On a multiple Čebyšev type functional defined by a generalized fractional integral operator

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Abstract

A new multiple Čebyšev type functional is introduced by making use of the generalized fractional integral operator with hypergeometric kernel and the notion of permanent of matrix analysis. Inequalities for this functional are established for synchronous functions.

Keywords: Fractional integral operator; Čebyšev type functional; synchronous (asynchronous) function

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1 Introduction

In recent years, a number of papers have appeared on the topic of the Čebyšev type functional defined by certain fractional integral operators. Mostly in such works the functional is associated with a special class of functions which are usually *synchronous* on a given interval.

Two functions $\varphi, \psi : I \to \mathbb{R}$ are said to be *synchronous* (*asynchronous*) if they satisfy the following condition:

$$[\varphi(t) - \varphi(s)][\psi(t) - \psi(s)] \geq (\leq)\, 0 \quad \text{for each } t, s \in I. \tag{1}$$

In the literature, two functions that satisfy (1) are also called *similarly ordered* (*oppositely ordered*). Obviously, two functions φ, ψ are synchronous if they are monotonic in the same sense, but not conversely.

Belarbi and Dahman [3] established some interesting inequalities for the Čebyšev type functional by making use of the well-known Riemann-Liouville integral operator. Subsequently, Saxena *et al.* [14] introduced the *Čebyšev-Saigo* and *multiple Čebyšev-Saigo functionals*:

$$T_{\mathrm{S}}(\varphi, \psi) := \mathcal{M}_{\mathrm{S}}(\varphi\psi) - \mathcal{M}_{\mathrm{S}}(\varphi)\,\mathcal{M}_{\mathrm{S}}(\psi) \tag{2}$$

and

$$T_{\mathrm{S}}(\varphi_1, \cdots, \varphi_n) := \mathcal{M}_{\mathrm{S}}\left(\prod_{j=1}^{n}\varphi_j\right) - \prod_{j=1}^{n}\mathcal{M}_{\mathrm{S}}(\varphi_j), \tag{3}$$

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where the integral mean $\mathcal{M}_{\mathrm{S}}$ is defined by

$$\mathcal{M}_{\mathrm{S}}(\varphi) := \mathcal{M}_{0,t}^{\alpha,\beta,\eta}(\varphi) = \frac{I_{0,t}^{\alpha,\beta,\eta}[\varphi]}{I_{0,t}^{\alpha,\beta,\eta}[1]},$$

and $I_{0,t}^{\alpha,\beta,\eta}$ denotes the Saigo fractional integral operator ([13]) defined by

$$I_{0,t}^{\alpha,\beta,\eta}[\varphi] := \frac{t^{-\beta}}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} {}_2F_1 \left[\begin{matrix} \alpha+\beta, -\eta \\ \alpha \end{matrix}; 1-x \right] \varphi(tx)\, \mathrm{d}x, \tag{4}$$

which has been studied in several diverse areas during past few decades, see for instance, the works in [5, 6], [11, 12], [16, 17, 18, 19, 20] and [21].

Purohit and Raina [10] also studied the same functionals and proposed their q-analogues. Further extensions using the operator introduced by Curiel and Galué [4] and their related inequalities was investigated by Baleanu *et al.* [1].

In the present paper, we study a new extension to the multiple Čebyšev-Saigo functional defined by (3). Our main purpose is to rebuilt the functional (3) by making use of a new fractional integral operator proposed in [7] and a multilinear mapping called *permanent*. The definition of the new functional is described in Section 2 and Section 3 is devoted to the main results which includes a theorem and a corollary.

2 Definitions and lemmas

Before stating our new functional, we first introduce some necessary notations and definitions. The Pochhammer symbol $(a)_k$ is defined (as usual) by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & (k=0;\ a \in \mathbb{C} \setminus \{0\}) \\ a(a+1)\cdots(a+k-1) & (k \in \mathbb{N};\ a \in \mathbb{C}), \end{cases}$$

where $\Gamma(a)$ is the familiar Gamma function. We adopt the convention of writing the finite sequence of parameters $a_1, \cdots, a_p$ by (a_p) and the product of p Pochhammer symbols by $((a_p))_k \equiv (a_1)_k \cdots (a_p)_k$, where an empty product $p=0$ is treated as unity.

