Novel orthogonal functions for solving differential equations of arbitrary order

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Abstract

Fractional calculus and the fractional differential equations have appeared in many physical and engineering processes. Therefore, an efficient and suitable method to solve them is very important. In this paper, novel numerical methods are introduced based on the fractional order of the Chebyshev orthogonal functions (FCF) with Tau and collocation methods to solve differential equations of the arbitrary (integer or fractional) order. The FCFs are obtained from the classical Chebyshev polynomials of the first kind. Also, the operational matrices of the fractional derivative and the product for the FCFs have been constructed. To show the efficiency and capability of these methods we have solved some well-known problems: the momentum, the Bagley-Torvik, and the Lane-Emden differential equations, then have compared our results with the famous methods in other papers.

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1 Introduction

In this section, summary of fractional calculus history and some basic definitions and theorems which are useful for our method have been introduced.

1.1 Summary of fractional calculus history

The fractional calculus is one of the oldest titles of classical calculus which we know today. The original ideas of fractional calculus can be traced back to the end of the 17th century, when the classical differential and integral calculus theories were created and developed by Newton and Leibniz in 1695 [1], but for many reasons were not used in sciences for many years, for example, the various definitions of the fractional derivative [2] and have no exact geometrical interpretation [3]. A review of some definitions and applications of fractional derivatives is given in [4] and [5]. In recent years, many physicists and mathematicians have undertaken studies on this subject. It was found that various applications can be modeled with the help of the fractional derivatives [6, 7]. For example, the nonlinear oscillation of earthquake [8], the fractional optimal control problems for dynamic systems [9, 10, 11, 12], and the fluid-dynamic models with fractional derivatives can eliminate the deficiency arising from the assumption of continuous traffic flow [13, 14, 15]. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations, the initial and boundary value problems, and dynamic systems containing fractional derivatives, such as Adomian's decomposition

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method [16, 17], fractional-order Legendre functions [18], fractional-order Chebyshev functions of the second kind [19], Homotopy analysis method [20], Bessel functions and spectral methods [21], Legendre and Bernstein polynomials [22], finite element methods [23], Legendre collocation [24], modified spline collocation [25], multiquadratic radial basis functions [26], and other methods [27, 28, 29, 30, 31, 32, 33].

1.2 Basical definitions

In this section, some basic definitions and theorems which are useful for our method have been introduced.

Definition 1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ, $\mu \in \Re$, and it is in the space C_μ^n if and only if $f^n \in C_\mu$, $n \in N$.

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense by the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as [34]

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D^m f(s) ds, \qquad \alpha > 0,$$

for $m-1 < \alpha \leq m$, $m \in N$, $t > 0$ and $f \in C_{-1}^m$. Some properties of the operator D^α are as follows: For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, $\gamma \geq -1, N_0 = \{0, 1, 2, ...\}$ and constant C:

$$(i)\ D^\alpha C = 0,$$

$$(ii)\ D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \tag{1.1}$$

$$(iii)\ D^\alpha t^\gamma = \begin{cases} 0 & \gamma \in N_0 \ and\ \gamma < \alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & Otherwise. \end{cases} \tag{1.2}$$

$$(iv)\ D^\alpha (\sum_{i=1}^n c_i f_i(t)) = \sum_{i=1}^n c_i D^\alpha f_i(t), \qquad where\ c_i \in R. \tag{1.3}$$

Definition 3. Suppose that $f(t), g(t) \in C(0, 1]$ and $w(t)$ is a weight function then we define

$$\begin{aligned} \| f(t) \|_w^2 &= \int_0^1 f^2(t) w(t) dt, \\ \langle f(t), g(t) \rangle_w &= \int_0^1 f(t) g(t) w(t) dt, \end{aligned}$$

Theorem 1. (*Generalized Taylor's formula*) Suppose that $f(t) \in C[0, 1]$ and $D^{k\alpha} f(t) \in C[0, 1]$, where $k = 0, 1, ..., m+1$, $0 < \alpha \leq 1$. Then we have

$$f(t) = \sum_{i=0}^m \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} D^{(m+1)\alpha} f(\xi), \tag{1.4}$$

with $0 < \xi \leq t$, $\forall t \in [0, 1]$. And thus

$$|f(t) - \sum_{i=0}^m \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+)| \leq M_\alpha \frac{t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}, \tag{1.5}$$

where $M_\alpha \geq |D^{(m+1)\alpha} f(\xi)|$.
Proof: See Ref. [35].

In case of $\alpha = 1$, the generalized Taylor's formula (1.4) reduces to the classical Taylor's formula.

Theorem 2. Suppose that $\{P_i(t)\}$ be a sequence of orthogonal polynomials, $w(t)$ is weight function for $\{P_i(t)\}$, and $q(t)$ is a polynomial of degree at most $n-1$, then for $p_n(t) \in \{P_i(t)\}$ we have: $\langle p_n(t), q(t)\rangle_w = 0$.
Proof: See the section 2.3 in Ref. [36].

1.3 The Chebyshev functions

The Chebyshev polynomials have frequently been used in numerical analysis including polynomial approximation, Gauss-quadrature integration, integral and differential equations and spectral methods. Chebyshev polynomials have many properties, for example orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many researchers have employed these polynomials in their research [37, 38, 39, 40, 41].

Using some transformations, some researchers extended Chebyshev polynomials to semi-infinite or infinite domain, for example by using $x = \frac{t-L}{t+L}, L > 0$ the rational Chebyshev functions on semi-infinite domain [42, 43, 44, 45, 46, 47, 48], by using $x = \frac{t}{\sqrt{t^2+L}}, L > 0$ the rational Chebyshev functions on infinite domain [49] are introduced.

In proposed work, by transformation $x = 1 - 2t^\alpha$, $\alpha > 0$ on the Chebyshev polynomials, the fractional order of the Chebyshev orthogonal functions in interval $[0, 1]$ have been introduced, that can use to solve differential equations of arbitrary order.

The aim of the paper is to present new numerical methods (Spectral methods using the FCFs) for approximating the solution of the differential equations of arbitrary (integer or fractional) order as follows:

$$\sum_{j=1}^{N_1} \lambda_j D^{\gamma_j} y(t) + \sum_{r=1}^{N_2} h_r(t)[y(t)]^{q_r} = f(t), \tag{1.6}$$

with these supplementary conditions:

$$y^{(i)}(t_0) = y_0, \quad i = 0, 1, ..., s-1, \quad s.t. \quad s-1 < max\{\gamma_j\} \leq s, \ s \in N, \tag{1.7}$$

where $0 \leq t < 1$, $\gamma_j > 0$ and $h_r, f \in L^2([0, 1))$ are known functions, $y(t)$ is the unknown function, D^{γ_j} are the Caputo fractional differentiation operator, λ_j are real numbers and N_1, N_2, q_r are positive integers.

The organization of the paper is expressed as follows: in section 2, the FCFs and their properties are obtained. Section 3 is devoted to applying the FCFs operational matrices of the fractional derivative and the product to solve differential equations of arbitrary order. In Section 4, the work methods are explained. Applications of the proposed methods are shown in section 5. Finally, a conclusion is provided.

2 The fractional order of the Chebyshev functions

In this section, first, the fractional order of the Chebyshev functions (FCFs) have been defined, and then some properties and convergence of them for our methods have been introduced.

2.1 The FCFs definition

The efficient methods have been used by many researchers to solve the differential equations (DE) is based on series expansion of the form $\sum_{i=0}^{n} c_i t^i$, such as Adomian's decomposition method [50] and Homotopy perturbation method [51]. But solution of many DEs and FDEs can't be estimated by polynomial basis, therefore we have decided to define a new basis for Spectral methods to solve them as follows:

$$\Phi_n(t) = \sum_{i=0}^{n} c_i t^{i\alpha}, \quad \alpha > 0.$$

Now by transformation $z = 1 - 2t^\alpha$, $\alpha > 0$ on classical Chebyshev polynomials of the first kind, the FCFs are defined in interval $[0,1]$, that be denoted by $FT_n^\alpha(t) = T_n(1-2t^\alpha)$.

