On the solutions of partial integrodifferential equations of fractional order

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Abstract

The main purpose of this paper is to study the existence of solutions for the nonlinear fractional partial integrodifferential equations with Dirichlet boundary condition. Under suitable assumption the results are established by using the Leray-Schauder fixed point theorem and Arzela-Ascoli theorem. An example is provided to illustrate the main result.

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1 Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. Now fractional calculus is undergoing rapid developments with more applications in the real world. Numerous applications of fractional calculus can be found in fluid dynamics, stochastic dynamical systems, plasma physics, nonlinear control theory, image processing, nonlinear biological systems and quantum mechanics, For more details on history and applications of fractional calculus see [22], [27] and [13] references therein.

Fractional derivatives provide more accurate models of real world problems than integer order derivatives. They also give an excellent instrument for the description of memory and properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer order models. The solvability of different types of fractional differential equations have been established by Lakshmikantham et al. in [14]. Wang and Xie [31] established the existence and uniqueness of solution for fractional differential equations involving Riemann-Liouville differential operators with integral boundary conditions by employing the monotone iterative method. Agarwal et al. [1] discussed the initial value problem for a class of fractional neutral functional differential equations and obtained the existence criteria from Krasnoselskii's fixed point theorem. Momani and Odibat [20] compared the solutions of the fractional order differential equations by homotopy perturbation method and variational iteration method. Ahmed et al. [2] introduced a new concept of the coupling of nonlocal integral conditions and proved the existence and uniqueness of solutions for a coupled system of fractional differential equations. They also verified the existence results by means of Leray-Schauder alternative and Schaufer's fixed point theorem, while uniqueness result was derived from Banach's contraction principle.

To model the process with delay, it is not sufficient to employ an ordinary or partial differential equation. An approach to resolve this problem is to use integrodifferential equations. Many

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mathematical formulations of physical phenomena lead to integrodifferential equations. There are few articles available in the literature for the study of fractional integrodifferential equations. For example, Balachandran et al. [4, 5] studied the existence results for several kinds of fractional integrodifferential equations in a Banach space using fixed point technique. In [32], Zhang et al. investigated the existence of nonnegative solutions for nonlinear fractional differential equations with nonlocal fractional integrodifferential boundary conditions on an unbounded domain by using the Leray-Schauder nonlinear alternative theorem. The differential transform method was applied to fractional integrodifferential equations in [3] to solve those equations analytically. The solutions of system of fractional partial differential equations has been found by Parthiban and Balachandran [25] by using Adomain decomposition method.

Another interesting area of research is the investigation of fractional partial differential equations. Because of their immense applications in scientific fields, fractional partial differential equations are found to be an effective tool to describe certain physical phenomena, such as diffusion processes [10] and viscoelasticity theories [12]. In recent years, increasing number of papers by many authors from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations. Some partial differential equations of fractional order type like one-dimensional time-fractional diffusion-wave equation were used for modeling relevant physical processes (see [26]). Regarding fractional partial differential equations, Luchko [18] used the Fourier transform method of the variable separation to construct a formal solution and under certain condition he showed that the formal solution is the generalized solution of the initial-boundary value problem. To prove the uniqueness he used the maximum principle for generalized time fractional diffusion equation [17]. By applying the energy inequality, Oussaeif and Bouziani [24] proved the existence and uniqueness of solution for parabolic fractional differential equations in a functional weighted Sobolev space with integral conditions. Joice Nirmala and Balachandran [28] determined the solution of time fractional telegraph equation by means of Adomain decomposition method and analysed the efficiency of this method. Using measure of noncompactness and Monch's fixed point theorem, the existence of solutions is studied by Guo and Zhang [9] for a class of impulsive partial hyperbolic differential equations. In this paper, we extend the results of [23] to fractional order partial integrodifferential equation.

