The spectrum operator of χ^2 sequence space defined by Musielak Orlicz function

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Abstract

In this paper we have examined various spectrum of the operator $D(p,q,r,s)$ on the sequence space χ^2 defined by Musielak Orlicz function.

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1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Avinoy Paul and B.C. Tripathy [6] , Turkmenoglu [7], Raj [8-9] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) \ .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} \left|x_{mn}\right|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$\left|x_{mn}\right|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n} \left\{ \left|x_{mn} - y_{mn}\right|^{\frac{1}{m+n}} : m, n : 1, 2, 3, ... \right\}, \tag{1.1}$$

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forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ & & & & \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [12] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1}\,|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1}x_{mn} - \Delta^{m-1}x_{mn+1} - \Delta^{m-1}x_{m+1n} + \Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}$.

2 Definitions and preliminaries

Let X and Y be Banach metric spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on X into itself is denoted by $B(X)$. The adjoint $T^* : X^* \to X^*$ of T is defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$. Clearly, T^* is a bounded linear operator on the dual space X^*.

Let $T : D(T) \to X$ a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T. Many properties of T_α and T_α^{-1} depend on á and spectral theory is concerned with those properties. We are interested in the set of all α in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another essential property. We also detemine $\alpha's$, for which the domain of T_α^{-1} is dense in X.

A regular value is a complex number α of T such that
(N_1) T_α^{-1} exists
(N_2) T_α^{-1} is bounded and
(N_3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $C \backslash \rho(T)$ in the complex plane C is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consists of those values of $\alpha \in C$, for which T_α is not invertible.

We discuss about the point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator.Some of them are motivated by applications to physics, in particular in quantum mechanics.

Definition 2.1. The point (discrete) spectrum $\sigma_p(T, X)$ is the set of complex number α such that T_α^{-1} does not exist. Further $\alpha \in \sigma_p(T, X)$ is called the eigen value of T.

Definition 2.2. The continuous spectrum $\sigma_c(T, X)$ is the set of complex number α such that T_α^{-1} exists and satisfies (N_3) but not (N_2) that is T_α^{-1} is unbounded.

Definition 2.3. The residual spectrum $\sigma_r(T, X)$ is the set of complex number α such that T_α^{-1} exists (and may be bounded or not) but not satisfy (N_3), that is the domain of T_α^{-1} is not dense in X.

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and equal to the empty set and the spectrum consists of only the point spectrum.

Given a bounded linear operator T in a Banach metric space X, we call a sequence $(x_{mn}) \in X$ as a sequence for T if

$$d(x, 0) = 1 \Longrightarrow \|x_{mn} - 0\| = 1 = \|x_{mn}\| \tag{1.2}$$

and

$$d(Tx, 0) = 1 \Longrightarrow \|Tx_{mn} - 0\| = \|Tx_{mn}\| \to 0 \; as \; m, n \to \infty. \tag{1.3}$$

Definition 2.4. The approximate point spectrum
$\sigma_{ap}(T, X) = \{\alpha \in C : \; there \; exists \; (2.1), (2.2) \; sequence \; for \; T - \alpha I\}$.

Definition 2.5. The defect spectrum $\sigma_d(T, X) = \{\alpha \in C : T - \alpha I \; is \; not \; surjective\}$.

Definition 2.6. The compression spectrum $\sigma_d(T, X) = \left\{\alpha \in C : \overline{R(T - \alpha I)} \neq X\right\}$.

Theorem 2.7. (**Goldberg's classification of spectrum**, see [13])
If X is Banach metric space and $T \in B(X)$, then there are three possibilities for $R(T)$:
(I) $R(T) = X$,
(II) $R(T) \neq \overline{R(T)} = X$,
(III) $\overline{R(T)} \neq X$. and
(1)T^{-1} exists and is continuous.
(2)T^{-1} exists but is discontinuous.
(3)T^{-1} does not exist.

Definition 2.8 (see [10]**.**] An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Lemma 2.9. Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

Definition 2.10 (see [11]**.**] Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \ldots, \alpha d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$;
(or)
(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_E = sup\left(|det(d_{mn}(x_{mn}, 0))|\right) =$$
$$sup\left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \ldots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \ldots & d_{2n}(x_{1n}, 0) \\ \vdots & & & \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \ldots & d_{nn}(x_{nn}, 0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Definition 2.11. Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k, \ell-$ th term of Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be non-negative if $a_{k\ell}^{mn}$ is non-negative.

Let E and F be two sequence spaces and $A = \left(a_{k,\ell}^{mn}\right)$ be an four dimensional infinite matrix of real or complex numbers $a_{k,\ell}^{mn}$, where $m,n \in \mathbb{N}$. Then $A : E \to F$, if for every sequence $x = (x_{mn})_{k\ell} \in E$ the sequence $Ax = \{(Ax)_{k\ell}\}$ is in F where $(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$, provided the right hand side converges for every $k,\ell \in \mathbb{N}$ and $x \in E$.
Consider the operator $D(p,q,r,s)$, where

$$D(p,q,r,s) = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ q & p & 0 & 0 & 0 & 0 & 0 & . & . & . \\ r & q & p & 0 & 0 & 0 & 0 & . & . & . \\ s & r & q & p & 0 & 0 & 0 & . & . & . \\ 0 & s & r & q & p & 0 & 0 & . & . & . \\ 0 & 0 & s & r & q & p & 0 & . & . & . \\ 0 & 0 & 0 & s & r & q & p & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \ddots & & \end{pmatrix}$$

Remark: In particular if we consider p=1,q=1,r=1,s=1 then $D(p,q,r,s) = \Delta_2$.

