_2 = \frac{1}{\Gamma(3/5)} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1036800\Gamma(4)}{\Gamma(43/5-i)} \sum_{v=1}^{8-j} \frac{(1/2)^{8-j-v}}{\Gamma(9-j-v)}$$

$$+\frac{1}{\Gamma(3/5)} \sum_{v=1}^{8} \frac{1}{\Gamma(43/5-v)} + \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1036800\Gamma(4)}{\Gamma(43/5-i)} \sum_{v=1}^{8-j} \frac{(1/2)^{8-j-v}}{\Gamma(9-j-v)} + \sum_{v=1}^{8} \frac{1}{\Gamma(43/5-v)}$$

$$M_3 = \frac{1}{\Gamma(3/5)} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1036800\Gamma(4)}{\Gamma(43/5-i)} \frac{\mathbf{B}(37/5-j,4/5)}{\Gamma(8-j)} + \frac{1}{\Gamma(43/5-n)} \frac{\mathbf{B}(37/5,4/5)}{\Gamma(38/5)}$$

$$+ \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1036800\Gamma(4)}{\Gamma(43/5-i)} \frac{\mathbf{B}(39/5-j,4/5)}{\Gamma(8-j)} + \frac{\mathbf{B}(37/5,4/5)}{\Gamma(38/5)}.$$

By Theorem 4.3, BVP(5.3) has at least one solution since

$$\lim_{r\to\infty} \frac{r}{M_1+M_2[|b_0|+|a_0|r^{1/3}]+M_3 \max\{|I_i|:i\in\mathbb{N}_1^8\}r^{1/3}} = +\infty > 1.$$

6 Conclusion and future studies

One important part of this paper is to present a new method for converting BVPs for impulsive fractional differential equations to integral equations and to establish existence results for three classes of two-point booundary value problems for hiigher order impulsive fractional differential equations involved the Riemann-Liouville fractional derivatives.

Another important part is to demonstrate the application of the powerful mathematical tool (fixed point theorems in Banach spaces) for solving nonlinear fractional differential models.

Some problems considered in this paper can be improved under weaker conditions on the functions f, g and I_k, J_k. Further studies are also located on seeking the numerical simulation of these models.

Impulsive fractional differential equations represent a real framework for mathematical modeling to real world problems. Significant progress has been made in the theory of impulsive fractional differential equations. Impulsive fractional differential equations is an important area of study [5].

This paper contributes within the domain of impulsive fractional differential equations. The author strongly believes that the article will highly be appreciated by the researchers working in the field of fractional calculus and on fractional differential models.

7 Appendix

Proof of Theorem 4.1. Let T_1 be defined by (4.12). From Lemma 4.1 and Lemma 4.3, we know that x is a solution of BVP(1.28) if and only if x is a fixed point of T_1, $T_1 : X \to X$ is completely continuous. We shall apply Lemma 2.1.

Let $\Omega_r = \{x \in X : ||x|| \le r\}$. For $x \in \Omega_r$. Then $||x|| \le r$, i.e., $|x(t)| \le r$ for all $t \in (0, 1]$. So (H1) implies

$$|f(t, x(t))| \le \left[b_f + a_f \left|\frac{x(t)}{(t-t_s)^{n-\alpha}}\right|^{\sigma}\right] t^p(1-t)^q \le [b_f + a_f r^{\sigma}]t^p(1-t)^q,$$

$$|I_j(t_s, x(t_s))| \le B_I + A_I \left|\frac{x(t_s)}{(t_s-t_{s-1})^{n-\alpha}}\right|^{\sigma} \le B_I + A_I r^{\sigma}.$$

