

Some results on uniqueness of meromorphic functions sharing a polynomial

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Abstract

In this paper, with the aid of weighted value sharing we study the uniqueness problems of meromorphic functions when certain nonlinear differential polynomials generated by them share a nonconstant polynomial with weight two. The result of the paper not only improves the results due to the present first author [Bull. Math. Anal. Appl., 2(2010), 106-118] and of Zhang and Xu [Comput. Math. Appl., 61(2011), 722-730], at the same time finds a possible answer of an open question posed by Zhang and Xu.

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1 Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We assume that the reader is familiar with the classical value distribution theory of meromorphic functions as described in, say, the standard monograph [6, 16]. For a nonconstant meromorphic function f , we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$ outside of a possible exceptional set E of finite linear measure. The meromorphic function $a(z)$ is called a small function of f if $T(r, a) = S(r, f)$.

Two nonconstant meromorphic functions f and g share a small function a CM (counting multiplicities) provided that $f - a$ and $g - a$ have the same set of zeros with the same multiplicities; f and g share a IM (ignoring multiplicities) if we do not consider the multiplicities. A finite value z_0 is called a fixed point of $f(z)$ if $f(z_0) = z_0$.

The following result is very well known in the value distribution theory (see [3, 4]).

Theorem A. Let f be a transcendental meromorphic function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many solutions.

Corresponding to Theorem A the following result was proved by Yang and Hua [14] in 1997.

Theorem B. Let f and g be two nonconstant meromorphic functions, and let $n(\geq 11)$ be an integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Regarding fixed point the following result was obtained by Fang and Qiu [5] in 2000.

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Theorem C. Let f and g be two nonconstant meromorphic functions, and let $n(\geq 11)$ be an integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t satisfying $t^{n+1} = 1$.

A new trend in this direction is to consider the uniqueness of a meromorphic functions concerning the value sharing of the k -th derivative of a linear expression of a meromorphic function. For the last decade a handful number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials which are mainly the k -th derivative of some linear expressions of f and g (see [8], [9], [10] and [11], for example). In 2010 Xu, Lu and Yi [12] proved the following results for meromorphic functions where an additional condition namely the sharing of poles by the meromorphic functions are taken into consideration.

Theorem D. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1 c_2)^n c^2 = -1$ or $f = tg$ for a constant t such that $t^n = 1$.

Theorem E. Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers with $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f = g$.

Observing the above results the following question is inevitable.

Question 1. Is it possible in any way to relax the nature of sharing the fixed point and in the same time reduce the lower bound of n in Theorems D and E ?

To proceed further we need the following notion of weighted sharing of values which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1. [7] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k , then z_0 is an a -point of f with multiplicity $m(\leq k)$ if and only if it is an a -point of g with multiplicity $m(\leq k)$ and z_0 is an a -point of f with multiplicity $m(> k)$ if and only if it is an a -point of g with multiplicity $n(> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Keeping in mind the notion of weighted sharing of values, the present first author [9] proved the following result in 2010 which deals with Question 1.

Theorem F. Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers satisfying $n > 3k + m + 7$. Suppose that $P(w) = a_m w^m + \dots + a_1 w + a_0$, where $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(z, 2)$; f and g share $(\infty, 0)$ then either $f = tg$ for a constant t such that $t^d = 1$, where $d =$

$\gcd(n+m, \dots, n+m-i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \dots, m$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^m + \dots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + \dots + a_1 w_2 + a_0). \quad (1.1)$$

It is now quite natural to ask the following question.

Question 2. What can be said if sharing fixed point in Theorem F is replaced by sharing a nonzero polynomial ?

In 2011 Zhang and Xu [18] obtained the following result in which they replace fixed point sharing by sharing a nonzero polynomial with degree ≤ 5 .