2 Preliminaries

In this section, we introduce some notations and basic facts of fractional calculus. Let $\Omega \subset \mathbb{R}$ and $C(J, \mathbb{R})$ is the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$. Let $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Riemann Liouville derivative and Caputo derivative are defined as follows:

Definition 2.1. [13] The partial Riemann-Liouville fractional integral operator of order $\alpha > 0$ with respect to t of a function $f(x, t)$ is defined by

$$I^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s)\, \mathrm{d}s.$$

Definition 2.2. [13] The partial Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f(x,t)$ with respect to t of the form

$$D^{\alpha} f(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(x,s)}{(t-s)^{\alpha}} \, \mathrm{d}s.$$

Definition 2.3. [13] The Caputo partial fractional derivative of order $\alpha > 0$ with respect to t of a function $f(x,t)$ is defined as

$$^{C}D^{\alpha} f(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{\partial f(x,s)}{\partial s} \, \mathrm{d}s.$$

To know more properties above fractional operators and historical aspects of they refer the books [19] and [28]. For more details on the geometric and physical interpretation for fractional derivatives of Caputo types see [6]. There has been a significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives in the past few years, for instance, see the papers of Gejji and Jafari [8], Furati and Tatar [7]. The Riemann Liouville and Caputo fractional derivatives are linked by the following relationship.

$$^{c}D^{\alpha} f(x,t) = D^{\alpha} f(x,t) - \frac{f(x,0)}{\Gamma(1-\alpha) t^{\alpha}}.$$

About the called Caputo derivative we must remark here that Liouville in [15] and [16] was the first that introduced formally the called fractional Caputo derivative of order $\frac{1}{2}$ with the objective to solve certain integral equation connected with the known Tautochrone problem.

In this paper, we consider the fractional partial integrodifferential equation of the form

$$^{c}D^{\alpha} u(x,t) = a(t) \Delta u(x,t) + f\left(t, u(x,t), \int_0^t g(t,s,u(x,s)) \, \mathrm{ds}\right), \quad t \in J, \tag{2.1}$$

where $0 < \alpha < 1$ and the nonlinear functions $g : J \times J \times \mathbb{R} \to \mathbb{R}$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The initial and boundary conditions are

$$\begin{aligned} u(x,0) &= \varphi(x), & x &\in \Omega, \\ u(x,t) &= 0, & (x,t) &\in \partial\Omega \times J. \end{aligned}$$

where $\varphi(x) \in L^1(\mathbb{R})$. In order to establish our result assume the following conditions.

(H_1) $f(t,u_1,u_2)$ is continuous with respect to u_1, u_2, Lebesgue measurable with respect to t and satisfies

$$\frac{\int_\Omega \Phi(x) f(t,u_1,u_2) \, \mathrm{d}x}{\int_\Omega \Phi(x) \, \mathrm{d}x} \leq f\left(t, \frac{\int_\Omega \Phi(x) u_1(x,t) \, \mathrm{d}x}{\int_\Omega \Phi(x) \, \mathrm{d}x}, \frac{\int_\Omega \Phi(x) u_2(x,t) \, \mathrm{d}x}{\int_\Omega \Phi(x) \, \mathrm{d}x}\right),$$

where $\Phi(x)$ is an eigenfunction.

(H_2) There exists an integrable function $m_1(t) : J \to [0, \infty)$ such that

$$\| f(t, u_1, u_2) \| \leq m_1(t) \sum_{i=1}^{2} \|u_i\|,$$

where $m_1(t) \geq 0$ and $\Big(\int\limits_0^t (m_1(s))^{\frac{1}{\beta}} \ \mathrm{d}s \Big)^\beta \leq l_1$, for some $\beta \in (0, \alpha)$.

(H_3) $g(t, s, u)$ is continuous with respect to u, Lebesgue measurable with respect to t and also satisfies the inequality

$$\frac{\int\limits_\Omega \Phi(x) g(t, s, u) \ \mathrm{d}x}{\int\limits_\Omega \Phi(x) \ \mathrm{d}x} \leq g\left(t, s, \frac{\int\limits_\Omega \Phi(x) u(x, t) \ \mathrm{d}x}{\int\limits_\Omega \Phi(x) \ \mathrm{d}x} \right).$$

(H_4) There exists an integrable function $m_2(t, s) : J \times J \to [0, \infty)$ such that

$$\| g(t, s, u) \| \leq m_2(t, s) \|u\|.$$

(H_5) $a(t)$ is continuous on J and for β as in $(H2)$, $\Big(\int\limits_0^t (a(s))^{\frac{1}{\beta}} \ \mathrm{d}s \Big)^\beta \leq l_2$.