We know $|M_{ij}| \leq \Gamma(n-k)$. By (4.12), we have

$$(t-t_s)^{n-\alpha}|(T_1x)(t)| \leq (t-t_s)^{n-\alpha}\left[\sum_{v=1}^{n-k}\frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{|M_{jv}|}{||M||}\sum_{w=1}^{m}\sum_{\sigma=1}^{n-(j-1)}\frac{|I_{\sigma x}(t_w)|(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)}\right.$$

$$+\sum_{v=1}^{n-k}\frac{t^{\alpha-v}}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{|M_{jv}|}{||M||}\int_0^1\frac{(1-s)^{\alpha-(j-1)-1}}{\Gamma(\alpha-(j-1))}|f_x(s)|ds$$

$$\left.+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{|I_{vx}(t_w)|}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_x(s)|ds\right]$$

$$\leq\sum_{v=1}^{n-k}\frac{1}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{\Gamma(n-k)}{||M||}\sum_{w=1}^{m}\sum_{\sigma=1}^{n-(j-1)}\frac{(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)}B_I+\sum_{v=1}^{n}\frac{mB_I}{\Gamma(\alpha-v+1)}$$

$$+\sum_{v=1}^{n-k}\frac{1}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{\Gamma(n-k)}{||M||}\frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))}b_f+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}b_f$$

$$+\left[\sum_{v=1}^{n-k}\frac{1}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{\Gamma(n-k)}{||M||}\sum_{w=1}^{m}\sum_{\sigma=1}^{n-(j-1)}\frac{(1-t_w)^{\alpha-(j-1)-\sigma}}{\Gamma(\alpha-(j-1)-\sigma+1)}A_I+\sum_{v=1}^{n}\frac{mA_I}{\Gamma(\alpha-v+1)}\right.$$

$$\left.+\sum_{v=1}^{n-k}\frac{1}{\Gamma(\alpha-v+1)}\sum_{j=1}^{n-k}\frac{\Gamma(n-k)}{||M||}\frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-(j-1))}a_f+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}a_f\right]r^\sigma.$$

It follows that

$$||T_1x|| \leq M_0 + M_1r^\sigma. \tag{7.1}$$

In order to use Lemma 3.4, from (7.1), we must choose $r > 0$ such that

$$M_0 + M_1r^\sigma \leq r. \tag{7.2}$$

Then $T_1\Omega_r \subseteq \Omega_r$. So T_1 has a fixed point in Ω_r. Then BVP(1.28) has a solution. We consider the following three cases:

Case 1. $\sigma < 1$.

Since $\lim_{r\to\infty}\frac{M_0+M_1r^\sigma}{r} = 0$, we can choose $r > 0$ sufficiently small such that (7.2) holds. Then $T_1\Omega_r \subseteq \Omega_r$. So T_1 has a fixed point in Ω_r. Then BVP(1.28) has a solution.

Case 2. $\sigma = 1$.

Since $\lim_{r\to\infty}\frac{M_0+M_1r^\sigma}{r} = M_1 < 1$, we can choose $r > 0$ sufficiently small such that (7.2) holds. Then $T_1\Omega_r \subseteq \Omega_r$. So T_1 has a fixed point in Ω_r. Then BVP(1.28) has a solution.

Case 3. $\sigma > 1$.

Choose $r = \left(\frac{M_0}{M_1(\sigma-1)}\right)^{1/\sigma}$. Then we have by the inequality in (iii) that

$$||T_1x|| \leq M_0 + M_1r^\sigma \leq r.$$

Then $T_1\Omega_r \subseteq \Omega_r$. So T_1 has a fixed point in Ω_r. Then BVP(1.28) has a solution.

The proof of Theorem 4.1 is completed. ∎

Proof of Theorem 4.3. Let X, Z L_1 and N_1 be defined by (4.14). By (a)-(c), (H3)-(H5), from Lemma 4.4, L_1 be a Fredholm operator of index zero and N_1 be L_1-compact on each closed nonempty set Ω centered at zero. We seek fixed point of the operator equation $L_1x = N_1x$. To apply Lemma 3.5, we should define an open bounded subset Ω of X centered at zero such that (i), (ii) and (iii) in Lemma 3.5 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1 = \{x \in X \cap D(L_1) \setminus \mathrm{Ker}L_1,\ L_1x = \lambda N_1x \text{ for some } \lambda \in (0,1)\}$. We prove that Ω_1 is bounded.

