Solving multi-dimensional fractional integro-differential equations with the initial and boundary conditions by using multi-dimensional Laplace Transform method

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Abstract

In this paper, the one-dimensional Laplace transform method is evolved to solve linear one-dimensional fractional Volterra integro-differential equations with initial conditions and extend this study by taking multi-dimensional Laplace transform method of the linear fractional multi-dimensional Volterra integro-differential equations to find solution of the initial and boundary value problems. It is noticed that the suggested methods is suitable to find solution for problems. The results of the noticed methods are supporter, easily and active.

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1 Introduction

Fractional integro-differential equations arise in mathematical modeling of various physical phenomena like heat conduction in materials with memory, diffusion processes etc.

Srivastava et al.[11] derived a number of interesting expressions for the composition of certain multi-dimensional fractional integral operators involving a general class of polynomials with essentially arbitrary coefficients. Systems of integral equations, appear in scientific applications in engineering, chemistry, physics and populations growth models. Studies of systems of integral equations have attracted much concern in applied sciences, A. M. Wazwaz[1].

In mathematical literature there are numerous avenues of applications of operators of fractional calculus in a wide variety of fields. H. M. Srivastava and R. K. Saxena[12] supported main on good generalizations of the fractional Volterra-type integro-differential equations in the kernel uses the confluent hypergeometric function in n complex variables.

Collocations method have been used as a numerical solution of fractional order Volterra type integro-differential equations by E. Rawashdeh[10]. H. M. Srivastava and R. K. Saxena[13] interested many of results by using the field of fractional calculus and its applications to present and consider the investigation for the historical account by various authors. Kilbas et al.[3] Maravall used the Laplace transform method to obtain the explicit solution of a certain kind of ordinary differential equations with fractional derivatives.
W. H. Wang[17] studied a modification of variational iteration method which was supported to solve

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fractional integro-differential equations. This method is an effective method. V. E. Tarasov[16] took the first order fractional integro-differential equations of electromagnetic and waves in wide class media and the result should be described by fractional differential equations. M. A. Mohammed and F. S. Fadhel[7] found solution of linear nonhomogeneous two-dimensional Volterra type fractional order integro-differential equations by using variational formulation. O. H. Mohammed[9] has employed method to get on the solution of one-dimensional fractional Volterra integro-differential equations in the Caputo sense. B. Ahmad and J. J. Nieto [4] studied the existence and uniqueness of solutions for the following nonlinear fractional integro-differential equation. A. M. Wazwaz[1] used Laplace transform method as important method with his properties for solving integral equations with Volterra type of the second kind, and gave some applications by using the linearity and the convolution properties to find the exact solution and to support the study.

A. Kader and N. Aziz [2] investigated the existence and uniqueness of solutions to fractional integro–differential equations involving the Caputo fractional derivative by using Banach and Schaefers fixed point methods. Also this leads us to extend some results gained by M. Benchohra and B. A. Slimani [5]. Where M. Benchohra and B. A. Slimani [5] established sufficient conditions for the existence of solutions with the initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative. H. M. Srivastava and Ž. Tomovski [14]considered the main results by introduced and investigated a fractional calculus with integral operator in many works to study which contains a certain family of generalized Mittag–Leffler functions in its kernel. Ž. Tomovski et al[15] were introduced and investigated in several earlier works to study a certain family of generalized fractional derivative operators of Riemann–Liouville type and approved solution of the fractional differential equations with constant and variable coefficients by using the Laplace transformation methods.

F. Mirzaee[8] derived the operational matrix of Euler functions for fractional derivative of order β in the Caputo sense and by this matrix they developed an efficient collocation method for solving nonlinear fractional Volterra integro-differential equations. Also, given examples to demonstrate the validity and applicability of the proposed method and the comparisons are made with the existing results. Khan et al.[6] they investigated the effects of an arbitrary wall shear stress on unsteady magneto hydro dynamic (MHD) flow of a Newtonian fluid with conjugate effects of heat and mass transfer.