The construction of our multiple Čebyšev type functional uses two new concepts. The first one is a generalized fractional integral operator introduced by the authors in [7] which is defined in the following manner.

Definition 2.1. Let $x, h, \nu \in \mathbb{R}_+$, $\delta, a, b, f_1, \cdots, f_r \in \mathbb{C}$ and $m_1, \cdots, m_r \in \mathbb{N}$. Also, let $A \equiv \delta + \nu(\mu+h)$, $\Re(\mu) > 0$ and $\varphi(s)$ be a suitable complex-valued function defined on $\mathbb{R}_+$. Then, the fractional integral of a function $\varphi(x)$ is defined by

$$\begin{aligned} (\mathcal{I}\varphi)(x) &\equiv \left(\mathcal{I}_{h;\nu,\delta:\ \ (f_r)}^{\mu;a,b:\ (f_r+m_r)} \varphi \right)(x) \\ &:= \frac{\nu x^{-A}}{\Gamma(\mu)} \int_0^x (x^\nu - s^\nu)^{\mu-1} {}_{r+2}F_{r+1} \left[\begin{matrix} a, b, & (f_r+m_r) \\ \mu, & (f_r) \end{matrix}; 1 - \frac{s^\nu}{x^\nu} \right] \varphi(s)\, s^{\nu h+\nu-1} \mathrm{d}s, \end{aligned} \tag{5}$$

where ${}_{r+2}F_{r+1}[z]$ denotes the generalized hypergeometric function (see [8]; see also [15]) defined by

$${}_pF_q \left[\begin{matrix} (a_p) \\ (b_q) \end{matrix}; z \right] := \sum_{k=0}^{\infty} \frac{((a_p))_k}{((b_p))_k} \frac{z^k}{k!}. \tag{6}$$

A useful equivalent form of (5) which is more suitable for our study is given by

$$(\mathcal{I}\varphi)(x) = \frac{x^{-\delta}}{\Gamma(\mu)} \int_0^1 (1-y)^{\mu-1} {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ \mu, & (f_r) \end{matrix}; 1-y\right] \varphi\left(xy^{1/\nu}\right) y^h \mathrm{d}y. \tag{7}$$

If we set $r = h = 0$, $\nu = 1$, $\delta = \beta$, $\mu = \alpha$, $a = \alpha + \beta$ and $b = -\eta$ in (7), we obtain the Saigo fractional integral operator defined above in (4). For $\varphi(s) = s^\lambda$, we have the following result ([7]):

$$\mathcal{I}x^\lambda = x^{\lambda-\delta} \sum_{k=0}^{m} \frac{A_k}{A_0} \frac{(a)_k (b)_k \Gamma(\mathfrak{c}_1(\lambda)) \Gamma(\mathfrak{c}_1(\lambda) + \mathfrak{p}_k)}{\Gamma(\mathfrak{c}_1(\lambda) + \mu - a) \Gamma(\mathfrak{c}_1(\lambda) + \mu - b)} \tag{8}$$

$$(\Re(\mu) > 0; \Re(\mathfrak{c}_1(\lambda)) > -\min[0, \Re(\mathfrak{p}_m)]),$$

where

$$\left.\begin{aligned} \mathfrak{c}_1(t) &:= 1 + h + \frac{t}{\nu}, \\ \mathfrak{p}_k &:= \mu - a - b - k, \end{aligned}\right\} \tag{9}$$

and the coefficients A_k $(0 \le k \le m := m_1 + \cdots + m_r)$ are given by

$$A_k = \sum_{j=k}^{m} \begin{Bmatrix} j \\ k \end{Bmatrix} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad A_m = 1,$$

with σ_j $(0 \le j \le m)$ generated by

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j.$$

Upon setting $\lambda = 0$ in the formula (8), we have

$$\mathcal{I}(1) = x^{-\delta} \sum_{k=0}^{m} \frac{A_k}{A_0} \frac{(a)_k (b)_k \Gamma(1+h) \Gamma(1+h+\mathfrak{p}_k)}{\Gamma(1+h+\mu-a) \Gamma(1+h+\mu-b)} \tag{10}$$

$$(\Re(\mu) > 0; 1 + h > -\min[0, \Re(\mathfrak{p}_m)]),$$

where $\mathfrak{p}_m$ is given by (9).