In fact, for $x \in \Omega_1$, we have $L_1x = \lambda N_1x$ and $N_1x \in \mathrm{Im}L_1$. Then

$$\begin{cases} D^{\alpha}_{0^+}x(t) = \lambda f(t, x(t)), \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0^m, \\ D^{\alpha-n+i}_{0^+}x(0) = 0, i \in \mathbb{N}_1^k, \\ D^{\alpha-n+j}_{0^+}x(1) = 0, j \in \mathbb{N}_1^{n-k}, \\ \lim\limits_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t) = \lambda I_n(t_s, x(t_s)),\ s \in \mathbb{N}_1^m, \\ \Delta D^{\alpha-n+j}_{0^+}x(t_s) = \lambda I_j(t_s, x(t_s)), j \in \mathbb{N}_1^{n-1}, s \in \mathbb{N}_1^m. \end{cases}$$

So

$$\begin{aligned} &-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_{vx}(t_w)(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_{vx}(t_w)(1-t_w)^{k-v}}{\Gamma(k-v+1)} \\ &-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}f_x(s)ds + \int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f_x(s)ds = 0. \end{aligned} \tag{7.3}$$

It follows from (H3) that

$$|f(t, x(t))| \le \prod_f\left(\frac{|x(t)|}{(t-t_s)^{n-\alpha}}\right)t^p(1-t)^q \le \prod_f(||x||)t^p(1-t)^q, t \in (t_s, t_{s+1}], s \in \mathbb{N}_0^m,$$

$$|I_j(t_s, x(t_s))| \le \prod_I(||x||), s \in \mathbb{N}_1^m, j \in \mathbb{N}_1^n.$$

It follows from (H4) and (4.26) that there exists $\bar{t} \in (t_s, t_{s+1}]$ (for some $s \in \mathbb{N}_0^m$) such that

$$|x(\bar{t})| \le M. \tag{7.4}$$

By similar method used in (4.22), we have

$$x(t) = \frac{I_{0+}^{n-\alpha}x(0)}{\Gamma(\alpha-n+1)}t^{\alpha-n} - \sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_{vx}(t_w)(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}t^{\alpha-i}$$

$$-\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}f_x(s)dst^{\alpha-i}$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_x(s)ds, t\in(t_s,t_{s+1}], s\in\mathbb{N}_0^m.$$

Then

$$x(\bar{t}) = \frac{I_{0+}^{n-\alpha}x(0)}{\Gamma(\alpha-n+1)}\bar{t}^{\alpha-n} - \sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_{vx}(t_w)(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}\bar{t}^{\alpha-i}$$

$$-\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(\alpha-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}f_x(s)ds\bar{t}^{\alpha-i}$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(\bar{t}-t_w)^{\alpha-v} + \int_0^{\bar{t}}\frac{(\bar{t}-s)^{\alpha-1}}{\Gamma(\alpha)}f_x(s)ds, \bar{t}\in(t_s,t_{s+1}].$$

We get

$$\frac{|I_{0+}^{n-\alpha}x(0)|}{\Gamma(\alpha-n+1)} \le |x(\bar{t})|\bar{t}^{n-\alpha} + \sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{|K_{j\ i}|}{\Gamma(\alpha-i+1)||K||}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{|I_{vx}(t_w)|(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}\bar{t}^{n-i}$$

$$+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{|K_{j\ i}|}{\Gamma(\alpha-i+1)||K||}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}|f_x(s)|ds\bar{t}^{n-i}$$

$$+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{|I_{vx}(t_w)|}{\Gamma(\alpha-v+1)}(\bar{t}-t_w)^{\alpha-v}\bar{t}^{n-\alpha} + \bar{t}^{n-\alpha}\int_0^{\bar{t}}\frac{(\bar{t}-s)^{\alpha-1}}{\Gamma(\alpha)}|f_x(s)|ds$$

$$\le M + \sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}\prod_I(||x||)$$

$$+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\frac{\mathbf{B}(n+q-j,p+1)}{\Gamma(n-j)}\prod_f(||x||)$$

$$+\sum_{v=1}^{n}\frac{m}{\Gamma(\alpha-v+1)}\prod_I(||x||) + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||).$$