P. Mokhtary et al.[18] used operational Tau method based on Mntz-Legendre polynomials to provide a computational technique for obtaining the numerical solutions of fractional differential equations using a sequence of matrix operations. By this method obtained the main result. X.-J. Yang et al.[19]Local fractional calculus has been successfully applied to describe the numerous widespread real-world phenomena in the fields of physical sciences and engineering sciences by the methods of integral transforms via local fractional calculus and their applications have been provides information on it. The methods have been used to solve various local fractional ordinary and partial differential equations.

2 The One-Dimensional Laplace Transform Method for Solving Initial Value Problems Linear One-Dimensional Fractional Order Volterra Integro-Differential Equations

Consider the initial value problem which consists of the linear one-dimensional fractional Volterra integro-differential equation with constant coefficient:

$$D^v y(x) = f(x) + \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} y(t)dt, x \geq 0$$

together with the initial conditions.

$$D^{-(i-1-n+v)} y(0) = A_i, \quad i = 1, 2, ..., n$$

where a_i is a known constant for each $i = 0, 1, 2, ..., n$. Such that a_n, f is a known function of $x, A_i$is a known constant for each $i = 0, 1, 2, ..., n$ and y is the unknown function that must be determined.

Then, taking the one-dimensional Laplace transform to both sides of the above Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$s^v Y(s) - \sum_{i=1}^{n} s^{n-i} D^{i-1-n+v} y(0) = s^{-v} Y(s).$$

Therefore

$$Y(s) = \frac{\sum_{i=1}^{n} s^{n-i} D^{i-1-n+v} y(0)}{s^v - s^{-v}}$$

where $L\{f(x)\} = F(s)$, $L\{y(x)\} = Y(s)$. Hence, taking the inverse Laplace transform to the above equation, one can have:

$$y(x) = L^{-1}\left\{ \frac{\sum_{i=1}^{n} s^{n-i} D^{i-1-n+v} y(0)}{s^v - s^{-v}} \right\}$$

is the solution of the above initial value problem.

To illustrate this approach, consider the following example.

Example 1:

Consider the initial value problem which consists of the fractional order $0 < D^{\frac{4}{3}} < 1$ one-dimensional linear Volterra integro-differential equation with constant coefficients:

$$D^{\frac{1}{2}} y(x) = 10 + \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{\frac{1}{2}-1} y(t)dt$$

together with the initial condition

$$D^{-(1-\frac{1}{2})} y(0) = 0.$$

Then, taking the Laplace transform to both sides of the above fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$L\left\{D^{\frac{1}{2}}y(x)\right\} = L\{10\} + L\left\{\frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{\frac{1}{2}}y(t)dt\right\}.$$

Therefore

$$\begin{aligned} s^{\frac{1}{2}}Y(s) - D^{-(1-\frac{1}{2})}y(0) &= \frac{10}{s} + s^{-\frac{1}{2}}Y(s) \\ Y(s) &= \frac{10}{s^{\frac{1}{2}}(s-1)} \end{aligned}$$

and hence

$$y(x) = 10t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)$$

is the solution of the above initial value problem.

Example 2:

Consider the initial value problem which consists of the fractional order $2 < D^{\frac{5}{2}} < 3$ one-dimensional linear Volterra integro-differential equation with constant coefficients:

$$D^{\frac{5}{2}}y(x) = x^{\frac{3}{4}} + \frac{1}{\Gamma(\frac{5}{2})}\int_0^x (x-t)^{\frac{5}{2}-1}t^{\mu}dt$$

together with the initial condition

$$D^{-(3-v)}y(0) - D^{-(2-v)}y(0) - D^{-(1-v)}y(0) = 0.$$

Then, taking the Laplace transform to both sides of the above fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$\begin{aligned} &\left\{s^{\frac{5}{2}}Y(s) - s^2D^{-(3-\frac{5}{2})}y(0) - sD^{-(2-\frac{5}{2})}y(0) - D^{-(1-\frac{5}{2})}y(0)\right\} \\ &\qquad = \frac{(\frac{3}{4})!}{s^{\frac{3}{4}+1}} + \frac{1}{\Gamma(\frac{5}{2})}\left\{\frac{(\frac{5}{2}-1)!}{s^{\frac{5}{2}}}.\frac{\mu!}{s^{\mu+1}}\right\}. \end{aligned}$$