By this definition the singular Sturm-Liouville differential equation of classical Chebyshev polynomials becomes:

$$\frac{\sqrt{1-t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt}\left[\frac{\sqrt{1-t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt}\ FT_n^\alpha(t)\right] + n^2\alpha^2\ FT_n^\alpha(t) = 0, \tag{2.1}$$

where $t \in [0,1]$ and the FCFs are the eigenfunctions of the Eq. (2.1).

The $FT_n^\alpha(t)$ can be obtained using the recursive relation as follows:

$$\begin{aligned}
&FT_0^\alpha(t) = 1 \quad , \quad FT_1^\alpha(t) = 1 - 2t^\alpha, \\
&FT_{n+1}^\alpha(t) = (2 - 4t^\alpha)\ FT_n^\alpha(t) -\ FT_{n-1}^\alpha(t), \qquad n = 1, 2, \cdots .
\end{aligned}$$

Fig. 1 shows graphs of the FCFs for various values of n and α.
The analytical form of $FT_n^\alpha(t)$ of degree $n\alpha$ given by

$$\begin{aligned}
FT_n^\alpha(t) &= \sum_{k=0}^{n} (-1)^k \frac{n 2^{2k}(n+k-1)!}{(n-k)!(2k)!} t^{\alpha k} \\
&= \sum_{k=0}^{n} \beta_{n,k}.t^{\alpha k}, \qquad t \in [0,1],
\end{aligned} \tag{2.2}$$

where

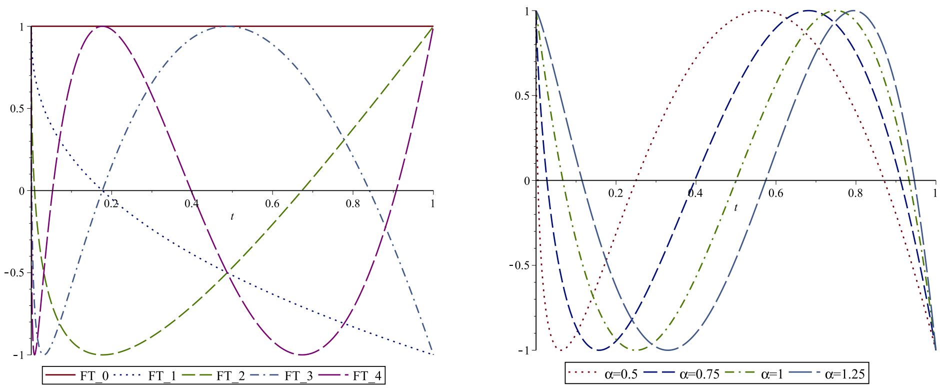
$$\beta_{n,k} = (-1)^k \frac{n 2^{2k}(n+k-1)!}{(n-k)!(2k)!} \qquad \textit{and} \qquad \beta_{0,k} = 1.$$

Note that $FT_n^\alpha(0) = 1$ and $FT_n^\alpha(1) = (-1)^n$.

The weight function for the FCFs is $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{1-t^\alpha}}$, and the FCFs with this weight function are orthogonal in interval $[0,1]$ that satisfy in following relation:

$$\int_0^1 FT_n^\alpha(t)\ FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}, \tag{2.3}$$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$. The Eq. (2.3) is provable using the property of orthogonality in the Chebyshev polynomials.



(a) Graph of the FCFs with $\alpha = 0.40$ and various values of n

(b) Graph of the FCFs with $n = 4$ and various values of α

FIGURE 1. Graphs of the FCFs for various values of n and α.

2.2 Approximation of functions

Any function $y(t)$, $t \in [0, 1]$, can be expanded as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n \; FT_n^{\alpha}(t),$$

where the coefficients a_n obtain by inner product:

$$\langle y(t), FT_n^{\alpha}(t)\rangle_w = \langle \sum_{n=0}^{\infty} a_n \; FT_n^{\alpha}(t), FT_n^{\alpha}(t)\rangle_w$$

and using the property of orthogonality in the FCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^1 \; FT_n^{\alpha}(t) y(t) w(t) dt, \qquad n = 0, 1, 2, \cdots .$$

In the numerical methods, we have to use first $(m+1)$-terms FCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m} a_n \; FT_n^{\alpha}(t) = A^T \Phi(t), \tag{2.4}$$

with

$$\begin{aligned} A &= [a_0, a_1, \cdots, a_m]^T, & (2.5)\\ \Phi(t) &= [FT_0^{\alpha}(t), \; FT_1^{\alpha}(t), \cdots, \; FT_m^{\alpha}(t)]^T. & (2.6) \end{aligned}$$

2.3 Convergence of method

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 3. Suppose that $D^{k\alpha}f(t) \in C[0,1]$ for $k = 0, 1, ..., m+1$, and E_m^α is the subspace is generated by $\{FT_0^\alpha(t),\ FT_1^\alpha(t), ...,\ FT_m^\alpha(t)\}$. If $f_m = A^T\Phi$ is the best approximation to f from E_m^α, then the error bound is presented as follows

$$\| f(t) - f_m(t) \|_w \leq \frac{M_\alpha}{2^{(m+1)}\Gamma((m+1)\alpha+1)}\sqrt{\frac{\pi}{\alpha.(m+1)!}},$$

where $M_\alpha \geq |D^{(m+1)\alpha}f(t)|, \quad t \in [0,1]$.

Proof. By theorem 1, we have $y = \sum_{i=0}^{m} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha}f(0^+)$ and

$$|f(t) - y(t)| \leq M_\alpha \frac{t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}.$$

Since the best approximation to f from E_m^α is $f_m(t) = A^T\Phi(t)$, and $y(t) \in E_m^\alpha$, thus

$$\begin{aligned}
\| f(t) - f_m(t) \|_w^2 &\leq \| f(t) - y(t) \|_w^2 \\
&\leq \frac{M_\alpha^2}{\Gamma((m+1)\alpha+1)^2}\int_0^1 \frac{t^{\frac{\alpha}{2}+2(m+1)\alpha-1}}{\sqrt{1-t^\alpha}}dt \\
&= \frac{M_\alpha^2}{\Gamma((m+1)\alpha+1)^2}\frac{\pi}{2^{2(m+1)}.\alpha.(m+1)!}.*
\end{aligned}$$

Theorem 4. The fractional order of the Chebyshev functions $FT_n^\alpha(t)$, has precisely n real zeros on interval $(0,1)$ in the form

$$t_k = \left(\frac{1-\cos(\frac{(2k-1)\pi}{2n})}{2}\right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, ..., n$$

Moreover, $\frac{d}{dt}FT_n^\alpha(t)$ has precisely $n-1$ real zeros on interval $(0,1)$ in the following points:

$$t'_k = \left(\frac{1-\cos(\frac{k\pi}{n})}{2}\right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, ..., n-1.$$

Proof. The Chebyshev polynomial $T_n(x)$ has n real zeros [52]:

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \qquad k = 1, 2, ..., n.$$

Therefore $T_n(x)$ can be written as

$$T_n(x) = (x - x_1)(x - x_2)...(x - x_n).$$

Using transformation $x = 1 - 2t^{\alpha}$ yields to

$$FT_n^{\alpha}(t) = ((1-2t^{\alpha}) - x_1)((1-2t^{\alpha}) - x_2)...((1-2t^{\alpha}) - x_n),$$

so, the real zeros of $FT_n^{\alpha}(t)$ are $t_k = (\frac{1-x_k}{2})^{\frac{1}{\alpha}}$.

Also, we know that, the real zeros of $\frac{d}{dt}T_n(t)$ occurs in the following points [52]:

$$x'_k = \cos(\frac{k\pi}{n}), \qquad k = 1, 2, ..., n-1.$$

Same as in the previous, the real zeros of $\frac{d}{dt}FT_n^{\alpha}(t)$ are $t'_k = (\frac{1-x'_k}{2})^{\frac{1}{\alpha}}$. $*$

3 Operational matrices of the FCFs

In this section, the operational matrices the fractional derivative and the product for the FCFs are constructed, these matrices can be used to solve the linear and nonlinear differential equations of arbitrary order.