(H_6) There exists an integrable function $m(t, s) = m_1(t) m_2(t, s)$ such that $\Big(\int\limits_0^t (m(s, \tau))^{\frac{1}{\beta}} \ \mathrm{d}s \Big)^\beta \leq l_3$, $0 < \beta < \alpha$.

It is easy to show that the initial value problem (2.1) is equivalent to the following equation

$$u(x, t) = \varphi(x) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha - 1} [a(s) \Delta u(x, s) + f(s, u(x, s), v(x, s))] \ \mathrm{d}s, \tag{2.2}$$

where $v(x, s) = \int\limits_0^s g(s, \tau, u(x, \tau)) \ \mathrm{d}\tau$, for $t \in J$.

3 Existence Results

Consider the following eigenvalue problem

$$\left.\begin{aligned} \Delta u + \lambda u &= 0, \quad (x, t) \in \Omega \times J, \\ u &= 0, \quad (x, t) \in \partial\Omega \times J, \end{aligned}\right\} \tag{3.1}$$

where λ is a constant not depending on the variables x and t. The theory of eigenvalue problems is well known by [30]. Thus, for $x \in \Omega$ the smallest eigenvalue λ_1 of the problem (3.1) is positive and the corresponding eigenfunction $\Phi(x) \geq 0$. Now we define the function $U(t)$ as

$$U(t) = \frac{\int\limits_\Omega u(x, t) \Phi(x) \ \mathrm{dx}}{\int\limits_\Omega \Phi(x) \ \mathrm{dx}}. \tag{3.2}$$

Theorem 3.1. Assume that there exists a $\beta \in (0, \alpha)$ for some $\alpha > 0$ such that (H1)-(H6) holds. For any constant $b > 0$, suppose that

$$r = \min\left\{ T, \left[\frac{\Gamma(\alpha) b}{(\|U(0)\| + b)(\lambda_1 l_1 + l_2 + l_3)} \left(\frac{\alpha - \beta}{1 - \beta} \right)^{1-\beta} \right]^{\frac{1}{\alpha-\beta}} \right\}. \tag{3.3}$$

Then there exists at least one solution for the initial value problem (2.1) on $\Omega \times [0, r]$.

Proof. First we have to prove the initial value problem (2.1) has a solution if and only if the equation

$$U(t) = U(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(s)\ \mathrm{ds} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, U(s), V(s))\ \mathrm{ds}, \tag{3.4}$$

where $V(t) = \int_0^t g(t, s, U(t))$, has a solution.

Step 1. The proof of sufficiency is similar to that of Lemma 3.1 [23]. To prove the necessary part, let $u(x, t)$ be a solution of (2.1). This implies $u(x, t)$ is a solution of (2.2). Now multiplying both sides of equation (2.2) by $\Phi(x)$ and integrating with respect to $x \in \Omega$, we get

$$\begin{aligned} \int_\Omega \Phi(x) u(x,t)\ \mathrm{dx} &= \int_\Omega \Phi(x)\varphi(x)\ \mathrm{dx} + \frac{1}{\Gamma(\alpha)} \int_\Omega \Phi(x) \int_0^t (t-s)^{\alpha-1} a(s) \Delta u(x,s)\ \mathrm{ds\ dx} \\ &+ \frac{1}{\Gamma(\alpha)} \int_\Omega \Phi(x) \int_0^t (t-s)^{\alpha-1} f(s, u(x,s), v(x,s))\ \mathrm{ds\ dx} \end{aligned}$$

Using Green's formula and assumption (H_1), we get

$$U(t) \le U(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) U(s)\ \mathrm{ds} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, U(s), V(s))\ \mathrm{ds}. \tag{3.5}$$

Let $K = \{U : U \in C(J,\ \mathbb{R}),\ \| U(t) - U(0) \| \le b\}$. Define an operator $F : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as

$$FU(t) = U(0) - \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) U(s)\ \mathrm{ds} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, U(s), V(s))\ \mathrm{ds}. \tag{3.6}$$

Clearly $U(0) \in K$. This means that K is nonempty. From our construction of K, we can say that K is closed and bounded. Now for any $U_1, U_2 \in K$ and for any $a_1, a_2 \ge 0$ such that $a_1 + a_2 = 1$,

$$\begin{aligned} \| a_1 U_1 + a_2 U_2 - U(0) \| &\le a_1 \| U_1 - U(0) \| + a_2 \| U_2 - U(0) \| \\ &\le a_1 b + a_2 b = b. \end{aligned}$$