Therefore

$$Y(s) = \frac{(\frac{3}{4})!}{s^{\frac{17}{4}}} + \frac{\Gamma(\mu+1)}{s^{\frac{15}{2}}}.$$

Hence

$$y(x) = (\frac{3}{2})!\frac{t^{\frac{17}{4}-1}}{\Gamma(\frac{17}{4})} + \Gamma(\frac{3}{2}+1)\frac{t^{\frac{15}{2}-1}}{\Gamma(\frac{15}{2})}$$

is the solution of the above initial value problem.

Example 3:

Consider the initial value problem which consists of the fractional order $1 < D^{\frac{5}{3}} < 2$ one-dimensional linear Volterra integro-differential equation with constant coefficients:

$$D^{\frac{5}{3}}y(x) = \frac{1}{\Gamma(\frac{5}{3})}\int_0^x (x-t)^{\frac{5}{3}-1}\cosh 2\sqrt{kt}dt$$

together with the initial condition

$$D^{-(2-\frac{5}{3})}y(0) - D^{-(1-\frac{5}{3})}y(0) = 0.$$

Then, taking the Laplace transform to both sides of the above fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$s^{\frac{5}{3}}Y(s) - sD^{-(2-\frac{5}{3})}y(0) - D^{-(1-\frac{5}{3})}y(0) = \frac{1}{\Gamma(\frac{5}{3})}\frac{(\frac{5}{3}-1)!}{s^{\frac{5}{3}}}.e^{k/s} = \frac{1}{s^{\frac{5}{3}}}e^{k/s}.$$

Therefore

$$Y(s) = \left(\frac{t}{k}\right)^{\frac{(\frac{10}{3}-1)}{2}} I_{(\frac{10}{3}-1)}(2\sqrt{kt})$$

is the solution of the above initial value problem.

Example 4:

Consider the initial value problem which consists of the fractional order $3 < D^{\frac{7}{2}} < 4$ one-dimensional linear Volterra integro-differential equation with constant coefficients:

$$D^{\frac{7}{2}}y(x) = \frac{1}{\Gamma(\frac{7}{2})}\int_0^x (x-t)^{\frac{7}{2}-1}.\frac{3\sqrt{\pi}}{4(t)^{\frac{3}{2}}}\left(1 + 6\sqrt{\frac{t}{\pi}} + 4t - 12.\frac{4(t)^{\frac{3}{2}}}{3\sqrt{\pi}}\right)dt$$

together with the initial condition

$$D^{-(4-\frac{7}{2})}y(0) - D^{-(3-\frac{7}{2})}y(0) - D^{-(2-\frac{7}{2})}y(0) = 0.$$

Then, taking the Laplace transform to both sides of the above fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$\begin{aligned}
\Rightarrow \quad & s^{\frac{7}{2}}Y(s) - s^3D^{-(4-\frac{7}{2})}y(0) - s^2D^{-(3-\frac{7}{2})}y(0) - sD^{-(2-\frac{7}{2})}y(0) \\
= \quad & s^{\frac{-7}{2}}(s^2\sqrt{s})\left[\frac{1}{s} + \frac{3}{s\sqrt{s}} - \frac{4}{s^2} - \frac{12}{s^2\sqrt{s}}\right] \\
= \quad & s^{\frac{-7}{2}}\left[\frac{1}{s\sqrt{s} + 3s - 4\sqrt{s} - 12}\right] \\
= \quad & s^{\frac{-7}{2}}\left[\frac{5}{(s-4)(\sqrt{s}+3)}\right].
\end{aligned}$$

Therefore

$$Y(s) = s^{\frac{14}{2}}\left(e^{4t}[3 - 2erf(2\sqrt{t})] - 3e^{9t}erfc(3\sqrt{t})\right)$$

is the solution of the above initial value problem.