The second concept is based on the notion of "permanent" used in the Matrix Analysis. Suppose $A = (a_{ij})$ be an $n \times n$ matrix, then the permanent of A written as $\mathrm{per}(A)$ is defined by ([22, p. 99])

$$\mathrm{per}(A) \equiv \underset{1 \le i,j \le n}{\mathrm{per}} (a_{ij}) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{11}$$

where S_n deontes the symmetric group of degree n, i.e., the group of all permutation of $\mathcal{N} \equiv \{1, 2, \cdots, n\}$. Thus, the permanent is the determinant without the alternating minus signs.

The combination of (7), (10) and (11) now suggests the following definition of our new multiple Čebyšev type functional.

Definition 2.2 (multiple Čebyšev type functional)**.** Let $\mu, h, \nu \in \mathbb{R}_+$, and let $a, b, f_1, \cdots, f_r \in \mathbb{R}$ and $m_1, \cdots, m_r \in \mathbb{N}$ be such that the inequalities $1 + h > -\min[0, \mathfrak{p}_m]$ and

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r + m_r) \\ \mu, & (f_r) \end{matrix}; 1 - y\right] > 0 \quad (y \in (0,1)) $$

hold true, where $\mathfrak{p}_m$ is given by (9). Let $\varphi_{ij} : \mathbb{R}_+ \to \mathbb{R}$ $(1 \le i, j \le n)$ be the functions such that $\mathcal{I}(\varphi_{ij})$ $(1 \le i, j \le n)$ exist. Then, for these functions, the multiple Čebyšev type functional is defined by

$$ \mathfrak{T}_{n\times n}(\varphi_{11}, \cdots, \varphi_{1n}, \cdots, \varphi_{n1}, \cdots, \varphi_{nn}) := \mathcal{M}\left(\underset{1\le i,j\le n}{\mathrm{per}}(\varphi_{ij})\right) - \underset{1\le i,j\le n}{\mathrm{per}}(\mathcal{M}(\varphi_{ij})), \tag{12} $$

where per $(\cdot)$ is defined by (11) and the integral mean $\mathcal{M}(\varphi)$ in terms of the integral operator (7) is given by

$$ \mathcal{M}(\varphi) := \frac{\mathcal{I}(\varphi)}{\mathcal{I}(1)}. \tag{13} $$

If we set $\varphi_{ij} = 0$ $(i \ne j; 1 \le i, j \le n)$, then the functional $\mathfrak{T}_{n\times n}$ reduces to

$$ \mathfrak{T}_n(\varphi_{11}, \varphi_{22}, \cdots, \varphi_{nn}) := \mathcal{M}\left(\prod_{i=1}^{n} \varphi_{ii}\right) - \prod_{i=1}^{n} \mathcal{M}(\varphi_{ii}), \tag{14} $$

which provides a generalization of the multiple Čebyšev-Saigo functional T_{S} defined in (3).

On the other hand, if we set $n = 2$ in (14), and denote $\varphi = \varphi_{11}$ and $\psi = \varphi_{22}$, we obtain

$$ \mathfrak{T}(\varphi, \psi) \equiv \mathfrak{T}_2(\varphi, \psi) := \mathcal{M}(\varphi\psi) - \mathcal{M}(\varphi)\mathcal{M}(\psi). $$

For the Čebyšev type functional $\mathfrak{T}$, we have the following useful result.

Lemma 2.3 ([7, Theorem 5.1])**.** Let φ, ψ be synchronous (asynchronous) on $\mathbb{R}_+$, then

$$ \mathfrak{T}(\varphi, \psi) \ge (\le)\, 0. $$

In this section, we now introduce the synchronicity between m functions of single variable (see [2, p. 644, Definition B.]; see also [9, p. 16, Definition 2.]).