Theorem G. Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $\deg(p) = l \leq 5$, n, k and m be three positive integers with $n > 3k + m + 7$. Let $P(w) = a_m w^m + \dots + a_1 w + a_0$ be a nonzero polynomial. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $p(z)$ CM, f and g share ∞ IM, then one of the following three cases hold:

- (i) $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m, \dots, n+m-i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \dots, m$;
- (ii) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1);
- (iii) $P(w)$ is reduced to a nonzero monomial, namely, $P(w) = a_i w^i \neq 0$ for some $i \in \{0, 1, \dots, m\}$; if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z) dz$, c_1, c_2 and c are three constants satisfying $a_i^2 (c_1 c_2)^{n+i} [(n+i)c]^2 = -1$, if $p(z)$ is a nonzero constant b , then $f(z) = c_3 e^{cz}$, $g(z) = c_4 e^{-cz}$, where c_3, c_4 and c are three constants such that $(-1)^k a_i^2 (c_3 c_4)^{n+i} [(n+i)c]^{2k} = b^2$.

In the same paper the authors posed the following question, as far as we are aware, this remains open.

Question 3. Is it possible in any way to remove the condition $\deg(p) = l \leq 5$ in Theorem G ?

In the paper our main concern is to find out the possible answer of the above question. The result of the paper improves Theorem F by reducing the lower bound of n and also improves Theorem G by relaxing the nature of sharing the polynomial as well as by reducing the lower bound of n . The following is the main result of the paper.

Theorem 1. Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l , and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers with $n > \max\{3k + m + 6, k + 2l\}$. Suppose that either k, l are co-prime or $k > l$ when $l \geq 2$. Let $P(w)$ be defined as in Theorem G. If $[f^n P(f)]^{(k)} - p$ and $[g^n P(g)]^{(k)} - p$ share $(0, 2)$; f and g share $(\infty, 0)$, then the following conclusions hold:

- (i) If $P(w) = a_m w^m + \dots + a_1 w + a_0$ is not a monomial, then either $f = tg$ for a constant t that satisfies $t^d = 1$, where $d = (n+m, \dots, n+m-i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \dots, m$; or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1).
- (ii) When $P(w) = c_0$ or $P(w) = a_m w^m$, then either $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$, or $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$, b_1, b_2 and b are three constants satisfying $c_0^2 (nb)^2 (b_1 b_2)^n = -1$ or $a_m^2 ((n+m)b)^2 (b_1 b_2)^{n+m} = -1$, where m^* is defined by

$$m^* = \begin{cases} m & \text{if } P(w) \neq c_0; \\ 0 & \text{if } P(w) = c_0. \end{cases}$$

We now explain the following definitions and notations which are used in the paper.

Definition 2. [7] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer p we denote by $N(r, a; f | \leq p)$ the counting function of those a -points of f (counted with proper multiplicities) whose multiplicities are not greater than p . By $\overline{N}(r, a; f | \leq p)$ we denote the corresponding reduced counting function.

Analogously we can define $N(r, a; f | \geq p)$ and $\overline{N}(r, a; f | \geq p)$.

Definition 3. Let a be any value in the extended complex plane, and let k be an arbitrary non-negative integer. We denote by $N_k(r, a; f)$ the counting function of a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4. [1] Let f and g be two nonconstant meromorphic functions that share $(a, 2)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a -point of f with multiplicity p and also an a -point of g with multiplicity q . We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a -points of f and g , where $p > q \geq 3$ ($q > p \geq 3$). Also we denote by $\overline{N}_E^{(3)}(r, a; f)$ the reduced counting function of those a -points of f and g , where $p = q \geq 3$.

Definition 5. [7] Let f and g be two nonconstant meromorphic functions sharing the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g . Clearly

$$\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).$$

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane \mathbb{C} . We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 1. [13] Let f be a nonconstant meromorphic function and let $a_n(z) (\not\equiv 0)$, $a_{n-1}(z)$, \dots , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \dots, n$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. [17] Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \quad (2.1)$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \quad (2.2)$$

Lemma 3. [6] Suppose that f is a nonconstant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S\left(r, \frac{f'}{f}\right),$$

then $f = e^{az+b}$, where $a(\neq 0)$, b are constants.