3.1 The fractional derivative operational matrix of the FCFs

In the next theorem, the operational matrix of the Caputo fractional derivative of order $\alpha > 0$ for FCFs is generalized, which can be expressed by:

$$D^{\alpha}\Phi(t) = D^{(\alpha)}\Phi(t). \tag{3.1}$$

Theorem 5. Let $\Phi(t)$ be FCFs vector in Eq. (2.6), and $D^{(\alpha)}$ be an $(m+1)\times(m+1)$ operational matrix of the Caputo fractional derivatives of order $\alpha > 0$, then for $i, j = 0, 1, ..., m$:

$$D_{i,j}^{(\alpha)} = \begin{cases} \frac{2}{\sqrt{\pi}c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k}\beta_{j,s}\frac{\Gamma(\alpha k+1)\Gamma(s+k-\frac{1}{2})}{\Gamma(\alpha k-\alpha+1)\Gamma(s+k)}, & i > j \\ 0 & otherwise \end{cases} \tag{3.2}$$

Proof. Using Eq. (3.1)

$$\begin{bmatrix} D_{0,0} & \cdots & D_{0,j} & \cdots & D_{0,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{i,0} & \cdots & D_{i,j} & \cdots & D_{i,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{m,0} & \cdots & D_{m,j} & \cdots & D_{m,m} \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_m \end{bmatrix} = \begin{bmatrix} D^{\alpha}\Phi_0 \\ \vdots \\ D^{\alpha}\Phi_i \\ \vdots \\ D^{\alpha}\Phi_m \end{bmatrix}.$$

By orthogonality property of the FCFs, and the Eqs. (1.2) and (2.2), for $i, j = 0, 1, ..., m-1$:

$$D_{i,j}^{(\alpha)} = \frac{2\alpha}{\pi c_j}\int_0^1 D^{\alpha}FT_i^{\alpha}(t)FT_j^{\alpha}(t)w(t)dt.$$

Since $D^{\alpha}FT_0^{\alpha}(t) = 0$, therefore $D_{0,j}^{(\alpha)} = \int_0^1 D^{\alpha}FT_0^{\alpha}(t)FT_j^{\alpha}(t)w(t)dt = 0$. And if $i \leq j$ then $deg(D^{\alpha}(FT_i^{\alpha}(t))) < deg(FT_j^{\alpha}(t))$, therefore by theorem 2, $D_{i,j}^{(\alpha)} = 0$ for any $i \leq j$. Now for $i > j$ we have:

$$
\begin{aligned}
D_{i,j}^{(\alpha)} &= \frac{2\alpha}{\pi c_j}\int_0^1 \sum_{k=1}^{i}\beta_{i,k}\frac{\Gamma(\alpha k+1)t^{\alpha k-\alpha}}{\Gamma(\alpha k-\alpha+1)}\sum_{s=0}^{j}\beta_{j,s}t^{\alpha s}\frac{t^{\frac{\alpha}{2}-1}}{\sqrt{1-t^{\alpha}}}dt \\
&= \frac{2\alpha}{\pi c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k}\beta_{j,s}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-\alpha+1)}\int_0^1 \frac{t^{\alpha(k+s-\frac{1}{2})-1}}{\sqrt{1-t^{\alpha}}}dt.
\end{aligned}
$$

Now, by integration of the above equation, the theorem can be proved. $*$

Remark 1: The fractional derivative operational matrix of the FCFs for $\alpha = 1$ is same as the derivative operational matrix of the shifted Chebyshev polynomials [57].

3.2 The product operational matrix of the FCFs

The following property of the product of two FCFs vectors will also be applied.

$$\Phi(t)\Phi(t)^T A \approx \widehat{A}\Phi(t), \tag{3.3}$$

where $\widehat{A}$ is an $(m+1)\times(m+1)$ product operational matrix for the vector $A = \{a_i\}_{i=0}^{m}$.

Theorem 6. Let $\Phi(t)$ be FCFs vector in Eq. (2.6) and A be a vector, then the elements of $\widehat{A}$ are obtained as

$$\widehat{A}_{ij} = \sum_{k=0}^{m} a_k \widehat{g_{ijk}}, \tag{3.4}$$

where

$$
\widehat{g}_{ijk} = \begin{cases} \frac{c_k}{2c_j} & i \neq 0 \ and \ j \neq 0 \ and \ (k = i+j \ or \ k = |i-j|) \\ \frac{c_k}{c_j} & (j = 0 \ and \ k = i) \ or \ (i = 0 \ and \ k = j) \\ 0 & otherwise \end{cases}
$$

Proof. Using Eq. (3.3)

$$
\begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_i \\ \vdots \\ \Phi_m \end{bmatrix}
[\Phi_0 \cdots \Phi_k \cdots \Phi_m]
\begin{bmatrix} a_0 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix}
\approx
\begin{bmatrix} \widehat{A}_{0,0} & \cdots & \widehat{A}_{0,m} \\ \vdots & \cdots & \vdots \\ \widehat{A}_{i,0} & \cdots & \widehat{A}_{i,m} \\ \vdots & \cdots & \vdots \\ \widehat{A}_{m,0} & \cdots & \widehat{A}_{m,m} \end{bmatrix}
\begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_m \end{bmatrix}
$$

By the orthogonality property Eq. (2.3) the elements $\{\widehat{A}_{ij}\}_{i,j=0}^{m}$ can be calculated from

$$\widehat{A}_{ij} = \frac{2\alpha}{\pi c_j}\sum_{k=0}^{m-1} a_k g_{ijk}, \tag{3.5}$$

where g_{ijk} is given by

$$g_{ijk} = \int_0^1 FT_i^{\alpha}(t)FT_j^{\alpha}(t)FT_k^{\alpha}(t)w(t)dt.$$

We use the following property:

$$FT_i^{\alpha}(t)FT_j^{\alpha}(t) = \frac{1}{2}(FT_{i+j}^{\alpha}(t) + FT_{|i-j|}^{\alpha}(t)), \tag{3.6}$$

and by substituting of the Eq. (3.6) in g_{ijk}:

$$g_{ijk} = \begin{cases} \frac{\pi c_k}{4\alpha} & i \neq 0 \ and \ j \neq 0 \ and \ (k = i + j \ or \ k = |i - j|) \\ \frac{\pi c_k}{2\alpha} & (j = 0 \ and \ k = i) \ or \ (i = 0 \ and \ k = j) \\ 0 & otherwise \end{cases}$$

now by using of Eq. (3.5), the theorem can be proved. $*$

Remark 2: The product operational matrix of the FCFs is the same as the product operational matrix of the shifted Chebyshev polynomials [57]. In whole, it can be said that the components of $\widehat{A}$ are independent of values of α.

4 Application of the Methods

In this section, the FCFs Tau and collocation methods to solve the differential equations of arbitrary order are applied.

4.1 Tau method

In this section, the FCFs Tau method to solve the differential equations of arbitrary order in the Eq. (1.6) with supplementary conditions in the Eq. (1.7) is applied.

First, unknown functions $y(t)$, $D^{\alpha}y(t)$ and known functions $f(t)$ are expanded as follows

$$y(t) \approx \sum_{n=0}^{m} a_n FT_n^{\alpha}(t) = A^T \Phi(t), \tag{4.1}$$

$$D^{\alpha}y(t) \approx \sum_{n=0}^{m} a_n D^{\alpha} FT_n^{\alpha}(t) = A^T D^{(\alpha)} \Phi(t), \tag{4.2}$$

$$f(t) \approx \sum_{n=0}^{m} f_n FT_n^{\alpha}(t) = F^T \Phi(t). \tag{4.3}$$

And it is easy to show by induction that:

$$[y(t)]^{q_r} \approx A^T (\widehat{A})^{q_r - 1} \Phi(t), \qquad for \ q_r = 1, 2, \cdots .$$

Also

$$[y(t)]^{q_r} h_r(t) \approx A^T (\widehat{A})^{q_r - 1} \Phi(t) \Phi^T(t) B_r = A^T (\widehat{A})^{q_r - 1} \widehat{B_r} \Phi(t), \tag{4.4}$$

where $h_r(t) \approx B_r^T \Phi(t)$ and $\widehat{A}$, $\widehat{B_r}$ are the product operational matrices of vectors A and B_r, respectively.