Thus $a_1U_1 + a_2U_2 \in K$. Therefore K is nonempty closed convex set. Next we have to prove the operator F maps K into itself.

$$\| FU(t) - FU(0) \| = \left\| \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) U(s) \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, U(s), V(s)) \, \mathrm{d}s \right\|$$

$$\leq \frac{\lambda_1}{\Gamma(\alpha)} (\|U(0)\| + b) \int_0^t (t-s)^{\alpha-1} \|a(s)\| \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, U(s), V(s))\| \, \mathrm{d}s.$$

Then by using Holder inequality and (H_6), we arrive

$$\begin{aligned}
\| FU(t) - FU(0) \| \quad &\leq \quad \frac{\lambda_1}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^t \left((t-s)^{\alpha-1} \right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \left(\int_0^t \|a(s)\|^{\frac{1}{\beta}} \, \mathrm{d}s \right)^{\beta} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t m_1(s)(t-s)^{\alpha-1} \left(\|U(s)\| + \|V(s)\| \right) \, \mathrm{d}s \\
&\leq \quad \frac{\lambda_1}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^t \left((t-s)^{\alpha-1} \right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \left(\int_0^t \|a(s)\|^{\frac{1}{\beta}} \, \mathrm{d}s \right)^{\beta}
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^t \left((t-s)^{\alpha-1} \right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m_1(s))^{\frac{1}{\beta}} \, \mathrm{d}s \right)^{\beta} \\
&+ \frac{1}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^t \left((t-s)^{\alpha-1} \right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \left(\int_0^t (m(s,\tau))^{\frac{1}{\beta}} \, \mathrm{d}s \right)^{\beta} \\
\leq \quad & \frac{(\|U(0)\| + b)\, \lambda_1 l_1}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} + \frac{(\|U(0)\| + b)\, l_2}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
& \frac{(\|U(0)\| + b)\, l_3}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
= \quad & \frac{(\|U(0)\| + b)\, (\lambda_1 l_1 + l_2 + l_3)}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} r^{\alpha-\beta} \\
\leq \quad & b.
\end{aligned}$$

Therefore F maps K into itself. Now define a sequence $\{U_k(t)\}$ in K such that

$$U_0(t) = U(0) \text{ and } U_{k+1}(t) = U_k(t), \quad k = 0, 1, 2, \ldots$$

Since K is closed, there exists a subsequence $\{U_{k_i}(t)\}$ of $U_k(t)$ and $\widetilde{U}(t) \in K$ such that

$$\lim_{k_i \to \infty} U_{k_i}(t) = \widetilde{U}(t).$$

Then Lebesgue's dominated convergence theorem yields that

$$\widetilde{U}(t)=\widetilde{U}(0)-\frac{\lambda_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}a(s)\widetilde{U}(s)\ \mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\widetilde{U}(s),\widetilde{V}(s))\ \mathrm{d}s,$$

where $\widetilde{V}(t)=\int_0^t g(t,s,\widetilde{U}(t))$. Next we claim that F is completely continuous.

Step 2. For that first we prove $T:K\to K$ is continuous. Let $\{U_m(t)\}$ be a converging sequence in K to $U(t)$. Then for any $\varepsilon>0$, let

$$\|U_m(t)-U(t)\|\leq\frac{\Gamma(\alpha)\varepsilon}{2\lambda_1 l_1}\left(\frac{\alpha-\beta}{1-\beta}\right)^{1-\beta}r^{\alpha-\beta}.$$

