Inclusion theorems of double Deferred Cesàro means II

Richard F. Patterson[1], Fatih Nuray[2] & Metin Başarir[3]

[1]Department of Mathematics and Statistics, University of North Florida Jacksonville, Florida, 32224
[2]Department of Mathematical Sciences, Afyon Kocatepe University, Afyonkarahisar, Turkey
[3]Department of Mathematics, Sakarya University, Sakakya, Turkey

E-mail: rpatters@unf.edu[1], fnuray@aku.edu.tr[2], basarir@sakarya.edu.tr[3]

Abstract

In 1932 R. P. Agnew present a definition for Deferred Cesàro mean. Using this definition R. P. Agnew present inclusion theorems for the deferred and none Deferred Cesàro means. This paper is part 2 of a series of papers that present extensions to the notion of double Deferred Cesàro means. Similar to part 1 this paper uses this definition and the notion of regularity for four dimensional matrices, to present extensions and variations of the inclusion theorems presented by R. P. Agnew in [2].

2010 Mathematics Subject Classification. **40B05**. 40C05
Keywords. Double Cesàro mean, Deferred Cesàro mean, Double sequence, *RH*-Regular Matrix, *P*-convergent sequences.

1 Introduction

This paper is part 2 of a series of papers characterization the inclusion between Cesàro means and double Deferred Cesàro means. In part 1[11] we presented the notion of double Deferred Cesàro means which is a multi-dimensional analog and Agnew's Deferred Cesàro means in [2]. Using this notions and as series of basic results in [11], this paper present a series of inclusion theorems similar to the following: *The double Cesáro mean includes* $D_{m-1,q_m,n-1,p_n}$ *be a Deferred Cesàro mean with* $q_m = m$, $p_n = n; m \neq \alpha_1, \alpha_2, \ldots$ *and* $n \neq \beta_1, \beta_2, \ldots$ *with*

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \ldots, \alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j + 1, 2, 3, \ldots, \beta_n$$

where $\{q_{\alpha_i}\}$ *and* $\{p_{\beta_j}\}$ *are increasing single dimensional sequences of integers such that* $\alpha_m > m$ *and* $\beta_n > n$.

2 Definitions, notations and preliminary results

The definitions, notations, and preliminary results are similar to those in Part 1 [11] which are restated here for the purpose of completeness.

Definition 2.1 (Pringsheim, 1900)**.** A double sequence $x = \{x_{k,l}\}$ has a **Pringsheim limit** L (denoted by P-lim $x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an $\{x\}$ is described more briefly as "P-convergent".

Tbilisi Mathematical Journal 9(2) (2016), pp. 15–23.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 14 March 2016.
Accepted for publication: 07 May 2016.

Definition 2.2 (Patterson, 2000)**.** A double sequence $\{y\}$ is a **double subsequence** of $\{x\}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then {y} is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [13] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix A is said to be **RH-regular** if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of bounded was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [3] and [13].

Theorem 2.4. (Hamilton [3], Robison [13]) The four-dimensional matrix A is RH-regular if and only if

RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l;
RH_2: P-$\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: P-$\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ for each l;
RH_4: P-$\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ for each k;
RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ is P-convergent;
RH_6: there exist finite positive integers Δ and Γ such that $\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta$.

The main goals of this paper includes the comparison of double Cesàro mean transformation

$$(C,1,1)_{m,n,k,l} := \begin{cases} \frac{1}{mn}, & \text{if} \quad k \le m \text{ and } l \le n \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

with the double Deferred Cesàro mean

$$D_{m,n,k,l} := \begin{cases} \frac{1}{(\alpha_m-\beta_m)(q_n-p_n)}, & \text{if} \quad \beta_m < k \le \alpha_m \text{ and } p_n < l \le q_n, \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

where $[p_n]$ $[q_n]$ $[\alpha_m]$, and $[\beta_m]$ are sequences of nonnegative integers satisfying

$$\alpha_m < \beta_m, \text{ and } p_n < q_n \text{ for } m,n = 1,2\ldots; \tag{1.1}$$

and

$$\lim_m \beta_m = +\infty, \text{ and } \lim_n q_n = +\infty. \tag{1.2}$$

Using these four dimensional transformations we shall present a catalog of inclusion theorems such as the following. *The four dimensional summability method M include $D_{p_n,\alpha_n,q_n,\beta_n}$ where p_n and q_n for almost all n is a give non-negative integer p if and only if α_n and β_n are almost all positive integers.*

3 Main results

Theorem 3.1. The Double Cesàro transformation includes every Double Deferred Cesàro mean of the form $D_{p_n,\alpha_n q_n,\beta_n}$ for which α_n and β_n contains almost all positive integers.