For $m \ge 2$, the functions $\varphi_i : I \to \mathbb{R}$ $(1 \le i \le m)$ are synchronous (or similarly ordered) if

$$ [\varphi_i(t) - \varphi_i(s)][\varphi_j(t) - \varphi_j(s)] \ge 0 \quad \text{for } 1 \le i, j \le m, \text{ all } t, s \in I. \tag{15} $$

When $m = 2$, the condition (15) is equivalent to the inequality (1). We notice that the condition (15) now provides m^2 inequalities. But not all of them are necessary for our definition. In fact, since only $\frac{1}{2}m(m-1)$ different inequalities are contained, the condition (15) can be replaced by

$$ [\varphi_i(t) - \varphi_i(s)][\varphi_j(t) - \varphi_j(s)] \ge 0 \quad \text{for } 1 \le i < j \le m, \text{ all } t, s \in I. \tag{16} $$

For the sake of brevity, we use below the notation $\langle \varphi_i, \varphi_j \rangle$, whenever, the functions φ_i, φ_j are synchronous. With this new notation, we can state that $\varphi_1, \cdots, \varphi_m$ are synchronous (or similarly ordered) if and only if each two of them are synchronous, i.e.,

$$ \langle \varphi_i, \varphi_j \rangle \quad (1 \le i < j \le m). $$

Lemma 2.4. If $\varphi_i : I \to \mathbb{R}_+$ $(1 \le i \le m)$ are synchronous, then

$$\langle \varphi_l, \varphi_{l+1}\varphi_{l+2}\cdots\varphi_m \rangle, \tag{17}$$

for $l \ge 1$. More generally, we have

$$\left\langle \varphi_l, \prod_{i=1, i\neq l}^{m} \varphi_i \right\rangle \quad (1 \le l \le m)\,. \tag{18}$$

Proof. The proof is based on the part (iv) of Lemma 1 in [2]. It asserts that if $\langle \varphi_1, \varphi_2 \rangle$ and $\langle \varphi_1, \varphi_3 \rangle$, then we have $\langle \varphi_1, \varphi_2\varphi_3 \rangle$. For arbitrarily chosen function φ_l $(l \ge 1)$ and from the definition of synchronicity, we have $\langle \varphi_l, \varphi_j \rangle$ $(l < j \le m)$, which provides

$$\langle \varphi_l, \varphi_{l+1} \rangle, \langle \varphi_l, \varphi_{l+2} \rangle, \langle \varphi_l, \varphi_{l+3} \rangle, \cdots, \langle \varphi_l, \varphi_m \rangle.$$

The first two angle brackets produce $\langle \varphi_l, \varphi_{l+1}\varphi_{l+2} \rangle$, which in the combination of the third one leads to $\langle \varphi_l, \varphi_{l+1}\varphi_{l+2}\varphi_{l+3} \rangle$. Continuing this process, we finally obtain the result (17). If $l = 1$, then we have $\langle \varphi_1, \varphi_2\varphi_3\cdots\varphi_m \rangle$. If $l > 1$, the definition also implies that $\langle \varphi_j, \varphi_l \rangle$ $(1 \le i < l)$, i.e.,

$$\langle \varphi_1, \varphi_l \rangle, \langle \varphi_2, \varphi_l \rangle, \cdots, \langle \varphi_{l-1}, \varphi_l \rangle. \tag{19}$$

Since notations $\langle \varphi, \psi \rangle$ and $\langle \psi, \varphi \rangle$ have the same meaning, (19) can be equivalently written in the form:

$$\langle \varphi_l, \varphi_1 \rangle, \langle \varphi_l, \varphi_2 \rangle, \cdots, \langle \varphi_l, \varphi_{l-1} \rangle,$$

which gives

$$\langle \varphi_l, \varphi_1\varphi_2\cdots\varphi_{l-1} \rangle. \tag{20}$$

Now combining (17) and (20), we obtain the desired result (18). Q.E.D.

3 Main results

Theorem 3.1. Let $\varphi_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ $(1 \le i, j \le n)$ be the functions such that $\mathcal{I}(\varphi_{ij})$ $(1 \le i, j \le n)$ exist and

$$\left\langle \varphi_{i\sigma(i)}, \varphi_{j\sigma(j)} \right\rangle \quad \text{for } 1 \le i < j \le n, \text{ all } \sigma \in S_n. \tag{21}$$

Then

$$\mathfrak{T}_{n\times n}(\varphi_{11}, \cdots, \varphi_{1n}, \cdots, \varphi_{n1}, \cdots, \varphi_{nn}) \ge 0.$$