So for $t \in (t_s, t_{s+1}]$, we have

$$(t-t_s)^{n-\alpha}|x(t)| \le (t-t_s)^{n-\alpha}\left[\frac{|I_{0+}^{n-\alpha}x(0)|}{\Gamma(\alpha-n+1)}t^{\alpha-n}\right.$$

$$+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{|K_{j\ i}|}{\Gamma(\alpha-i+1)||K||}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{|I_{vx}(t_w)|(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}t^{\alpha-i}+\sum_{w=1}^{s}\sum_{v=1}^{n}\frac{|I_{vx}(t_w)|}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}$$

$$\left.+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{|K_{j\ i}|}{\Gamma(\alpha-i+1)||K||}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}|f_x(s)|dst^{\alpha-i}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_x(s)|ds\right]$$

$$\le \frac{M}{\Gamma(\alpha-n+1)}+\left[\frac{1}{\Gamma(\alpha-n+1)}\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}\right.$$

$$\left.+\frac{1}{\Gamma(\alpha-n+1)}\sum_{v=1}^{n}\frac{m}{\Gamma(\alpha-v+1)}+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}+\sum_{v=1}^{n}\frac{m}{\Gamma(\alpha-v+1)}\right]\prod_I(||x||)$$

$$+\left[\frac{1}{\Gamma(\alpha-n+1)}\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\frac{\mathbf{B}(n+q-j,p+1)}{\Gamma(n-j)}+\frac{1}{\Gamma(\alpha-n+1)}\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\right.$$

$$\left.+\sum_{i=1}^{n-k-1}\sum_{j=1}^{n-k-1}\frac{\Gamma(n-k)}{\Gamma(\alpha-i+1)||K||}\frac{\mathbf{B}(n+q-j,p+1)}{\Gamma(n-j)}+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\right]\prod_f(||x||).$$

It follows that

$$||x|| \le M_1 + M_2\prod_I(||x||) + M_3\prod_f(||x||). \tag{7.4}$$

Since $\lim\limits_{r\to\infty}\frac{r}{M_1+M_2\prod_I(r)+M_3\prod_f(r)} > 1$ (see (4.23)), we get from (7.4) that there exists a constant $M_4 > 0$ independent of λ such that $||x|| \le M_4$. It follows that Ω_1 is bounded.

Step 2. Let $\Omega_2 = \{ct^{\alpha-n} \in \mathrm{Ker}L_1 :\ N_1(ct^{\alpha-n}) \in \mathrm{Im}L_1\}$. We prove that Ω_2 is bounded. For $ct^{\alpha-n} \in \Omega_2$, we have

$$N_1(ct^{\alpha-n}) = \begin{pmatrix} f(t, ct^{\alpha-n}) \\ I_1(t_s, ct_s^{\alpha-n}) : s \in \mathbb{N}_1^m \\ I_2(t_s, ct_s^{\alpha-n}) : s \in \mathbb{N}_1^m \\ \cdots \\ I_n(t_s.ct_s^{\alpha-n}) : s \in \mathbb{N}_1^m \end{pmatrix}.$$

So

$$-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)}+\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)}$$

$$-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1}\frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\int_0^1\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}f(s, cs^{\alpha-n})ds+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s, cs^{\alpha-n})ds = 0.$$

From (H5), we get that $|c| \leq M_0$. This shows Ω_2 is bounded.

Step 3. If the first inequality in (H5) holds and

$$\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) ds > 0,$$

or the second inequality in (H5) holds and

$$\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) ds < 0,$$

we prove that $\Omega_3 = \{c \in \text{Ker } L_1 : \ \lambda \wedge (c) + (1-\lambda) Q N_1(c) = 0, \ \lambda \in [0,1]\}$ is bounded, where $\wedge : \text{Ker} L \to Z/\text{Im} L_1$ is the isomorphism given by $\wedge(ct^{\alpha-n}) = (c, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)$.