3 The Multi-Dimensional Fractional Volterra Integro-Differential equation of $y(x_1, x_2, ..., x_n)$ of order $v_i, i = 1, 2, ..., n$ with Multi-Dimensional Laplace Transforms

Consider Fractional order for multi-dimensional Volterra integro-differential equation with constant coefficient:

$$\frac{\partial^{\sum_{i=1}^{n} v_i} u(x_1, x_2, ..., x_n)}{\prod_{i=1}^{n} \partial x_i^{v_i}} = f(x_1, x_2, ..., x_n) +$$

$$\frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}} ... \int_{c_1}^{x_1} \prod_{i=1}^{n} (x_i - t_i)^{v_i - 1} u(t_1, t_2, ..., t_n) dt_1 dt_2 ... dt_n, x_i \geq 0.$$

for each $i = 1, 2, ..., n$. together with appropriate initial and boundary conditions where $a_{i_1, i_2, ..., i_n}$ is a known constant for each $i_1, i_2, ..., i_n = 0, 1, ..., m$ such that $0 \leq \sum_{i=1}^{n} v_i \leq m$ and $a_{i_1, i_2, ..., i_n} \neq 0$ for each $i_1, i_2, ..., i_n$. such that $0 \leq \sum_{i=1}^{n} v_i \leq m, f$ is a known function of $x_1, x_2, ..., x_n$, and y is the unknown function that must be determined.

Then, taking the multi-dimensional Laplace transforms to both sides of the above multi-dimensional fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$U(s_1, s_2, ..., s_n) = \frac{F(s_1, s_2, ..., s_n) - (-1)^n \prod_{i=1}^{n} \sum_{l_i=1}^{m_i} s_i^{l_i - 1} \frac{\partial^{-\sum_{i=1}^{n}(m_i - l_i + v_i)} u(0,0,...,0)}{\prod_{i=1}^{n} \partial x_i^{-(m_i - l_i + v_i)}}}{\prod_{i=1}^{n} s_i^{v_i} - \prod_{i=1}^{n} s_i^{-v_i}} -$$

$$\frac{\sum_{k=1}^{n-1}(1)^{n-k} \sum_{\substack{i_1, i_2, ..., i_k = 1 \\ i_1, i_2, ..., i_n}}^{n} \prod_{j=1}^{k} s_{ij}^{m_{ij}} \prod_{\substack{i=1 \\ i \neq i_j}}^{n} \sum_{l_1=1}^{m_i} s_i^{l_1 - 1}}{\prod_{i=1}^{n} s_i^{v_i} - \prod_{i=1}^{n} s_i^{-v_i}}.$$

$$L_k\left\{\frac{\partial^{\sum_{\substack{i=1 \\ i \neq i_j}}^{n}(m_i - l_i + v_i)} u(0, 0, ..., 0, x_{i1}, 0, ..., 0, x_{i2}, 0, ..., 0, x_{ik}, 0, ..., 0)}{\prod_{\substack{i=1 \\ i \neq i_j}}^{n}(m_i - l_i + v_i)}\right\}$$

where

$$L_n\left\{\frac{\partial^{\sum_{i=1}^{n} v_i} u(x_1, x_2, ..., x_n)}{\prod_{i=1}^{n} \partial x_i^{v_i}}\right\} = (-1)^n \prod_{i=1}^{n}\sum_{l_i=1}^{m_i} s_i^{l_i - 1} \frac{\partial^{-\sum_{i=1}^{n}(m_i - l_i + v_i)} u(0, 0, ..., 0)}{\prod_{i=1}^{n} \partial x_i^{-(m_i - l_i + v_i)}} -$$