Proof. By making use of the definition of (11) of the permanent, we have

$$\mathcal{M}\left(\underset{1\le i,j\le n}{\operatorname{per}}(\varphi_{ij})\right) = \sum_{\sigma\in S_n} \mathcal{M}\left(\prod_{i=1}^{n} \varphi_{i\sigma(i)}\right).$$

We thus only need to prove the assertion that for each (fixed) $\sigma \in S_n$,

$$\mathcal{M}\left(\prod_{i=1}^{n} \varphi_{i\sigma(i)}\right) \ge \prod_{i=1}^{n} \mathcal{M}\left(\varphi_{i\sigma(i)}\right). \tag{22}$$

For given $\sigma \in S_n$, the functions $\varphi_{1\sigma(1)}, \cdots, \varphi_{n\sigma(n)}$ are different, and hence can be relabelled as $\psi_1, \cdots, \psi_n$. Also, according to the condition (21), each two of them are synchronous, so we have

$$\langle \psi_i, \psi_j \rangle \quad (1 \leq i < j \leq n).$$

We note that this condition also means that $\psi_1, \cdots, \psi_n$ are all synchronous. Therefore, it suffices to prove that

$$\mathcal{M}\left(\prod_{i=1}^{n} \psi_i\right) \geq \prod_{i=1}^{n} \mathcal{M}(\psi_i). \tag{23}$$

We may proceed by induction on n. For $n = 1$, (23) is trivially true and for $n = 2$, (23) reduces to the inequality $\mathcal{M}(\psi_1\psi_2) \geq \mathcal{M}(\psi_1)\mathcal{M}(\psi_2)$, which is also true due to Lemma 2.3. Now suppose that the inequality (23) holds true for $n \geq 2$. Let $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(1 \leq i \leq n+1)$ be the functions such that $\mathcal{I}(\psi_i)$ $(1 \leq i \leq n+1)$ exist and satisfy

$$\langle \psi_i, \psi_j \rangle \quad (1 \leq i < j \leq n+1). \tag{24}$$

The condition (24) means that all $\psi_1, \cdots, \psi_{n+1}$ are synchronous, and the following $\frac{1}{2}(n+1)n$ inequalities hold true:

$$[\psi_i(t) - \psi_i(s)][\psi_j(t) - \psi_j(s)] \geq 0 \quad (1 \leq i < j \leq n+1; t, s \in \mathbb{R}_+). \tag{25}$$

Define

$$F(y) := \prod_{i=1}^{n} \psi_i(y) \quad \text{and} \quad G(y) := \psi_{n+1}(y).$$

We now show that

$$[F(t) - F(s)][G(t) - G(s)] \geq 0 \quad (t, s \in \mathbb{R}_+), \tag{26}$$

namely, $F(y)$ and $G(y)$ are synchronous. The inequality (26) obviously holds for those values of t, s for which $G(t) = G(s)$. Let

$$E_> := \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : G(t) > G(s)\} \quad \text{and} \quad E_< := \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : G(t) < G(s)\}.$$

If $(t, s) \in E_>$, then in view of (25), we have $\psi_i(t) \geq \psi_i(s)$ $(1 \leq i < n+1)$, and thus

$$F(t) \geq F(s).$$

Similarly, it follows that

$$F(t) \leq F(s),$$

if $(t, s) \in E_<$. Therefore, the inequality (26) holds for all $t, s \in \mathbb{R}_+$.

Now, by using the Lemma 2.3 and the induction assumption, we have

$$\mathcal{M}\left(\prod_{i=1}^{n+1} \psi_i\right) = \mathcal{M}(FG) \geq \mathcal{M}(F)\mathcal{M}(G) \geq \prod_{i=1}^{n} \mathcal{M}(\psi_i)\mathcal{M}(G) = \prod_{i=1}^{n+1} \mathcal{M}(\psi_i).$$

This completes the proof. Q.E.D.

Alternatively, we can also prove Theorem 3.1 in the following manner.

We first observe that the inequality (23) can be established by the repeated applications of Lemma 2.3, provided that

$$\langle \psi_i, \psi_{i+1}\psi_{i+2}\cdots\psi_n \rangle \quad (1 \le i \le n-1). \tag{27}$$

However, since $\langle \psi_i, \psi_j \rangle$ $(1 \le i < j \le n)$, (27) clearly holds in view of the first result (17) of Lemma 2.4.