Proof. Let $[x_{k,l}]$ be summable by $D_{p_n,\alpha_n,q_n,\beta_n}$ (say to L) such that $\text{P}-\lim_{m,n} D_{m,n} = L$ and choose two integers K and L large such that $[p_m]$ and $[q_n]$ contains all integers greater than K and L, respectively. Thus let $i_1 = i_2 = i_3 = \cdots = i_K = 1$ and $j_1 = j_2 = j_3 = \cdots = j_L = 1$ and determine for $m > K$ and $n > L$ index i_m and j_n is such that $p_{i_m} = m$ and $q_{i_n} = n$. Since $\lim_m i_m = +\infty$ and $\lim_n j_n = +\infty$, it follows

$$\text{P} - \lim_{m,n} D_{m,n} = L \text{ and } \text{P} - \lim_{m,n} D_{i_m,j_n} = L.$$

Therefore $[x]$ is summable by $D_{p_m,m,q_n,n}$ to L. The result follows from Lemma 3.3 of [11]. Q.E.D.

Theorem 3.2. The Double Cesàro transformation fails to contain includes $D_{p_n,\alpha_n,q_n,\beta_n}$ if there exists an Pringsheim increasing sequence double sequence $[\alpha_{k,l}]$ of integers whose elements belong to neither $[p_n]$ nor $[q_n]$.

Proof. Let us consider the following

$$\bar{M}_{m,n} = \begin{cases} 0, & \text{if} \quad (m,n) \neq (\alpha_m, \beta_n);\ m,n = 1,2,3,\ldots \\ x_{m,n}, & \text{if} \quad (m,n) = (\alpha_m, \beta_n);\ m,n = 1,2,3,\ldots \end{cases}$$

where $[x]$ is a P-divergent double sequence. Let $[s_{m,n}]$ be double sequence that is mapped by M into $\bar{M}$. Condition 3.2, $p_m \neq \alpha_m$, and $q_n \neq \beta_n$ asure us that $D_{p_n,\alpha_n,q_n,\beta_n}$ sum $[x]$ to zero. Since M fails to sum $[x]$. Q.E.D.

The following theorem follows from Theorem 3.1 and 3.2.

Theorem 3.3. The four dimensional summability method M include $D_{p_n,\alpha_n,q_n,\beta_n}$ where p_n and q_n for almost all n is a give non-negative integer p if and only if α_n and β_n are almost all positive integers.

Theorem 3.4. The four dimensional summability method M include $D_{m-1,q_m,n-1,\beta_n}$ where $q_m - m$ and $p_n - n$ both increases monotonically with m and n, respectively if and only if $q_m - m$ and $p_n - n$ both are both bounded.

Proof. To establish to sufficiency part not that $q_m - m$ and $p_n - n$ must have a limit, say α and β, respectively and that $q_m - m = \alpha$ and $p_n - n = \beta$ for almost all m and n. Thus $\{q_m\}$ and $\{p_n\}$ contains almost all positive integers and Theorem 3.1 grants us the results.

To established the necessary part, suppose $q_m - m$ and $p_n - n$ increases monotonically with m and n are both unbounded. The goal now is to show that the set of double sequences that are double Cesàro summable are not summable by the double Deferred Cesàro mean. Let $m_1 = n_1 = 1$ and m_2 and n_2 are the smallest integers such that

$$q_m - m > q_{m_1} - m_1 \text{ and } p_n - n > p_{n_1} - n_1$$

Then choose m_3 and n_3 to be the smallest integers m and n such that

$$q_m - m > q_{m_2} - m_2 \text{ and } p_n - n > p_{n_2} - n_2.$$

Thus having chosen

$$m_1 < m_2 < \cdots < m_\alpha \text{ and } n_1 < n_2 < \cdots < n_\beta.$$

We then choose $m_{\alpha+1}$ and $n_{\beta+1}$ to be the smallest integers such that

$$q_m - m > q_{m_\alpha} - m_\alpha \text{ and } p_n - n > p_{n_\beta} - n_\beta.$$