$$\sum_{k=1}^{n-1}(1)^{n-k}\sum_{\substack{i_1,i_2,\ldots,i_k=1\\ i_1,i_2,\ldots,i_n}}^{n}\prod_{j=1}^{k}s_{ij}^{m_{ij}}\prod_{\substack{i=1\\ i\neq i_j}}^{n}\sum_{l_1=1}^{m_i}s_i^{l_1-1}$$

$$L_k\left\{\frac{\partial^{\sum_{\substack{i=1\\ i\neq i_j}}^{n}(m_i-l_i+v_i)}u(0,0,...,0,x_{i1},0,...,0,x_{i2},0,...,0,x_{ik},0,...,0)}{\prod_{\substack{i=1\\ i\neq i_j}}^{n}(m_i-l_i+v_i)}\right\}$$
$$+\prod_{i=1}^{n}s_i^{v_i}L_n u(x_1,x_2,...,x_n)L_n\left\{f(x_1,x_2,...,x_n)\right\}=F(s_1,s_2,...,s_n).$$

Now, taking the inverse multi-dimensional Laplace transform to both sides of the above equation, one can get solution of the above initial and boundary value problem.

Example 5:

Consider the initial and boundary value problem which consists of the fractional order linear three-dimensional Volterra integro-differential equation of the second kind:

$$\begin{aligned}\frac{\partial^{\frac{8}{3}}u(x_1,x_2,x_3)}{\partial x_1^{\frac{1}{3}}\partial x_2^{\frac{2}{3}}\partial x_3^{\frac{5}{3}}} &= z+\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{5}{3})}\\ &\times\int_0^z\int_0^y\int_0^x(x-t_1)^{(\frac{1}{3})-1}(y-t_2)^{(\frac{2}{3})-1}(z-t_3)^{(\frac{5}{3})-1}u(t_1,t_2,t_3)dt_1dt_2dt_3\end{aligned}$$

together with the initial and boundary conditions

$$\begin{aligned}\frac{\partial^{\frac{-7}{3}}u(x_1,0,0)}{\partial x_2^{\frac{-2}{3}}\partial x_3^{\frac{-5}{3}}} &= \frac{\partial^{-2}u(0,x_2,0)}{\partial x_1^{\frac{-1}{3}}\partial x_3^{\frac{-5}{3}}}=\frac{\partial^{-1}u(0,0,x_3)}{\partial x_1^{\frac{-1}{3}}\partial x_2^{\frac{-2}{3}}}\\ &= \frac{\partial^{\frac{-5}{3}}u(x_1,x_2,0)}{\partial x_3^{\frac{-5}{3}}}=\frac{\partial^{\frac{-2}{3}}u(x_1,0,x_3)}{\partial x_2^{\frac{-2}{3}}}=\frac{\partial^{\frac{-1}{3}}u(0,x_2,x_3)}{\partial x_1^{\frac{-1}{3}}}\\ &= \frac{\partial^{\frac{-8}{3}}u(0,0,0)}{\partial x_1^{\frac{1}{3}}\partial x_2^{\frac{2}{3}}\partial x_3^{\frac{5}{3}}}=0.\end{aligned}$$

Then, taking the three-dimensional Laplace transforms to both sides of the above three-dimensional fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$L_3\left\{\frac{\partial^{\frac{8}{3}}u(x_1,x_2,x_3)}{\partial x_1^{\frac{1}{3}}\partial x_2^{\frac{2}{3}}\partial x_3^{\frac{5}{3}}}\right\}=L_3(z)+\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{5}{3})}$$
$$L_3\left\{\int_0^z\int_0^y\int_0^x(x-t_1)^{(\frac{1}{3})-1}(y-t_2)^{(\frac{2}{3})-1}(z-t_3)^{(\frac{5}{3})-1}u(t_1,t_2,t_3)dt_1dt_2dt_3\right\}.$$