Lemma 4. [2] Let F, G be two nonconstant meromorphic functions sharing $(1, 2)$ and (∞, k) where $0 \leq k < \infty$. If $H \neq 0$, then

- (i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) - m(r, 1; G) - \overline{N}_E^{(3)}(r, 1; F) - \overline{N}_L(r, 1; G) + S(r, F) + S(r, G);$
- (ii) $T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) - m(r, 1; F) - \overline{N}_E^{(3)}(r, 1; G) - \overline{N}_L(r, 1; F) + S(r, F) + S(r, G).$

Lemma 5. [15] Suppose that F and G be two nonconstant meromorphic functions and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F} \right) - \left(\frac{G'}{G-1} - \frac{G'}{G} \right). \quad (2.3)$$

If F, G share $(\infty, 0)$ and if $V = 0$, then $F = G$.

Lemma 6. Let f and g be two nonconstant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers. Suppose that V is given as in (2.3), where $F = \frac{(f^n P(f))^{(k)}}{p(z)}$, $G = \frac{(g^n P(g))^{(k)}}{p(z)}$, $P(w), p(z)$ are defined as in Theorem 1. If $V \neq 0$, F and G share $(1, 2)$, f and g share ∞ IM, then the poles of F and G are zeros of V and

$$\begin{aligned} (n+m+k-1)\overline{N}(r, \infty; f | \geq 1) &= (n+m+k-1)\overline{N}(r, \infty; g | \geq 1) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

Proof. We note that the order of the possible poles of F and G are at least $n+m+k$ as f and g share $(\infty, 0)$. Thus F and G share $(\infty, n+m+k-1)$. Now using Milloux theorem (see [6], p. 55) and Lemma 1, we obtain from the definition of V that

$$m(r, V) = S(r, f) + S(r, g).$$

Therefore

$$\begin{aligned} (n+m+k-1)\overline{N}(r, \infty; f | \geq 1) &= (n+m+k-1)\overline{N}(r, \infty; g | \geq 1) \\ &= (n+m+k-1)\overline{N}(r, \infty; F | \geq n+m+k) \\ &\leq N(r, 0; V) \\ &\leq T(r, V) + O(1) \\ &\leq N(r, \infty; V) + m(r, V) + O(1) \\ &\leq N(r, \infty; V) + S(r, f) + S(r, g) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

This proves the lemma.

Q.E.D.

Lemma 7. Let f and g be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ be three integers. Suppose that F and G are defined as in Lemma 6. If there exist two nonzero constants c_1 and c_2 such that $\bar{N}(r, c_1; F) = \bar{N}(r, 0; G)$ and $\bar{N}(r, c_2; G) = \bar{N}(r, 0; F)$, then $n \leq 3k + m + 3$.

Proof. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, 0; F) + \bar{N}(r, c_1; F) + \bar{N}(r, \infty; F) + S(r, F) \\ &\leq \bar{N}(r, 0; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; F) + S(r, F). \end{aligned} \quad (2.4)$$

Using (2.1), (2.2), (2.4) and Lemma 1 we obtain

$$\begin{aligned} (n+m)T(r, f) &\leq T(r, F) - \bar{N}(r, 0; F) + N_{k+1}(r, 0; f^n P(f)) \\ &\quad + O\{\log r\} + S(r, f) \\ &\leq \bar{N}(r, 0; G) + N_{k+1}(r, 0; f^n P(f)) + \bar{N}(r, \infty; f) \\ &\quad + O\{\log r\} + S(r, f) \\ &\leq N_{k+1}(r, 0; f^n P(f)) + N_{k+1}(r, 0; g^n P(g)) + \bar{N}(r, \infty; f) \\ &\quad + k\bar{N}(r, \infty; g) + O\{\log r\} + S(r, f) + S(r, g) \\ &\leq (k+m+2)T(r, f) + (2k+m+1)T(r, g) \\ &\quad + O\{\log r\} + S(r, f) + S(r, g). \end{aligned} \quad (2.5)$$

Similarly

$$\begin{aligned} (n+m)T(r, g) &\leq (k+m+2)T(r, g) + (2k+m+1)T(r, f) \\ &\quad + O\{\log r\} + S(r, f) + S(r, g). \end{aligned} \quad (2.6)$$

Combining (2.5), (2.6) and noting that $\log r = o(T(r, f)) = o(T(r, g))$ we get

$$(n-3k-m-3)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which gives $n \leq 3k + m + 3$. This completes the proof of the lemma. Q.E.D.

The following lemma can be proved in the line of the proof of Lemma 9 [18].