For any $\gamma_j, \ \ j = 1, ..., N_1$, we can choose the value of α such that, $\gamma_j, \ \ j = 1, ..., N_1$ be multiples of α, and by using the properties of the operator D^{α}, we can calculate the values $D^{(\gamma_j)}$:

$$D^{\gamma_j} y(t) \approx \sum_{n=0}^{m} a_n D^{\gamma_j} FT_n^{\alpha}(t) = A^T D^{(\gamma_j)} \Phi(t). \tag{4.5}$$

Now, by substituting the approximations above into Eq. (1.6) we obtain:

$$\sum_{j=1}^{N_1} \lambda_j A^T D^{(\gamma_j)} \Phi(t) + \sum_{r=1}^{N_2} A^T (\widehat{A})^{q_r - 1} \widehat{B_r} \Phi(t) = F^T \Phi(t). \tag{4.6}$$

And, by multiplying two sides of Eq. (4.6) in $\Phi^T(t)$ and then integration in the interval $[0, 1]$, according to the orthogonality of the FCFs we get (Tau method):

$$\sum_{j=1}^{N_1} \lambda_j A^T D^{(\gamma_j)} + \sum_{r=1}^{N_2} A^T (\widehat{A})^{q_r - 1} \widehat{B_r} = F^T. \tag{4.7}$$

which is a linear or nonlinear system of algebraic equations. By solving this system and using the initial conditions in the Eq. (1.7), we can obtain the approximate solution of Eq. (1.6) according to Eq. (4.1).

The residual error function has been defined as follows:

$$Res(t) = \left(\sum_{j=1}^{N_1} \lambda_j A^T D^{(\gamma_j)} + \sum_{r=1}^{N_2} A^T (\widehat{A})^{q_r - 1} \widehat{B_r} - F^T \right) \Phi(t). \tag{4.8}$$

4.2 Collocation method

In this section, the FCFs collocation method to solve the differential equations of arbitrary order in the Eq. (1.6) with supplementary conditions (1.7) is applied.

To apply the collocation method, we construct the residual function by substituting $y_m(t)$ in Eq. (2.4) for $y(t)$ in the Eq. (1.6):

$$Res(t) = \sum_{j=1}^{N_1} \lambda_j A^T D^{(\gamma_j)} \Phi(t) + \sum_{r=1}^{N_2} h_r(t) [y_m(t)]^{q_r} - f(t), \tag{4.9}$$

The equations for obtaining the coefficient $\{a_i\}_{i=0}^m$ in the Eq. (2.5) arise from equalizing $Res(t)$ to zero on $(m+1)$ collocation points:

$$Res(t_i) = 0, \qquad i = 0, 1, ..., m. \tag{4.10}$$

In this study, the roots of the FCFs in the interval $[0, 1]$ (Theorem 4), as collocation points are used. By solving the obtained set of equations and using supplementary conditions, we have the approximating function $y_m(t)$.

5 Illustrative examples

In this section, by using the present methods, some well-known linear and non-linear examples of arbitrary order differential equations are solved. To show the efficiency and capability of the Spectral methods based on the FCFs, the obtained results with the corresponding analytical or numerical solutions are compared.

Example 1. The first example is an inhomogeneous linear equation

$$D^{\alpha}y(t)+y(t)=\frac{3\sqrt{\pi}}{4}t^{\frac{3}{2}-\alpha}\left(\frac{5t}{2\Gamma(\frac{7}{2}-\alpha)}+\frac{1}{\Gamma(\frac{5}{2}-\alpha)}\right)+t^{\frac{5}{2}}+t^{\frac{3}{2}},$$

$$y(0)=0,\quad 0<\alpha\leq 1.$$

The exact solution of this problem is $y(t)=t^{\frac{5}{2}}+t^{\frac{3}{2}}$.

Tau method: By applying the technique described in the last section, $m+1$ of algebraic equations can be generated as

$$\begin{aligned} A^T[D^{(\alpha)}+I] &= F^T \\ A^T\Phi(0) &= 0. \end{aligned}$$

Collocation method: By applying the technique described in the last section, $m+1$ of algebraic equations can be generated as

$$\begin{aligned} Res(t_i)=A^TD^{(\alpha)}\varphi(t_i)+y_m(t_i)-(\frac{3\sqrt{\pi}t_i^{\frac{3}{2}-\alpha}}{4}(\frac{5t_i}{2\Gamma(\frac{7}{2}-\alpha)}+\frac{1}{\Gamma(\frac{5}{2}-\alpha)})+t_i^{\frac{5}{2}}+t_i^{\frac{3}{2}}) &= 0 \\ A^T\Phi(0) &= 0. \end{aligned}$$

for $i=0,1,...,m-1$.

By solving the above equation using both methods, we obtained the same results, with $m=5$ and $\alpha=0.50$, we can obtain the exact solution with

$$A^T=[0.55859375,\ -0.87890625,\ 0.421875,\ -0.119140625,\ 0.01953125,\ -0.001953125].$$

Also for $m=10$ and $\alpha=0.25$, we can obtain the exact solution with

$$\begin{aligned} A^T = {} & [0.4017829895019531 25, -0.7070770263671875, 0.4819679260253906 25, -0.255279541015625, \\ & 0.1061553955078125, -0.035430908203125, 0.0097293853759765625, -0.00217437744140625, \\ & 0.362396240234375e-3, -0.3814697265625e-4, 0.19073486328125e-5]. \end{aligned}$$

Example 2. Next, we consider an inhomogeneous multi-term fractional differential equation (the Bagley-Torvik equation)[53]

$$ED^{\gamma}y(t)+BD^{\beta}y(t)+Cy(t)=f(t),\qquad y(0)=y_0,\ y'(0)=y_1. \tag{5.1}$$

where $E\neq 0$ and $B,C\in R$.

Tau method: By applying the technique described in the last section, the equations are obtained as

$$\begin{aligned} A^T\left[ED^{(\gamma)}+BD^{(\beta)}+CI\right] &= F^T, \\ y(0)\approx A^T\Phi(0) &= y_0, \\ y'(0)\approx A^TD^{(1)}\Phi(0) &= y_1. \end{aligned}$$

Now, we must choose the value of α such that, γ and β be multiples of α. Using the definition 3, we can calculate the values $D^{(\gamma)}$ and $D^{(\beta)}$.

Collocation method: By applying the technique described in the last section, $m+1$ of algebraic equations can be generated as

$$\begin{aligned} Res(t_i) = E.A^T D^{(\gamma)}\varphi(t_i) + B.A^T D^{(\beta)}\varphi(t_i) + C.y_m(t_i) - f(t_i) &= 0, \\ A^T\Phi(0) &= y_0, \\ A^T D^{(1)}\Phi(0) &= y_1. \end{aligned}$$

for $i = 0, 1, ..., m-2$.

For $\gamma = 2,\ \beta = \frac{3}{2}$, $E = B = C = 1$ and $f(t) = 1 + t^{\eta},\ 1 \leq \eta < \frac{3}{2}$, Eq. (5.1) has been reduced to the inhomogeneous Bagley-Torvik equation originally proposed in [18, 54, 55]. This equation arises in the modeling of a rigid plate immersed in a Newtonian fluid.

$$D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = 1 + t^{\eta}, \qquad y(0) = 1,\ y'(0) = \lfloor 1 - \eta \rfloor + 1.$$

The exact solution of this problem is $y(t) = 1 + t^{\eta}$.

By solving the above equation using both methods, we obtained the same results. For instance:

Case 1: if $\eta = 1$

Now, we must choose the value of α such that 2, $\frac{3}{2}$ and 1 be multiples of α. i.e. $\alpha = 0.5$ or $\alpha = 0.25$ or $\alpha = \frac{1}{6}$.