For $ct^{\alpha-n} \in \text{Ker } L_1$, one sees that

$$-\lambda \wedge (ct^{\alpha-n}) = -\lambda(c, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)$$

$$= (1-\lambda) \left(\overline{Q}_{ct^{\alpha-n}, I_i(t_j, ct_j^{\alpha-n})}, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m \right),$$

where

$$\overline{Q}_{ct^{\alpha-n}, I_i(t_j, ct_j^{\alpha-n})} = \left[\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) ds \right]^{-1} \times$$

$$\left[- \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \sum_{w=1}^{m} \sum_{v=1}^{n-j} \frac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m} \sum_{v=1}^{k} \frac{I_v(t_w, ct^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)} \right.$$

$$\left. + \int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) f(s, cs^{\alpha-n}) ds \right].$$

Then

$$-\lambda c^2 = (1-\lambda) \left[\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) ds \right]^{-1} \times$$

$$c \left[- \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \sum_{w=1}^{m} \sum_{v=1}^{n-j} \frac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m} \sum_{v=1}^{k} \frac{I_v(t_w, ct^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)} \right.$$

$$\left. + \int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k} \sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|} \frac{(1-s)^{n-j-1}}{\Gamma(n-j)} \right) f(s, cs^{\alpha-n}) ds \right].$$

If $\lambda = 1$, we get $c = 0$. If $\lambda \in [0, 1)$, and $|c| > M_0$, we get

$$0 \geq -\lambda c^2 = (1-\lambda)\left[\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) ds\right]^{-1} \times$$

$$c\left[-\sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\sum_{w=1}^{m}\sum_{v=1}^{n-j}\frac{I_v(t_w, ct_w^{\alpha-n})(1-t_w)^{n-j-v}}{\Gamma(n-j-v+1)} + \sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w, ct^{\alpha-n})(1-t_w)^{k-v}}{\Gamma(k-v+1)}\right.$$

$$\left. + \int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) f(s, cs^{\alpha-n})ds\right] > 0,$$

a contradiction. Then $|c| \leq M_0$. Then Ω_3 is bounded.

If the first inequality in (H5) holds and

$$\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) ds < 0,$$

or the second inequality in (H5) holds and

$$\int_0^1 \left(\frac{(1-s)^{k-1}}{\Gamma(k)} - \sum_{i=1}^{k}\sum_{j=1}^{n-k-1} \frac{K_{j\ i}}{\Gamma(k-i+1)|K|}\frac{(1-s)^{n-j-1}}{\Gamma(n-j)}\right) ds > 0,$$

we can prove that $\Omega_3 = \{c \in \text{Ker } L_1 : \ \lambda \wedge (c) - (1-\lambda)QN_1(c) = 0, \ \lambda \in [0,1]\}$ is bounded, where $\wedge : \text{Ker}L \to Z/\text{Im}L_1$ is the isomorphism given by $\wedge(ct^{\alpha-n}) = (c, 0 : i \in \mathbb{N}_0^{n-1}, j \in \mathbb{N}_1^m)$.

Step 4. We shall show that all conditions of Lemma 2.2 are satisfied.

Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$. By Lemma 4.4, L_1 is a Fredholm operator of index zero and N_1 is L_1-compact on $\overline{\Omega}$. By the definition of Ω, we have

(a). $L_1(x) \neq \lambda N(x)$ for $x \in (D(L_1) \setminus \text{Ker}L_1) \cap \partial\Omega$ and $\lambda \in (0, 1)$;

(b). $N_1(x) \notin \text{Im}L_1$ for $x \in \text{Ker}L_1 \cap \partial\Omega$.

(c). $\deg(QN_1|_{\text{Ker}L_1}, \ \Omega \cap \text{Ker}L_1, 0) \neq 0$. In fact, let $H(x, \lambda) = \pm\lambda \wedge (x) + (1-\lambda)QN_1(x)$. According the definition of Ω, we know $H(x, \lambda) \neq 0$ for $x \in \partial\Omega \cap \text{Ker}L_1$, thus by homotopy property of degree,

$$\deg(QN_1|_{\text{Ker}L_1}, \Omega \cap \text{Ker}L_1, 0) = \deg(H(\cdot, 0), \Omega \cap \text{Ker}L_1, 0)$$

$$= \deg(H(\cdot, 1), \Omega \cap \text{Ker}L_1, 0) = \deg(\wedge, \Omega \cap \text{Ker}L_1, 0) \neq 0.$$

Thus by Lemma 3.5, $L_1x = N_1x$ has at least one solution in $D(L_1) \cap \overline{\Omega}$. Then x is a solution of BVP(1.30). The proof is completed. ∎

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