Hence

$$s_1^{\frac{1}{3}}s_2^{\frac{2}{3}}s_3^{\frac{5}{3}}L_3\left\{\frac{\partial^{\frac{-1}{3}}u(x_1,x_2,x_3)}{\partial x_1^{\frac{-2}{3}}\partial x_2^{\frac{-1}{3}}\partial x_3^{\frac{2}{3}}}\right\}-s_1^{\frac{1}{3}}L_1\left\{\frac{\partial^{\frac{-7}{3}}u(x_1,0,0)}{\partial x_2^{\frac{-2}{3}}\partial x_3^{\frac{-5}{3}}}\right\}-s_2^{\frac{2}{3}}L_1\left\{\frac{\partial^{-2}u(0,x_2,0)}{\partial x_1^{\frac{-1}{3}}\partial x_3^{\frac{-5}{3}}}\right\}-$$

$$s_3^{\frac{5}{3}} L_1 \left\{ \frac{\partial^{-1} u(0,0,x_3)}{\partial x_1^{\frac{-1}{3}} \partial x_2^{\frac{-2}{3}}} \right\} - s_1^{\frac{1}{3}} s_2^{\frac{2}{3}} L_2 \left\{ \frac{\partial^{\frac{-5}{3}} u(x_1,x_2,0)}{\partial x_3^{\frac{-5}{3}}} \right\} - s_1^{\frac{1}{3}} s_3^{\frac{5}{3}} L_2 \left\{ \frac{\partial^{\frac{-2}{3}} u(x_1,0,x_3)}{\partial x_2^{\frac{-2}{3}}} \right\} -$$

$$s_2^{\frac{2}{3}} s_3^{\frac{5}{3}} L_2 \left\{ \frac{\partial^{\frac{-1}{3}} u(0,x_2,x_3)}{\partial x_1^{\frac{-1}{3}}} \right\} - \left\{ \frac{\partial^{\frac{-8}{3}} u(0,0,0)}{\partial x_1^{\frac{1}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{5}{3}}} \right\}.$$

Therefore

$$U(s_1,s_2,s_3) = \frac{1}{s_1^{\frac{2}{3}} s_2^{\frac{1}{3}} s_3^{\frac{1}{3}}} - \frac{1}{s_1^{\frac{5}{3}} s_2^{\frac{4}{3}} s_3^{\frac{4}{3}}}.$$

Now, taking the inverse three-dimensional Laplace transform to both sides of the above equation, one can get:

$$L_3^{-1}\left\{U(s_1,s_2,s_3)\right\} = L_3^{-1}\left\{ \frac{1}{s_1^{\frac{2}{3}} s_2^{\frac{1}{3}} s_3^{\frac{1}{3}}} - \frac{1}{s_1^{\frac{5}{3}} s_2^{\frac{4}{3}} s_3^{\frac{4}{3}}} \right\}.$$

Hence

$$u(x,y,z) = \left(\frac{x^{\frac{-1}{3}}}{\Gamma(\frac{2}{3})} . \frac{y^{\frac{-2}{3}}}{\Gamma(\frac{1}{3})} . \frac{z^{\frac{-2}{3}}}{\Gamma(\frac{1}{3})} \right) - \left(\frac{x^{\frac{2}{3}}}{\Gamma(\frac{5}{3})} . \frac{y^{\frac{1}{3}}}{\Gamma(\frac{4}{3})} . \frac{z^{\frac{1}{3}}}{\Gamma(\frac{4}{3})} \right)$$

is the solution of the above initial and boundary value problem.