For $\alpha = 0.5$ and $m = 2$ we have $D^{(2)} = (D^{(0.5)})^4$ and $D^{(\frac{3}{2})} = (D^{(0.5)})^3$, and the unknown coefficients a_i can be obtained as $A^T = [1.375,\ -0.5,\ 0.125]$ which is obtained the exact solution of this problem. Also for $\alpha = 0.25$ and $m = 4$ we obtain the exact solution of this problem with $A^T = [1.2734375,\ -0.4375,\ 0.218750,\ -0.06250,\ 0.0078125]$, and for $\alpha = \frac{1}{6}$ and $m = 6$ we obtain the exact solution of this problem with

$$A^T = [1.2255859375,\ -0.38671875,\ 0.24169921875,\ -0.107421875,\ 0.0322265625,\ -0.005859375,\ 0.00048828125].$$

Case 2: if $\eta = \frac{5}{4}$

Now, we must choose the value of α such that 2, $\frac{3}{2}$ and $\frac{5}{4}$ be multiples of α. i.e. $\alpha = 0.25$ or $\alpha = \frac{1}{8}$.

For $\alpha = 0.25$ and $m = 5$ we obtain the exact solution of this problem with

$$A^T = [1.24609375,\ -0.41015625,\ 0.234375,\ -0.087890625,\ 0.01953125,\ -0.001953125],$$

and for $\alpha = \frac{1}{8}$ and $m = 10$ we obtain the exact solution of this problem with

$$\begin{aligned} A^T \quad = \quad & [1.176197052001953125,\ -0.3203582763671875,\ 0.240268707275390625,\ -0.147857666015625, \\ & 0.0739288330078125,\ -0.029571533203125,\ 0.0092411041259765625,\ -0.00217437744140625, \\ & 0.362396240234375e-3,\ -0.3814697265625e-4,\ 0.19073486328125e-5]. \end{aligned}$$

Example 3. Consider the following linear initial value problem [18, 56]

$$D^{\alpha} y(t) - y(t) = 1, \qquad y(0) = 0, \quad 0 < \alpha \leq 1. \tag{5.2}$$

The exact solution of this problem is $y(t) = \sum_{k=1}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}$.

Tau method: We can generate a set of linear algebraic equations as follows:

$$\begin{aligned} A^T(D^{(\alpha)} - I) &= F^T, \\ A^T\Phi(0) &= 0, \end{aligned}$$

where $F^T = [1, 0, ..., 0]$.

Collocation method: We have

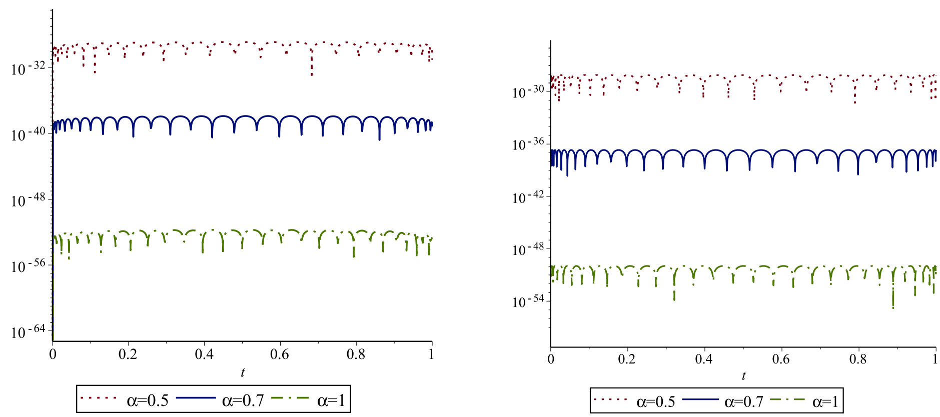
$$\begin{aligned} Res(t_i) = A^T D^{(\alpha)}\varphi(t_i) - y_m(t_i) - 1 &= 0, \\ A^T\Phi(0) &= 0, \end{aligned}$$

for $i = 0, 1, ..., m-1$.

The approximate solution of this problem is achieved. The absolute errors and the residual errors with $m = 30$ and various values of α, by using the Tau and collocation methods are displayed in Figs. 1.2(a), 1.2(b) and 1.3(a), 1.3(b), respectively.

Tables 1 and 2 show the absolute and residual errors for various values of α and $m = 30$, by using the Tau and collocation methods, respectively.

We can see that in this example, the Tau method is accurate.

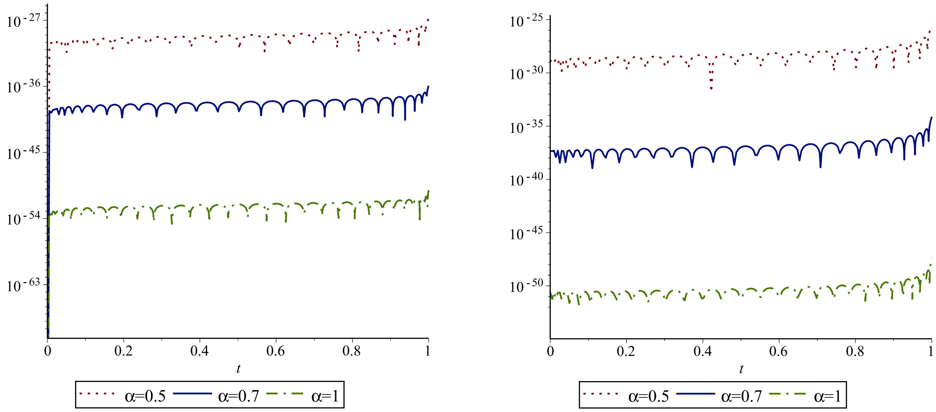


(a) Graph of the absolute errors (b) Graph of the residual errors

FIGURE 2. Graphs of Log absolute errors and Log residual errors for example 3 with $m = 30$ and the various values of α, using the Tau method.

Example 4. Consider the nonlinear Abel differential equation of arbitrary order of the first kind with $0 < \alpha \leq 1$ [21]:

$$D^{\alpha}y(t) = sin(t)y^3(t) - ty^2(t) + t^2y(t) - t^3, \qquad 0 < t \leq 1, \tag{5.3}$$



(a) Graph of the absolute errors (b) Graph of the residual errors

FIGURE 3. Graphs of Log absolute errors and Log residual errors for example 3 with $m = 30$ and various values of α, using the collocation method.

TABLE 1. The absolute and residual errors by the present method for example 3, using the Tau method (with various values of α and $m = 30$)

t	$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 1$	
	Abs. error	Res(t)	Abs. error	Res(t)	Abs. error	Res(t)
0.1	7.811e-30	2.360e-29	1.744e-40	1.839e-37	3.577e-53	9.704e-51
0.2	6.761e-30	8.102e-29	8.412e-39	4.405e-38	9.172e-53	9.704e-51
0.3	4.892e-30	7.804e-29	6.906e-39	1.053e-37	2.220e-52	1.054e-50
0.4	8.253e-30	1.525e-29	1.165e-38	4.823e-39	2.015e-52	1.048e-50
0.5	1.047e-29	7.862e-29	1.203e-38	1.575e-37	5.048e-53	1.080e-50
0.6	8.900e-30	1.317e-29	1.973e-39	2.014e-37	2.087e-52	1.048e-50
0.7	9.543e-30	7.989e-29	7.991e-39	9.716e-38	5.560e-52	1.054e-50
0.8	4.016e-30	4.338e-29	9.854e-39	3.242e-38	9.764e-52	9.704e-51
0.9	4.754e-30	3.351e-29	8.300e-40	1.844e-37	1.700e-51	9.704e-51
1.0	8.917e-32	8.106e-29	7.094e-40	2.086e-37	2.699e-51	1.080e-50

with initial condition:

$$y(0) = 0. \tag{5.4}$$

Tau method: The problem can be converted to

$$\begin{aligned} A^T D^{(\alpha)} - A^T \widehat{A}^2 \widehat{C} + A^T \widehat{A}\widehat{E} - A^T \widehat{F} &= -G^T \\ A^T \Phi(0) &= 0. \end{aligned}$$

where $\widehat{A}$ is obtained from Eq. (3.4) and $sin(t) \approx C^T\Phi(t)$, $t \approx E^T\Phi(t)$, $t^2 \approx F^T\Phi(t)$ and $t^3 \approx G^T\Phi(t)$.