We than define a double sequence $\{s_{k,l}\}$ as follows:

$$s_{k,l} = \begin{cases} q_{m_i} p_{n_j}, & \text{if} \quad k = q_{m_i} \text{ and } l = p_{n_j}; i,j = 1,2,3,\ldots \\ kl, & \text{if} \quad k \neq q_{m_i} \text{ and/or } l \neq p_{n_j}; i,j = 1,2,3,\ldots \end{cases}$$

Note $D_{m,n}$ maps $\{s_{k,l}\}$ into 1. for all (m,n). Thus $\{s_{k,l}\}$ is D-summable to 1. Also $\{s_{k,l}\}$ is not M-summable, since P$-\lim_{k,l} \frac{s_{k,l}}{k,l} \neq 0$. Thus the double Cesàro mean is contained in the double Deferred Cesàro mean. Q.E.D.

Theorem 3.5. Let $D_{m-1,q_m,n-1,p_n}$ be a Deferred Cesàro mean with $q_m = m$, $p_n = n; m \neq \alpha_1, \alpha_2, \ldots$ and $n \neq \beta_1, \beta_2, \ldots$ with

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1,2,3,\ldots,\alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j + 1,2,3,\ldots,\beta_n$$

where $\{q_{\alpha_i}\}$ and $\{p_{\beta_j}\}$ are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$. Then D is included in M if and only if $\frac{q_m}{m}$ and $\frac{p_n}{n}$ are bounded for all m and n.

Proof. Note $D_{m-1,m,n-1,n}$ is the identity transformation. Let us consider the ordered pair (m,n) and observe that for each pair (m,n), let

$$i = i_m \text{ and } j = j_n$$

be such that $\alpha_i \leq m < \alpha_{i+1}$ and $\beta_j \leq n < \beta_j$. Let $\{s_{m,n}\}$ be a given double sequence and consider the transformation

$$M_{m,n} = \frac{1}{mn} \begin{bmatrix} s_{1,1} & + & s_{1,2} & + & s_{1,3} & + & \cdots & + & s_{1,n} \\ s_{2,1} & + & s_{2,2} & + & s_{2,3} & + & \cdots & + & s_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{m,1} & + & s_{m,2} & + & s_{m,3} & + & \cdots & + & s_{m,n} \end{bmatrix}.$$

Using the definition of double Deferred Cesàro mean we obtain the following

$$
\begin{array}{ccc}
\begin{bmatrix} s_{1,1} & + & \cdots & + & s_{1,\beta_1-1} \\ s_{2,1} & + & \cdots & + & s_{2,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_1-1,1} & + & \cdots & + & s_{\alpha_1-1,\beta_1-1} \end{bmatrix} & + \quad \cdots \quad + & \begin{bmatrix} s_{1,\beta_j} & + & \cdots & + & s_{1,\beta_{j+1}-1} \\ s_{2,1} & + & \cdots & + & s_{2,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_1-1,\beta_j} & + & \cdots & + & s_{\alpha_1-1,\beta_{j+1}-1} \end{bmatrix} \\
\begin{bmatrix} s_{\alpha_1,1} & + & \cdots & + & s_{\alpha_1,\beta_1-1} \\ s_{\alpha_1+1,1} & + & \cdots & + & s_{\alpha_1+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_2-1,1} & + & \cdots & + & s_{\alpha_2-1,\beta_1-1} \end{bmatrix} & + \quad \cdots \quad + & \begin{bmatrix} s_{\alpha_1,\beta_j} & + & \cdots & + & s_{\alpha_1,\beta_{j+1}-1} \\ s_{\alpha_1+1,1} & + & \cdots & + & s_{\alpha_1+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_2-1,\beta_j} & + & \cdots & + & s_{\alpha_2-1,\beta_{j+1}-1} \end{bmatrix} \\
\begin{bmatrix} s_{\alpha_2,1} & + & \cdots & + & s_{\alpha_2,\beta_1-1} \\ s_{\alpha_2+1,1} & + & \cdots & + & s_{\alpha_2+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_3-1,1} & + & \cdots & + & s_{\alpha_3-1,\beta_1-1} \end{bmatrix} & + \quad \cdots \quad + & \begin{bmatrix} s_{\alpha_2,\beta_j} & + & \cdots & + & s_{\alpha_2,\beta_{j+1}-1} \\ s_{\alpha_2+1,1} & + & \cdots & + & s_{\alpha_2+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_3-1,\beta_j} & + & \cdots & + & s_{\alpha_3-1,\beta_{j+1}-1} \end{bmatrix} \\
\vdots & \vdots & \vdots \\
\begin{bmatrix} s_{\alpha_i,1} & + & \cdots & + & s_{\alpha_i,\beta_1-1} \\ s_{\alpha_i+1,1} & + & \cdots & + & s_{\alpha_i+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_{i+1}-1,1} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_1-1} \end{bmatrix} & + \quad \cdots \quad + & \begin{bmatrix} s_{\alpha_i,\beta_j} & + & \cdots & + & s_{\alpha_i,\beta_{j+1}-1} \\ s_{\alpha_i+1,1} & + & \cdots & + & s_{\alpha_i+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_{i+1}-1,\beta_j} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_{j+1}-1} \end{bmatrix}
\end{array}.
$$