Example 6:

Consider the initial and boundary value problem which consists of the fractional order linear three-dimensional Volterra integro-differential equation of the second kind:

$$\left(\frac{\partial}{\partial y}\right)^{\frac{13}{6}} u(x,y,z) = -\frac{1}{(\frac{2}{3})!(\frac{1}{4})!(\frac{5}{4})!} x^{(\frac{2}{3})} y^{(\frac{1}{4})} z^{(\frac{5}{4})} + \frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}$$
$$\int_0^z \int_0^y \int_0^x (x-t_1)^{(\frac{2}{3})-1} (y-t_2)^{(\frac{1}{4})-1} (z-t_3)^{(\frac{5}{4})-1} e^{at_1} e^{bt_2} e^{ct_3} dt_1 dt_2 dt_3.$$

together with the initial and boundary conditions

$$\frac{\partial^{\frac{-5}{6}} u(x,0,z)}{\partial y^{\frac{-5}{6}}} = \frac{\partial^{\frac{5}{6}} u(x,0,z)}{\partial y^{\frac{5}{6}}} = \frac{\partial^{\frac{13}{6}} u(x,0,z)}{\partial y^{\frac{13}{6}}} = 0.$$

Then, taking the three-dimensional Laplace transforms of both sides of the above three-dimensional fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$L_3\left\{ \left(\frac{\partial}{\partial y}\right)^{\frac{13}{6}} u(x,y,z) \right\} = -\frac{1}{(\frac{2}{3})!(\frac{1}{4})!(\frac{5}{4})!} L_3\left\{ x^{(\frac{2}{3})} y^{(\frac{1}{4})} z^{(\frac{5}{4})} \right\} + \frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}$$
$$L_3\left\{ \int_0^z \int_0^y \int_0^x (x-t_1)^{(\frac{2}{3})-1} (y-t_2)^{(\frac{1}{4})-1} (z-t_3)^{(\frac{5}{4})-1} e^{at_1} e^{bt_2} e^{ct_3} dt_1 dt_2 dt_3 \right\}.$$

Hence

$$s_2^{\frac{13}{6}} U(s_1,s_2,s_3) - L_3\left\{ \frac{\partial^{\frac{-5}{6}} u(x,0,z)}{\partial y^{\frac{-5}{6}}} \right\} - s_2 L_3\left\{ \frac{\partial^{\frac{5}{6}} u(x,0,z)}{\partial y^{\frac{5}{6}}} \right\} - s_2^2 L_3\left\{ \frac{\partial^{\frac{13}{6}} u(x,0,z)}{\partial y^{\frac{13}{6}}} \right\} =$$
$$-\frac{1}{(\frac{2}{3})!(\frac{1}{4})!(\frac{5}{4})!} \left\{ \frac{\frac{2}{3}!}{s^{\frac{5}{3}}} \frac{\frac{1}{4}!}{s^{\frac{5}{3}}} \frac{\frac{5}{4}!}{s^{\frac{9}{4}}} \right\} + \frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} \frac{1}{s_1^{\frac{2}{3}} s_2^{\frac{1}{4}} s_3^{\frac{5}{4}}} . \frac{1}{(s_1-a)(s_2-b)(s_3-c)}.$$

Therefore

$$U(s_1,s_2,s_3)=-\frac{1}{s_1^{\frac{5}{3}}s_2^{\frac{41}{12}}s_3^{\frac{9}{4}}}+\frac{1}{s_1^{\frac{2}{3}}(s_1-a)s_2^{\frac{29}{12}}(s_2-b)s_3^{\frac{5}{4}}(s_3-c)}.$$

Now, taking the inverse three-dimensional Laplace transform to both sides of the above equation, one can get:

$$L_3^{-1}\left\{U(s_1,s_2,s_3)\right\}=L_3^{-1}\left\{-\frac{1}{s_1^{\frac{5}{3}}s_2^{\frac{41}{12}}s_3^{\frac{9}{4}}}+\frac{1}{s_1^{\frac{2}{3}}(s_1-a)s_2^{\frac{29}{12}}(s_2-b)s_3^{\frac{5}{4}}(s_3-c)}\right\}.$$

Hence

$$u(x,y,z)=\frac{x^{\frac{-1}{3}}}{\Gamma(\frac{2}{3})}.\frac{y^{\frac{29}{12}}}{\Gamma(\frac{41}{12})}.\frac{z^{\frac{5}{4}}}{\Gamma(\frac{9}{4})}+x^{\frac{2}{3}}E_{1,(\frac{5}{3})}(ax).y^{\frac{29}{12}}E_{1,(\frac{41}{12})}(by).z^{\frac{5}{4}}E_{1,(\frac{9}{4})}(cz).$$

is the solution of the above initial and boundary value problem.