Collocation method: We have for $i = 0, 1, ..., m-1$:

TABLE 2. The absolute and residual errors by the present method for example 3 by using the collocation method (with various values of α and $m = 30$)

t	$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 1$	
	Abs. error	Res.	Abs. error	Res.	Abs. error	Res.
0.1	2.068e-31	2.015e-29	1.866e-39	5.562e-38	2.222e-52	1.342e-51
0.2	3.464e-30	4.359e-30	1.123e-39	7.188e-38	4.810e-52	2.960e-51
0.3	4.893e-30	6.906e-30	3.641e-39	7.977e-38	6.974e-52	7.198e-52
0.4	6.833e-30	4.300e-29	3.145e-39	1.034e-37	8.655e-52	1.820e-52
0.5	1.863e-30	3.751e-29	1.394e-39	1.038e-37	1.008e-51	4.678e-84
0.6	1.244e-29	5.754e-29	8.395e-39	2.410e-38	1.127e-51	2.732e-52
0.7	7.762e-31	8.523e-29	1.327e-39	1.056e-37	1.224e-51	1.681e-51
0.8	1.991e-29	1.625e-29	1.832e-38	1.991e-37	1.214e-51	1.186e-50
0.9	3.297e-29	1.068e-29	1.601e-38	3.607e-37	9.312e-52	1.215e-50
1.0	1.840e-27	2.740e-26	1.242e-36	7.645e-35	6.150e-51	4.139e-48

$$
\begin{aligned}
Res(t_i) = A^T D^{(\alpha)} \varphi(t_i) - sin(t_i) y_m^3(t_i) + t_i y_m^2(t_i) - t_i^2 y_m(t_i) + t_i^3 &= 0, \\
A^T \Phi(0) &= 0,
\end{aligned}
$$

Fig. 4 shows the graphs of residual errors of the Eq. (5.3) with $m = 20$ for various values of α, using the Tau and collocation methods, respectively. We can see that in this example, the collocation method is more accurate.

Fig. 5 shows the graphs of approximate solutions of the Eq. (5.3) with $m = 20$ for various values of α.

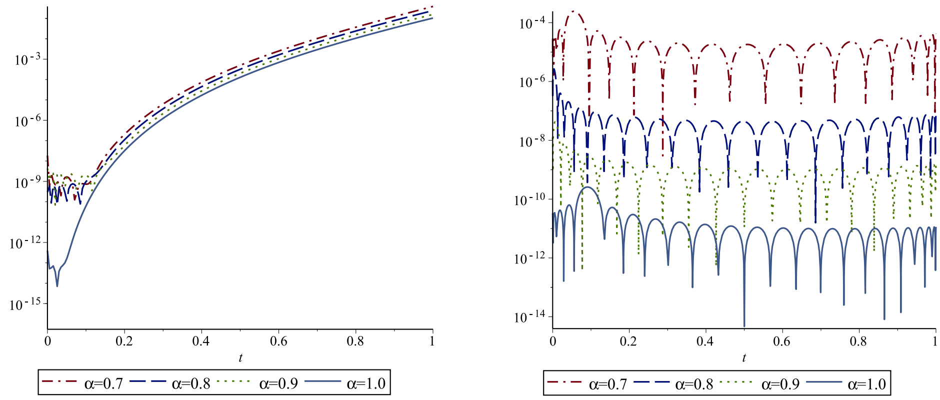
A class of Abel FDEs has been solved by Xu and He [58], and Parand and Nikarya [21].

Table 3 shows the approximate solutions and residual errors with $m = 20$ for various values of α and t.

Table 4 shows the comparison between the approximate solutions using Parand and Nikarya [21], and the present collocation method.

TABLE 3. Obtained values of $y(t)$ and the residual errors by the present method for example 4, using the collocation method (with $m = 20$ and various values of α)

t	$\alpha = 1.0$		$\alpha = 0.9$		$\alpha = 0.8$		$\alpha = 0.7$	
	$y_m(t)$	Res(t)	$y_m(t)$	Res(t)	$y_m(t)$	Res(t)	$y_m(t)$	Res(t)
0.1	-2.500e-5	2.36e-10	-3.655e-5	2.79e-09	-5.330e-5	6.71e-08	-5.881e-5	2.16e-05
0.2	-4.004e-4	2.56e-11	-5.465e-4	1.66e-09	-7.443e-4	4.39e-08	-9.985e-4	1.47e-05
0.3	-2.032e-3	7.40e-13	-2.666e-3	7.72e-10	-3.492e-3	2.64e-08	-4.558e-3	1.00e-05
0.4	-6.459e-3	1.36e-11	-8.248e-3	1.06e-09	-1.052e-2	3.00e-08	-1.340e-2	1.59e-05
0.5	-1.591e-2	3.28e-50	-1.992e-2	1.03e-11	-2.495e-2	4.39e-08	-3.125e-2	1.74e-05
0.6	-3.346e-2	1.03e-11	-4.132e-2	9.61e-10	-5.108e-2	2.06e-08	-6.329e-2	1.86e-05
0.7	-6.327e-2	3.98e-13	-7.740e-2	6.98e-10	-9.499e-2	2.73e-08	-1.171e-1	1.96e-05
0.8	-1.111e-1	7.80e-12	-1.353e-1	9.78e-10	-1.659e-1	4.53e-08	-2.056e-1	1.48e-05
0.9	-1.855e-1	6.36e-12	-2.263e-1	8.10e-10	-2.796e-1	6.13e-08	-3.535e-1	1.97e-05
1.0	-2.999e-1	1.13e-11	-3.701e-1	1.49e-09	-4.693e-1	7.88e-08	-6.297e-1	4.21e-05



(a) Graph of the residual errors using the Tau methods

(b) Graph of the residual errors using the collocation methods

FIGURE 4. Graphs of Log residual errors for example 4 with $m = 20$ and various values of α, using the Tau and collocation methods.

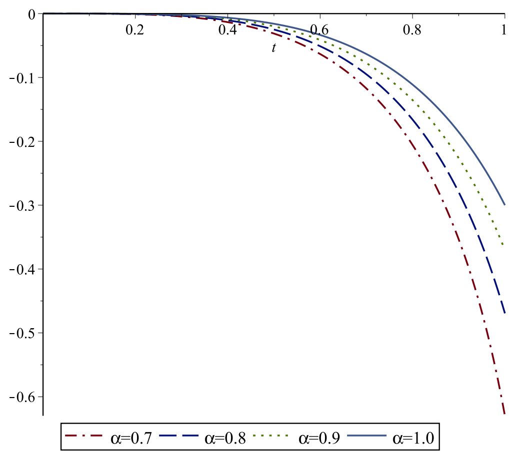


FIGURE 5. Graphs of the approximation solutions for example 4 with $m = 20$ and various values of α.

Example 5. Consider the Bessel differential equation [59, 60]

$$ty''(t) + y'(t) + ty(t) = 0, \tag{5.5}$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{5.6}$$

TABLE 4. Obtained values of $y(t)$ by Parand [21] and the present collocation method for example 4 with $\alpha = 0.8,\ 0.9$, and $m = 20$.

t	$\alpha = 0.9$			$\alpha = 0.8$		
	Parand	$y_m(t)$	Res(t)	Parand	$y_m(t)$	Res(t)
0.1	-0.0000365	-0.0000365563	2.79e-09	-0.000053344	-0.0000533072	6.71e-08
0.2	-0.0005465	-0.0005465187	1.66e-09	-0.000744347	-0.0007443232	4.39e-08
0.3	-0.0026670	-0.0026667458	7.72e-10	-0.003492701	-0.0034926797	2.64e-08
0.4	-0.0082482	-0.0082471655	1.06e-09	-0.010522097	-0.0105220716	3.00e-08
0.5	-0.0199300	-0.0199158605	1.03e-11	-0.024954200	-0.0249541768	4.39e-08
0.6	-0.0413235	-0.0412367140	9.61e-10	-0.051085888	-0.0510858674	2.06e-08
0.7	-0.0774071	-0.0769938176	6.98e-10	-0.094999267	-0.0949992321	2.73e-08
0.8	-0.1353508	-0.1336610538	9.78e-10	-0.165925912	-0.1659259489	4.53e-08
0.9	-0.2263126	-0.2201541109	8.10e-10	-0.279684833	-0.2796838014	6.13e-08
1.0	-0.3701646	-0.3490293830	1.49e-09	-0.469341539	-0.4693420981	7.88e-08

A solution known as the Bessel function of the first kind of order zero denoted by $J_0(t)$ is

$$J_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} (\frac{t}{2})^{2k}.$$

Tau method: The problem can be converted to

$$\begin{aligned} A^T[D^{(2)}\widehat{B} + D^{(1)} + \widehat{B}] &= 0, \\ A^T\Phi(0) &= 1, \\ A^T D^{(1)}\Phi(0) &= 0, \end{aligned}$$

where $t \approx B^T\Phi(t)$ and $\widehat{B}$ is obtained from Eq. (3.4) for vector B.