Lemma 8. Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l , and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers with $n > k + 2l$. If

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = p^2(z),$$

where $P(w)$ is defined as in Theorem G, then $P(w)$ is reduced to a nonzero monomial, namely $P(w) = a_i w^i \neq 0$ for some $i \in \{0, 1, \dots, m\}$.

Lemma 9. Let f and g be two nonconstant meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l , and let n, m and k be three positive integers with $n > k + 2l$. Further assume that either k, l are coprime or $k > l$ when $l \geq 2$. If

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = p^2(z), \quad (2.7)$$

where $P(z) = a_m z^m$ or $P(z) = c_0$, then $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2 ((n+m)b)^2 (b_1 b_2)^{n+m} = -1$ or $c_0^2 (nb)^2 (b_1 b_2)^n = -1$ and $Q(z)$ is same as in Theorem 1.

Proof. Let $P(z) = a_m z^m$. The case $P(z) = c_0$ can be proved similarly. First we assume that $k = 1$. Then (2.7) becomes

$$(a_m f^{n+m})'(a_m g^{n+m})' = p^2(z). \quad (2.8)$$

Noting that $n > k + 2l$, we deduce from (2.8) that f and g have no zeros. We put

$$f = e^\alpha, \quad g = e^\beta, \quad (2.9)$$

where α and β are two nonconstant entire functions. Therefore from (2.8) we get

$$a_m^2 (n+m)^2 \alpha' \beta' e^{(n+m)(\alpha+\beta)} = p^2(z).$$

From this it follows that α, β must be polynomials and $\alpha + \beta \equiv k_1$, where k_1 is a constant. Thus $\deg(\alpha) = \deg(\beta)$. Therefore $\alpha' + \beta' \equiv 0$ and

$$a_m^2 (n+m)^2 \alpha' \beta' e^{(n+m)k_1} = p^2(z).$$

Simplifying we obtain $\alpha' = bp(z)$ and $\beta' = -bp(z)$, where $b(\neq 0)$ is a constant. This gives $\alpha = bQ(z) + d_1$ and $\beta = -bQ(z) + d_2$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$ and d_1, d_2 are constants. Therefore

$$f = b_1 e^{bQ(z)}, \quad g = b_2 e^{-bQ(z)},$$

where b_1, b_2 and b are three constants satisfying

$$a_m^2 ((n+m)b)^2 (b_1 b_2)^{n+m} = -1.$$

If $k \geq 2$ then (2.7) becomes

$$(a_m f^{n+m})^{(k)} (a_m g^{n+m})^{(k)} = p^2(z). \quad (2.10)$$

Since f and g are transcendental meromorphic function, from (2.10) we obtain

$$N(r, 0; (a_m f^{n+m})^{(k)}) = O\{\log r\}.$$

From this and (2.9) we get

$$N(r, \infty; a_m f^{n+m}) + N(r, 0; a_m f^{n+m}) + N(r, 0; (a_m f^{n+m})^{(k)}) = O\{\log r\}.$$

Suppose that α is a transcendental entire function. Then by Lemma 3 we deduce that α is a polynomial, a contradiction. Next we assume that α, β are polynomials of degree p_1 and p_2 respectively. If $p_1 = p_2 = 1$, then

$$f = e^{Az+B}, \quad g = e^{Cz+D},$$

where $A(\neq 0), B, C(\neq 0)$ and D are constants. So from (2.10) we obtain

$$a_m^2 (AC)^k (n+m)^{2k} e^{(n+m)\{(A+C)z+(B+D)\}} = p^2(z),$$

which is impossible. Thus $\max\{p_1, p_2\} > 1$. We assume that $p_1 > 1$. Then $(a_m f^{n+m})^{(k)} = Q_1 e^{(n+m)\alpha}$ and $(a_m g^{n+m})^{(k)} = Q_2 e^{(n+m)\beta}$, where Q_1, Q_2 are polynomials of degree $k(p_1 - 1)$ and $k(p_2 - 1)$ respectively. Therefore from (2.10) we obtain $\alpha + \beta \equiv k_2$, a constant, and therefore $p_1 = p_2$ and $k(p_1 - 1) = l$. This shows that $l \geq k \geq 2$, contradicting with the assumption that k, l are prime to each other. This proves the lemma. Q.E.D.