By assumption (H_1),

$$f\left(t,U_m(t),\int_0^s g(t,s,U_m(\tau))\ \mathrm{d}s\right)\longrightarrow f\left(t,U(t),\int_0^s g(t,s,,U(t))\ \mathrm{d}s\right),$$

for each $t\in[0,r]$ and since

$$\left\|f\left(t,U_m(t),\int_0^s g(t,s,U_m(t))\ \mathrm{d}s\right)-f\left(t,U(t),\int_0^s g(t,s,U(t))\ \mathrm{d}s\right)\right\|\ \leq\ \frac{\Gamma(\alpha)\varepsilon}{2r^\alpha}\ \left(\frac{\alpha-\beta}{1-\beta}\right)^{1-\beta},$$

we have

$$\begin{aligned}\|FU_m(t)-FU(t)\| &\leq \frac{\lambda_1 l_1}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\|U_m(t)-U(t)\|+\frac{1}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\\ &\quad\left\|f\left(t,U_m(s),\int_0^\tau g(s,\tau,U_m(\tau))\ \mathrm{d}\tau\right)-f\left(t,U(s),\int_0^\tau g(s,\tau,U(\tau))\ \mathrm{d}\tau\right)\right\|\\ &\leq \varepsilon.\end{aligned}$$

Taking limit $m\to\infty$, the right hand side of the above inequality tends to zero. Therefore F is continuous.

Step 3. Moreover, for $U\in K$,

$$\begin{aligned}\|\ FU(t)\ \| &\leq \|U(0)\|+\frac{\lambda_1 l_1+l_2+l_3}{\Gamma(\alpha)}(\|U(0)\|+b)\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}r^{\alpha-\beta}\\ &\leq \|U(0)\|+b.\end{aligned}$$

Hence FK is uniformly bounded. Now it remains to show that F maps K into an equicontinuous family.

Step 4. Now let $U\in K$ and $t_1,t_2\in J$. Then if $0<t_1<t_2\leq r$, by the assumptions $(H1)-(H6)$

we obtain

$$\begin{aligned}
\| FU(t_1) - FU(t_2) \| \leq\ & \frac{\lambda_1}{\Gamma(\alpha)} (\|U(0)\| + b) \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right) \|a(s)\| \, \mathrm{d}s \\
& + \frac{\lambda_1}{\Gamma(\alpha)} (\|U(0)\| + b) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|a(s)\| \, \mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big) f(s, U(s), V(s)) \, \mathrm{d}s \right\| \\
& + \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, U(s), V(s)) \, \mathrm{d}s \right\| \\
\leq\ & \frac{\lambda_1 l_1}{\Gamma(\alpha)} \left(\int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \\
& + \frac{\lambda_1 l_1}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_2}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_2}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_3}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right)^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} \\
& + \frac{l_3}{\Gamma(\alpha)} (\|U(0)\| + b) \left(\int_{t_1}^{t_2} ((t_2 - s)^{\alpha-1})^{\frac{1}{1-\beta}} \, \mathrm{d}s \right)^{1-\beta} .
\end{aligned}$$

The right hand side is independent of $U \in K$. Since $0 < \beta < \alpha < 1$, the right hand side of the above inequality goes to zero as $t_1 \to t_2$. Thus, F maps K into an equicontinuous family of functions. In the view of Ascoli-Arzela theorem, F is completely continuous. Then applying Leray-Schauder fixed point theorem, we deduce that F has a fixed point in K, which is a solution of (2.1). Q.E.D.

Example

Consider the partial fractional intgrodifferential equation

$$ {}^{C}D^{\frac{1}{2}}u(x,t) = t^2\Delta u(x,t) + t + u(x,t) + \frac{1}{1+t^2}\int\limits_0^t su(x,s)\ \mathrm{d}s, \quad (x,t)\in\Omega\times J \tag{3.7} $$

with the initial condition

$$ u(x,0) = u_0,\ \ x\in\Omega $$

and the boundary condition

$$ u(x,t) = 0,\ \ (x,t)\in\partial\Omega\times J, $$

where $J=[0,1]$ and $\Omega=[0,\pi/2]$. Here $a(t)=t^2$, $\int\limits_0^t g(t,s,u(x,s))\ \mathrm{d}s = \frac{1}{1+t^2}\int\limits_0^t su(x,s)\ \mathrm{d}s$ and

$$ f(t,u(x,t),\int_0^t g(t,s,u(x,s))\ \mathrm{d}s) = t + u(x,t) + \frac{1}{1+t^2}\int\limits_0^t su(x,s)\ \mathrm{d}s. \tag{3.8} $$

Since the eigenfunctions of the Laplacian operator are $\sin mx$ and $\cos mx$ where $\lambda = m^2$, we note that the assumptions (H1)-(H6) of Theorem 3.3 are satisfied for some $\beta\in(0,1/2)$. Hence the problem (3.7) has a solution.

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