Collocation method: For satisfying the initial conditions, we satisfy conditions (5.6) by multiplying the operator the Eq, (2.4) by t^2 and adding it to 1, therefore, we have $\widehat{y_m}(t) = 1+t^2 y_m(t)$ and for $i = 0, 1, ..., m$:

$$Res(t_i) = t_i.\widehat{y_m}''(t_i) + \widehat{y_m}'(t_i) + t_i.\widehat{y_m}(t_i) \quad = \quad 0. \tag{5.7}$$

Fig. 6 shows the absolute errors of approximate solution with the exact solution and residual errors for $\alpha = 1$ and $m = 15$, using the Tau and collocation methods, respectively

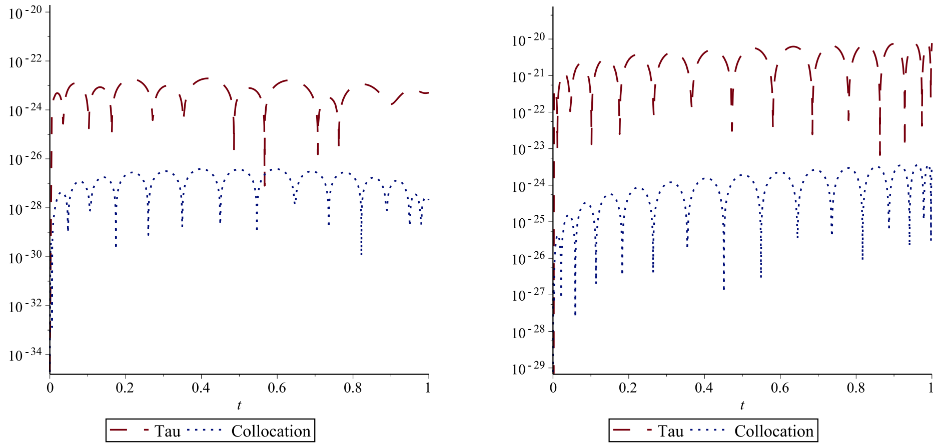
To show the convergence of the Tau method to solve this example with $\alpha = 1$, in the Fig. 7, we showed that, the absolute error decreases with increasing m.

In Table 5, a comparison is made between the approximate values using Razzaghi [59], Yousefi [60], and the Tau method together with the solution of $J_0(t)$.

It is noted that the maximum error for this problem, obtained in Razzaghi [59] is 10^{-6} and Yousefi [60] is 10^{-14}; but in the Tau method, for $m = 15$ the maximum error is 10^{-23}.

Example 6. The Lane-Emden differential equations are very important in astrophysics, for this reason, writing articles to review them [61, 62, 63, 64, 65]. Now, we consider the linear Lane-Emden equation of fractional order as follows

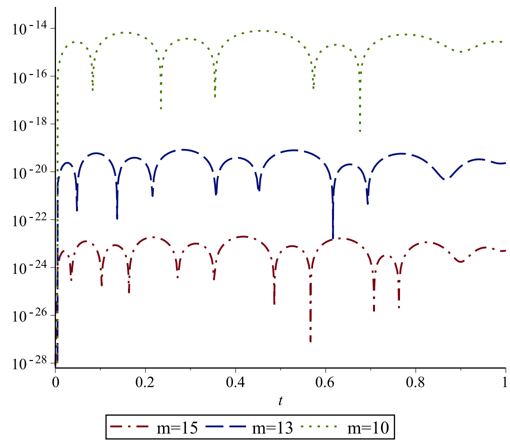
$$D^{\gamma}y(t) + \frac{k}{t^{\gamma-\beta}} D^{\beta}y(t) + \frac{1}{t^{\gamma-2}} y(t) = f(t), \tag{5.8}$$



(a) Graph of the absolute errors (b) Graph of the residual errors

FIGURE 6. Graphs of the absolute errors for example 5 with $m = 15$ and $\alpha = 1$, by using the Tau and collocation methods.

FIGURE 7. The absolute errors for example 5 with $\alpha = 1$ and $m = 10,\ 13,\ 15$ to show the convergence of the Tau method.



with the initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \tag{5.9}$$

where

$$f(t) = t^{2-\gamma}\left(6t\left(\frac{t^2}{6} + \frac{\Gamma(4-\beta) + k\Gamma(4-\gamma)}{\Gamma(4-\beta)\Gamma(4-\gamma)}\right) - 2\left(\frac{t^2}{2} + \frac{\Gamma(3-\beta) + k\Gamma(3-\gamma)}{\Gamma(3-\beta)\Gamma(3-\gamma)}\right)\right),$$

TABLE 5. Obtained values of $y(t)$ by Razzaghi [59], Yousefi [60], the Tau method, and exact solution for example 5 with $m = 15$

t	Razzaghi	Yousefi	Tau method	Exact solution	Abs. Er.
0.1	0.997502	0.9975015620660412	0.99750156206604003228128	0.99750156206604003228128	1.3e-24
0.2	0.990024	0.9900249722395755	0.99002497223957639081752	0.99002497223957639081750	1.6e-23
0.3	0.977625	0.9776262465382969	0.97762624653829608756973	0.97762624653829608756974	8.9e-24
0.4	0.960396	0.9603982266595635	0.96039822665956345034418	0.96039822665956345034416	1.7e-23
0.5	0.938468	0.9384698072408127	0.93846980724081290422840	0.93846980724081290422840	4.4e-24
0.6	0.912004	0.9120048634972114	0.91200486349721077595490	0.91200486349721077595489	1.1e-23
0.7	0.881200	0.8812008886074042	0.88120088860740528083880	0.88120088860740528083880	1.7e-24
0.8	0.846285	0.8462873527504811	0.84628735275048026608921	0.84628735275048026608920	8.8e-24
0.9	0.807524	0.8075237981225438	0.80752379812254477730241	0.80752379812254477730240	1.7e-24
1.0	0.765197	0.7651976865579627	0.76519768655796655144972	0.76519768655796655144971	5.0e-24

and $\gamma = \frac{3}{2},\ \beta = \frac{1}{2},\ k = 2$. The exact solution of Eqs. (5.8) and (5.9) is given as $y(t) = t^3 - t^2$ in [61, 63].

Tau method: The problem can be converted to

$$\begin{aligned} A^T D^{(\frac{3}{2})}\widehat{C} + 2A^T D^{(\frac{1}{2})} + A^T \widehat{E} &= F^T \\ A^T \Phi(0) &= 0, \\ A^T D^{(1)} \Phi(0) &= 0, \end{aligned}$$

where $t \approx C^T \Phi(t),\ t^{\frac{3}{2}} \approx E^T \Phi(t)$ and $t.f(t) \approx F^T \Phi(t)$ and $\widehat{C},\ \widehat{E}$ are obtained from the Eq. (3.4).

Collocation method: We have

$$\begin{aligned} Res(t_i) = t_i A^T D^{(\frac{3}{2})}\varphi(t_i) + 2A^T D^{(\frac{1}{2})}\varphi(t_i) + t_i^{\frac{3}{2}} y_m(t_i) - t_i f(t_i) &= 0, \\ A^T \Phi(0) &= 0, \\ A^T D^{(1)} \Phi(0) &= 0, \end{aligned}$$

for $i = 0, 1, ..., m-2$.