3 Proof of the theorem

Proof of Theorem 1. We discuss the following three cases separately.

Case (i) Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ is not a monomial. Suppose that F and G are defined as in Lemma 6. Then F, G are transcendental meromorphic functions that share $(1, 2)$ and $(\infty, 0)$. Therefore

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F | \geq n + m + k) = \overline{N}(r, \infty; f | \geq 1).$$

If possible, we assume that $H \neq 0$. Then $F \neq G$. So from Lemma 5 we have $V \neq 0$. From Lemma 1 and (2.1) we obtain

$$\begin{aligned} N_2(r, 0; F) &\leq N_2\left(r, 0; (f^n P(f))^{(k)}\right) + S(r, f) \\ &\leq T\left(r, (f^n P(f))^{(k)}\right) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) \\ &\quad + S(r, f) \\ &\leq T(r, F) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) \\ &\quad + O\{\log r\} + S(r, f). \end{aligned} \tag{3.1}$$

Similarly

$$\begin{aligned} N_2(r, 0; G) &\leq T(r, G) - (n + m)T(r, g) + N_{k+2}(r, 0; g^n P(g)) \\ &\quad + O\{\log r\} + S(r, g). \end{aligned} \tag{3.2}$$

Again by (2.2) we have

$$N_2(r, 0; F) \leq N_{k+2}(r, 0; f^n P(f)) + k\overline{N}(r, \infty; f) + S(r, f) \tag{3.3}$$

and

$$N_2(r, 0; G) \leq N_{k+2}(r, 0; g^n P(g)) + k\overline{N}(r, \infty; g) + S(r, g). \tag{3.4}$$

From (3.1) and (3.2) we get

$$\begin{aligned} (n + m)\{T(r, f) + T(r, g)\} &\leq T(r, F) + T(r, G) + N_{k+2}(r, 0; f^n P(f)) \\ &\quad + N_{k+2}(r, 0; g^n P(g)) - N_2(r, 0; F) - N_2(r, 0; G) \\ &\quad + O\{\log r\} + S(r, f) + S(r, g). \end{aligned} \tag{3.5}$$

Then using Lemma 1, Lemma 4, (3.3) and (3.4) we obtain from (3.5)

$$\begin{aligned}
(n+m)\{T(r, f) + T(r, g)\} &\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
&\quad + 2\overline{N}(r, \infty; G) + 2\overline{N}_*(r, \infty; F, G) \\
&\quad + N_{k+2}(r, 0; f^n P(f)) + N_{k+2}(r, 0; g^n P(g)) \\
&\quad - \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + O\{\log r\} \\
&\quad + S(r, f) + S(r, g) \\
&\leq 2N_{k+2}(r, 0; f^n P(f)) + 2N_{k+2}(r, 0; g^n P(g)) \\
&\quad + (k+2)\overline{N}(r, \infty; f) + (k+2)\overline{N}(r, \infty; g) \\
&\quad + 2\overline{N}_*(r, \infty; F, G) - \overline{N}_*(r, 1; F, G) \\
&\quad + O\{\log r\} + S(r, f) + S(r, g) \\
&\leq 2(k+m+2)\{T(r, f) + T(r, g)\} \\
&\quad + (k+2)(\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)) \\
&\quad + 2\overline{N}_*(r, \infty; F, G) - \overline{N}_*(r, 1; F, G) \\
&\quad + O\{\log r\} + S(r, f) + S(r, g).
\end{aligned}$$

Using Lemma 2, Lemma 6 and noting that f and g are transcendental meromorphic functions we obtain from above

$$\begin{aligned}
&(n-3k-m-6)\{T(r, f) + T(r, g)\} \\
&\leq \frac{2}{n+m+k-1}[\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G)] \\
&\quad - \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\leq \frac{2}{n+m+k-1}[N_{k+1}(r, 0; f^n P(f)) + k\overline{N}(r, \infty; f) \\
&\quad + N_{k+1}(r, 0; g^n P(g)) + k\overline{N}(r, \infty; g) + \overline{N}_*(r, 1; F, G)] \\
&\quad - \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
&[(n-3k-m-6)(n+m+k-1) - (4k+2m+2)]\{T(r, f) + T(r, g)\} \\
&\leq S(r, f) + S(r, g),
\end{aligned}$$

a contradiction as $n > \max\{3k+m+6, k+2l\}$.