Example 7:

Consider the initial and boundary value problem which consists of the fractional order $v=\frac{3}{2}, 1<v\leq 2$, linear three-dimensional Volterra integro-differential equation of the second kind:

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\frac{3}{2}}u(x,y) &= \frac{-1}{\frac{3}{2}!}\left(\frac{1}{x^{\frac{3}{2}}y}\right)-\frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}\int_0^y\int_0^x (x-t_1)^{(\frac{3}{4})-1}(y-t_2)^{(\frac{3}{4})-1}e^{a^2t_1}\\ &\quad\times erfc(a\sqrt{t_1})e^{\frac{-k^2}{4t_1}}\frac{k}{2\sqrt{\pi t_1^3}}\left(\frac{1}{\sqrt{\pi t_2}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t_2}\int_0^{a\sqrt{t_2}}e^{\tau^2}d\tau\right)dt_1dt_2\end{aligned}$$

together with the initial and boundary conditions

$$\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}}u(0,y)-\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}}u(0,y)=0.$$

Then, taking the three-dimensional Laplace transforms of both sides of the above three-dimensional fractional order Volterra integro-differential equation and using the linearity and the convolution properties, one can obtain:

$$\begin{aligned}&L_2\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{3}{2}}u(x,y)\right\}=\frac{-2}{3}!L_2\left\{x^{\frac{-3}{2}}y\right\}-\frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}\\ &L_2\left\{\int_0^y\int_0^x(x-t_1)^{(\frac{3}{4})-1}(y-t_2)^{(\frac{3}{4})-1}e^{a^2t_1}\right.\\ &\qquad\left.\times\, erfc(a\sqrt{t_1})e^{\frac{-k^2}{4t_1}}\frac{k}{2\sqrt{\pi t_1^3}}.\left(\frac{1}{\sqrt{\pi t_2}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t_2}\int_0^{a\sqrt{t_2}}e^{\tau^2}d\tau\right)dt_1dt_2\right\}.\end{aligned}$$

Hence

$$s^{\frac{3}{2}}{}_1U(s_1,s_2)-s_1L_2\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}}u(0,y)\right\}-L_2\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}}u(0,y)\right\}=\frac{1}{s^{\frac{-1}{2}}{}_1}\frac{e^{-k\sqrt{s_1}}}{(\sqrt{s_1}(\sqrt{s_1}+a))}\left(\frac{\sqrt{s_2}}{s_2+a^2}\right).$$

Therefore

$$U(s_1,s_2)=\frac{1}{s_1}\frac{e^{-k}}{(\sqrt{s_1}+a)}\left(\frac{\sqrt{s_2}}{s_2+a^2}\right).$$

Now, taking the inverse three-dimensional Laplace transform to both sides of the above equation, one can get:

$$L_2^{-1}\left\{U(s_1,s_2)\right\}=L_2^{-1}\left\{\frac{1}{s_1}\frac{e^{-k}}{(\sqrt{s_1}+a)}\left(\frac{\sqrt{s_2}}{s_2+a^2}\right)\right\}.$$

Hence

$$u(x,y)=\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\tau^2}d\tau-e^{a^2t}e^{ak}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+erfc\left(\frac{k}{2\sqrt{t}}\right).$$

4 Conclusion

In this work, we offered the one-dimensional Laplace transforms method for solving fractional Volterra integro-differential equations with initial conditions and offered the multi-dimensional Laplace Transforms method for solving fractional multi-dimensional Volterra integro-differential equations with the initial and boundary value problems. It is shown that, the method is power and active tool to find the solution of the initial and boundary value problems. The 1.D.L.T.M and M.D.L.T.M have been successfully and effectively applied to find solution of the initial and boundary value problems.

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