By solving the above equation using both methods, we obtained the same results, with $m = 6$ and $\alpha = 0.50$, we can obtain the exact solution with

$$A^T = [\ -0.0478515625,\ 0.05078125,\ 0.02294921875,\ -0.044921875,\ 0.0244140625,\ -0.005859375,\ 0.00048828125].$$

The residual errors with $m = 6$ are displayed in Fig. 8.

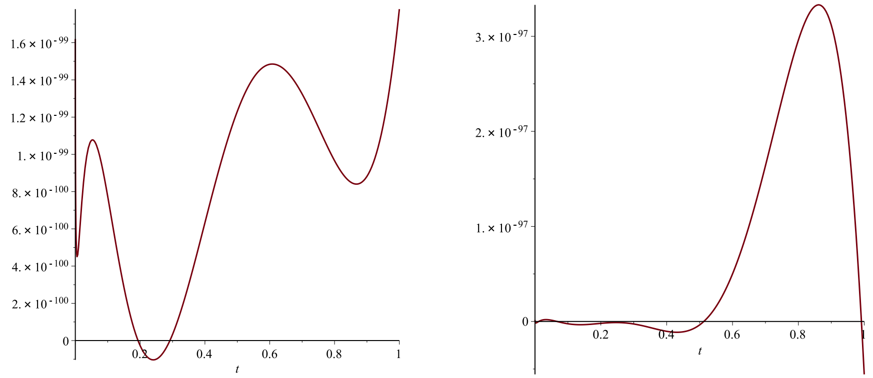
Table 6 shows the comparison of the absolute error obtained by the present Tau method, the reproducing kernel method (RKM) [61] and the collocation method in Ref. [63].

Example 7. Now, we consider the fourth-order, nonlinear, momentum differential equation as follows [66, 67, 68]

$$y''''(t) - S\left(ty'''(t) + 3y''(t) - 2y(t)y''(t)\right) - M^2 y''(t) = 0, \tag{5.10}$$

with the initial conditions

$$\begin{aligned} y(0) &= a, & y'(0) &= 0, \\ y(1) &= \frac{1}{2}, & y'(1) &= 0, \end{aligned} \tag{5.11}$$



(a) Graph of the residual error by using the Tau methods

(b) Graph of the residual error by using the collocation methods

FIGURE 8. Graphs of the residual errors for example 6 with $m = 10$ and $\alpha = 0.50$, by using the Tau and collocation methods

TABLE 6. Comparison of the present Tau method, the reproducing kernel method [61], and the collocation method [63] of the absolute errors for example 6

t	Akgul [61]	Mechee [63]	Tau method	Akgul [61]	Mechee [63]	Tau method
	$m = 5$			$m = 10$		
0.25	8.7370e-4	1.3345e-3	1.3791e-3	8.4636e-6	1.3232e-5	3.029e-95
0.50	9.9000e-4	1.5000e-3	4.0754e-3	2.9000e-6	2.6000e-5	1.318e-94
0.75	7.6702e-4	5.0673e-3	2.7006e-3	8.5754e-6	1.5634e-6	3.220e-94
1.00	5.4736e-4	3.6339e-3	8.6079e-4	5.4345e-6	4.1443e-5	6.162e-94

where S is the squeeze number, M is the Hartman number, a is a constant. $a > 0$ corresponds to suction and $a < 0$ corresponds to injection at the lower stationary disk in momentum problem [67] .

Tau method: The problem can be converted to

$$\begin{aligned} A^T D^{(4)} - S.A^T\left(D^{(3)}\widehat{B} + 3D^{(2)} - 2D^{(2)}\widehat{A}\right) - M^2.A^T D^{(2)} &= 0,\\ A^T\Phi(0) &= a,\\ A^T D^{(1)}\Phi(0) &= 0,\\ A^T\Phi(1) &= \frac{1}{2},\\ A^T D^{(1)}\Phi(1) &= 0, \end{aligned}$$

where $t \approx B^T\Phi(t)$ and $\widehat{B}$, $\widehat{A}$ are obtained from the Eq. (3.4).

Collocation method: For satisfying the initial conditions, we have $\widehat{y_m}(t) = a + 0.5(3-6a)t^2 + (-1+2a)t^3 + t^2(t-1)^2.y_m(t)$ and for $i = 0, 1, ..., m$:

$$Res(t_i) = \widehat{y_m}''''(t_i) - S\left(t_i\widehat{y_m}'''(t_i) + 3\widehat{y_m}''(t_i) - 2y(t_i)\widehat{y_m}''(t_i)\right) - M^2\widehat{y_m}''(t_i) = 0,$$

Tables 7 and 8 show the comparison of the obtained values by the present Tau method, the present collocation method, and the variational iteration method (VIM) [66] with $S = 0.1,\ M = 0.2$, and $a = 0.1$.

Fig. 9 shows the effect of the Hartman number M on $y'(t)$ when there is suction or injection into squeeze film while $S = 0.01$.

TABLE 7. Obtained values of $y'(t)$ by Khan [66], Tau method ($\alpha = 1, m = 20$) and collocation method ($\alpha = 0.5, m = 20$), example 7 with $S = 0.1, M = 0.2, a = 0.1$.

t	Khan [66]	Tau method	Collocation method
0.2	0.384801	0.38480128029055731	0.38480128029055732
0.4	0.575554	0.57555414305433113	0.57555414305433115
0.6	0.575174	0.57517467556439543	0.57517467556439545
0.8	0.384040	0.38404043290258845	0.38404043290258847
Extremum point $y'(t)$		0.49958498632719809	0.49958498632719819

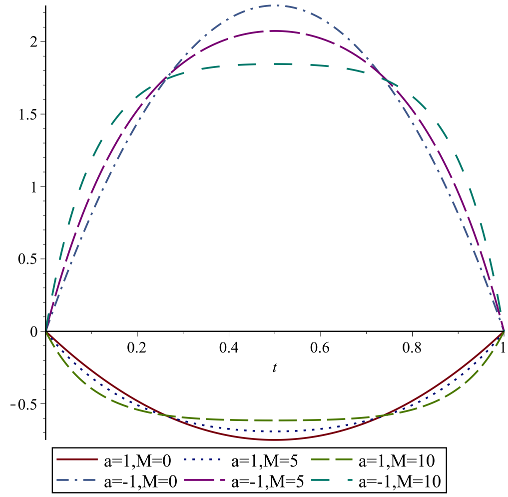
TABLE 8. Obtained values of $y(t)$ by the Tau method ($\alpha = 1, m = 20$) and the collocation method ($\alpha = 0.5, m = 20$), example 7 with $S = 0.1, M = 0.2, a = 0.1$.

t	Tau method	Res. Err.	Collocation method	Res. Err.
0.1	0.111259419974244944	8.759e-18	0.111259419974244944	4.700e-13
0.2	0.141752942532303441	1.887e-16	0.141752942532303441	5.193e-13
0.3	0.186604279882068437	1.217e-16	0.186604279882068437	2.817e-13
0.4	0.240990261007359037	5.088e-17	0.240990261007359037	4.628e-13
0.5	0.300123812543702849	8.016e-19	0.300123812543702848	4.590e-13
0.6	0.359237998438074101	5.230e-17	0.359237998438074101	4.143e-13
0.7	0.413570592630439735	1.224e-16	0.413570592630439734	4.105e-13
0.8	0.458348679966845107	1.878e-16	0.458348679966845107	5.395e-13
0.9	0.488772783363147929	9.669e-18	0.488772783363147929	6.654e-13

6 Conclusion

In this paper, the fractional order of the Chebyshev functions (FCFs) of the first kind have been introduced. Then the operational matrices of the fractional derivative and the product of these orthogonal functions have been obtained. Since the solution of many differential equations and fractional differential equations can't be estimated by polynomial basis, therefore we have decided to use new basis namely the FCF and their operational matrices for Spectral methods. As shown,

FIGURE 9. Influence of the Hartman number M on $y'(t)$ with suction and injection when $S = 0.01$ for example 7.



the methods are convergent, and the accuracy and stability them are good, and the error decreases with increasing m. Illustrative examples show that these methods have good results, and these could be due to the choice of fractional basis.

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