Definition 2.12. Let f be an sequence of Musielak Orlicz functions and a sequence of spectrum operator is defined as following :
$\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = lim_{m,n\to\infty}$
$\left\{\left[f\left(\sigma(D(p,q,r,s))((m+n)!|x_{mn}|)^{(1/m)+n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right] = 0\right\}$.

$\left[\Lambda_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$sup_{mn}\left\{\left[f\left(\sigma(D(p,q,r,s))|x_{mn}|^{(1/m)+n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right] < \infty\right\}$.

3 Main results

Theorem 3.1. If $\alpha,\beta,\gamma = p,q,r$, then $\alpha,\beta,\gamma \in$
$III_1\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)$.

Proof. If $\alpha = p, \beta = q, \gamma = r$ then the operator $D(p,q,r,s) - \alpha I - \beta I - \gamma I = D(0,0,0,s)$. Since $R(D(0,0,0,s)) \neq$
$\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. It is not invertible and hence $\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \in III_1$. Therefore we have
$\left\|\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]\right\| = \frac{s}{2}d(x,0)$. It is bounded below and it has a bounded inverse. Hence $\alpha,\beta,\gamma \in$
$III_1\left[\chi_f^2(\sigma(D(p,q,r,s))), \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right.$ This completes the proof. ∎

Lemma 3.2. $\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}.$

Theorem 3.3. $\left[\chi_f^2\left(\sigma_{ap}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}\setminus\left\{p\right\},\left\{q\right\},\left\{r\right\}$

Proof. We have $\left[\chi_f^2\left(\sigma_{ap}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\setminus$
$III_1\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right),$
$\left[\chi_f^2\left(\sigma_{ap}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}\setminus\left\{p\right\},\left\{q\right\},\left\{r\right\}$ is obtained by Lemma (3.2) and Theorem (3.1). This completes the proof. ∎

Lemma 3.4. $\left[\chi_f^2\left(\sigma_p\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) = \phi$

Theorem 3.5. $\left[\chi_f^2\left(\sigma_d\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}\setminus\left\{p\right\},\left\{q\right\},\left\{r\right\}$

Proof. We have $\left[\chi_f^2\left(\sigma_d\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\setminus$
$I_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right).$
Now, $I_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$II_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$III_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left[\chi_f^2\left(\sigma_p\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) = \phi$ by Lemma (3.4). Hence
$\left[\chi_f^2\left(\sigma_d\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right).$ ∎

Lemma 3.6. $\left[\chi_f^2\left(\sigma_r\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}.$

Theorem 3.7. $\left[\chi_f^2\left(\sigma_{co}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$\left\{\alpha,\beta,\gamma\in\mathbb{C}:\left|\alpha-p\right|,\left|\beta-q\right|,\left|\gamma-r\right|\leq\left|s\right|\right\}\setminus\left\{p\right\},\left\{q\right\},\left\{r\right\}$

Proof. $\left[\chi_f^2\left(\sigma_{co}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right) =$
$III_1\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\bigcup$

$III_2\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\bigcup$
$III_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$. Now,
$III_1\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\bigcup$
$III_2\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=$
$\left[\chi_f^2\left(\sigma_r\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=$
$\{\alpha,\beta,\gamma\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|\leq|s|\}$ is obtained by Lemma (3.6). Again,
$III_3\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=$
$\left[\chi_f^2\left(\sigma_p\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=\phi$ is obtained by Lemma (3.4). Hence,
$\left[\chi_f^2\left(\sigma_{co}\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=$
$\{\alpha,\beta,\gamma\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|\leq|s|\}\setminus\{p\},\{q\},\{r\}$. This completes the proof. ∎

Lemma 3.8. The adjoint operator T^* of T is onto if and only if T has a bounded inverse.

Theorem 3.9. If $\alpha,\beta,\gamma=p,q,r$, then $\alpha,\beta,\gamma\in$
$III_1\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$.

Proof. By Lemma (3.6), $\alpha,\beta,\gamma\in$
$III_1\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$ whenever $\alpha=p,\beta=q,\gamma=r$. By Lemma (3.8), $\alpha=p,\beta=q,\gamma=r$ is not in
$\left[\chi_f^2\left(\sigma_p\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$ and hence
$\left(\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)-\alpha I-\beta I-\gamma I\right)^{-1}$ exists. We have to prove that
$\left(\left[\chi_f^2\left(\sigma\left(D\left(p,q,r,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)-\alpha I-\beta I-\gamma I\right)^{-1}$ must be continuous, hence it is show that
$\left[\chi_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)^*-\alpha I-\beta I-\gamma I=$
$\left[\chi_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)^*$ is onto by Lemma (3.8). Given $y=(y_{mn})\in\left[\Lambda_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$ we can find $x=(x_{mn})\in$
$\left[\Lambda_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$
such that $\left[\chi_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)^*=$
$\left[\Lambda_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$. Therefore we have $x_{mn}=\frac{1}{s}y_{m-1,n-2}$ which shows that $\left[\chi_f^2\left(\sigma\left(D\left(0,0,0,s\right)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)^*$ is onto. This completes the proof. ∎

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