Next we assume that $H = 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.6}$$

where $A(\neq 0)$, B are constants. Now we consider the following three subcases.

Subcase (i) Let $B \neq 0$ and $A = B$. Then from (3.6) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \quad (3.7)$$

If $B = -1$, then from (3.7) we obtain

$$FG = 1,$$

i.e.,

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = p^2(z), \quad (3.8)$$

a contradiction by Lemma 8.

If $B \neq -1$, from (3.7), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r, \infty; G) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Using (2.1) and (2.2) we obtain from above inequality

$$\begin{aligned} T(r, G) &\leq N_{k+1}(r, 0; f^n P(f)) + k\overline{N}(r, \infty; f) + T(r, G) + N_{k+1}(r, 0; g^n P(g)) \\ &\quad - (n+m)T(r, g) + \overline{N}(r, \infty; g) + O\{\log r\} + S(r, g). \end{aligned}$$

Hence

$$(n+m)T(r, g) \leq (2k+m+1)T(r, f) + (k+m+2)T(r, g) + S(r, g).$$

In a similar manner we can get

$$(n+m)T(r, f) \leq (k+m+2)T(r, f) + (2k+m+1)T(r, g) + S(r, g).$$

Combining the above two inequality we obtain

$$(n-3k-m-3)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

a contradiction as $n > \max\{3k+m+6, k+2l\}$.

Subcase (ii) Let $B \neq 0$ and $A \neq B$. Then from (3.6) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding as in Subcase (i) we arrive at a contradiction.

Subcase (iii) Let $B = 0$ and $A \neq 0$. Then from (3.6) $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. So by Lemma 7 we have $n \leq 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F = G$. Then

$$[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}.$$

Integrating we get

$$[f^n P(f)]^{(k-1)} = [g^n P(g)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 7 we deduce that $n \leq 3k + m$, a contradiction. Thus $c_{k-1} = 0$. Repeating k -times, we obtain

$$f^n P(f) = g^n P(g).$$

Then

$$f^n(a_m f^m + \dots + a_1 f + a_0) = g^n(a_m g^m + \dots + a_1 g + a_0). \quad (3.9)$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (3.9) we get

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \dots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = \gcd(n+m, \dots, n+m-i, \dots, n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \dots, m\}$. Thus $f = tg$ for a constant t such that $t^d = 1$, $d = \gcd(n+m, \dots, n+m-i, \dots, n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \dots, m\}$.

If h is not a constant, then from (3.9) we see that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1).

Case (ii) Now we assume that $P(z) = a_m z^m$, where $a_m (\neq 0)$ is a complex constant. Let $F = \frac{(a_m f^{n+m})^{(k)}}{p(z)}$ and $G = \frac{(a_m g^{n+m})^{(k)}}{p(z)}$. Then F and G are transcendental meromorphic functions that share $(1, 2)$ and $(\infty, 0)$. Proceeding in the like manner as Case (i) above we obtain either $FG = 1$ or $F = G$.

If $FG = 1$, then

$$(a_m f^{n+m})^{(k)} (a_m g^{n+m})^{(k)} = p^2(z).$$

So by Lemma 9 we obtain $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2 ((n+m)b)^2 (b_1 b_2)^{n+m} = -1$ and $Q(z)$ is same as in Theorem 1. If $F = G$, then using Lemma 7 and proceeding similarly as in Case (i) we obtain $f = tg$ for a constant t such that $t^{n+m} = 1$.

Case (iii) Let $P(z) = c_0$. Taking $F = \frac{(c_0 f^n)^{(k)}}{p(z)}$, $G = \frac{(c_0 g^n)^{(k)}}{p(z)}$ and arguing similarly as in Case (ii) we obtain either $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $c_0^2 (nb)^2 (b_1 b_2)^n = -1$, $Q(z)$ is same as in Theorem 1 or $f = tg$ for a constant t satisfying $t^n = 1$. This completes the proof of the theorem. Q.E.D.

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