Remark 3.2. It may be point out that Theorem 3.1 remains valid if we replace the condition (21) with the condition that all functions φ_{ij} $(1 \le i, j \le n)$ are synchronous, namely,

$$\langle \varphi_{jk}, \varphi_{il} \rangle \quad (1 \le j, k, i, l \le n). \tag{28}$$

We then show that the conditions (21) and (28) are not equivalent in general. To see this, we consider, for example, the case $n = 2$. When $n = 2$, the condition (28) means that

$$\langle \varphi_{11}, \varphi_{12} \rangle, \langle \varphi_{11}, \varphi_{21} \rangle, \langle \varphi_{11}, \varphi_{22} \rangle, \langle \varphi_{12}, \varphi_{21} \rangle, \langle \varphi_{12}, \varphi_{22} \rangle \quad \text{and} \quad \langle \varphi_{21}, \varphi_{22} \rangle,$$

while the condition (21) only gives

$$\langle \varphi_{11}, \varphi_{22} \rangle \quad \text{and} \quad \langle \varphi_{12}, \varphi_{21} \rangle,$$

and it does not confirm if φ_{11}, φ_{12} are synchronous. Finally, it is worth mentioning that our condition (21) is wider than the monotonicity condition imposed in [14, p. 677, Theorem 2.12], since we can derive the synchronicity between m functions from the fact that they are monotonic in the same sense, but not conversely.

Corollary 3.3. Let $\varphi_{ij}, \psi_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ $(1 \le i, j \le n)$ be the functions such that $\mathcal{I}(\varphi_{ij})$ and $\mathcal{I}(\psi_{ij})$ exist. Let $\Phi_{ij} := \varphi_{ij}/\psi_{ij}$ such that $\mathcal{I}(\Phi_{ij})$ exists and

$$\left\langle \Phi_{i\sigma(i)}, \Phi_{j\sigma(j)} \right\rangle \qquad (1 \le i < j \le n; \forall \sigma \in S_n). \tag{29}$$

Also, we assume that each pair Φ_{ij}, ψ_{ij} is asynchronous (oppositely ordered). Then, there holds the inequality that

$$\mathcal{M}\left(\operatorname*{per}_{1 \le i,j \le n} (\Phi_{ij}) \right) \ge \operatorname*{per}_{1 \le i,j \le n} \left(\frac{\mathcal{M}(\varphi_{ij})}{\mathcal{M}(\psi_{ij})} \right). \tag{30}$$

Proof. The application of Theorem 3.1 to the functions Φ_{ij} gives the inequality that

$$\mathfrak{T}_{n\times n}(\Phi_{11}, \cdots, \Phi_{1n}, \cdots, \Phi_{n1}, \cdots, \Phi_{nn}) \ge 0,$$

which is equivalent to the following form:

$$\mathcal{M}\left(\operatorname*{per}_{1 \le i,j \le n} (\Phi_{ij}) \right) \ge \operatorname*{per}_{1 \le i,j \le n} \left(\mathcal{M}(\Phi_{ij}) \right). \tag{31}$$

Now since Φ_{ij} and ψ_{ij} are asynchronous, we can apply Lemma 2.3 to get $\mathfrak{T}(\Phi_{ij}, \psi_{ij}) \le 0$, which implies that

$$\mathcal{M}(\varphi_{ij}) \le \mathcal{M}(\Phi_{ij})\, \mathcal{M}(\psi_{ij}).$$

Finally, using the definition (11) of the permanent, we obtain that

$$\operatorname*{per}_{1\le i,j\le n}\left(\mathcal{M}\left(\Phi_{ij}\right)\right)=\sum_{\sigma\in S_n}\prod_{i=1}^{n}\mathcal{M}\left(\Phi_{i\sigma(i)}\right)\geq\sum_{\sigma\in S_n}\prod_{i=1}^{n}\frac{\mathcal{M}\left(\varphi_{i\sigma(i)}\right)}{\mathcal{M}\left(\psi_{i\sigma(i)}\right)}=\operatorname*{per}_{1\le i,j\le n}\left(\frac{\mathcal{M}\left(\varphi_{ij}\right)}{\mathcal{M}\left(\psi_{ij}\right)}\right),$$

which proves the Corollary 3.3. Q.E.D.

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