Let us denote the above sum by $\Omega_{m,n}$ and the sum below by $\Lambda_{m,n}$

$$
\begin{array}{ccccccccccc}
 & & & & & & s_{1,n+1} & + & \cdots & + & s_{1,\beta_{j+1}-1} \\
 & & & & & & s_{2,n+1} & + & \cdots & + & s_{2,\beta_{j+1}-1} \\
 & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & s_{m,n+1} & + & \cdots & + & s_{m,\beta_{j+1}-1} \\
s_{m+1,1} & + & \cdots & + & s_{m+1,n+1} & + & s_{m+1,n+1} & + & \cdots & + & s_{m+1,\beta_{j+1}-1} \\
s_{m+2,1} & + & \cdots & + & s_{m+2,n+1} & + & s_{m+2,n+1} & + & \cdots & + & s_{m+2,\beta_{j+1}-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s_{\alpha_{i+1}-1,1} & + & \cdots & + & s_{\alpha_{i+1}-1,n+1} & + & s_{\alpha_{i+1}-1,n+1} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_{j+1}-1}
\end{array}.
$$

Therefore $M_{m,n} = \frac{1}{mn}(\Omega_{m,n} - \Lambda_{m,n})$. It is important to observe that if $m = \alpha_{i+1} - 1$ and/or $n = \beta_{j+1} - 1$ then the terms in $\Lambda_{m,n}$ will not exist that is if $m = \alpha_{i+1} - 1$ and/or $n = \beta_{j+1} - 1$ then the terms in the rows and/or columns will not exists. Let us also denote the following sum by

$\bar{\Omega}_{m,n}$

$$\begin{array}{ccccccc}
\frac{\sum_{k,l=1,1}^{\alpha_1,\beta_1} s_{k,l}}{mn} & + & \frac{\alpha_1(\beta_2-\beta_1)}{mn}D_{0,1} & + & \cdots & + & \frac{\alpha_1(\beta_{j+1}-\beta_j)}{mn}D_{0,j} \\
\frac{(\alpha_2-\alpha_1)\beta_1}{mn}D_{1,0} & + & \frac{(\alpha_2-\alpha_1)(\beta_2-\beta_1)}{mn}D_{1,1} & + & \cdots & + & \frac{(\alpha_2-\alpha_1)(\beta_{j+1}-\beta_j)}{mn}D_{1,j} \\
\frac{(\alpha_3-\alpha_2)\beta_1}{mn}D_{2,0} & + & \frac{(\alpha_3-\alpha_2)(\beta_2-\beta_1)}{mn}D_{2,1} & + & \cdots & + & \frac{(\alpha_3-\alpha_2)(\beta_{j+1}-\beta_j)}{mn}D_{2,j} \\
+ & + & + & + & + & + & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & + & + & + & + & + & + \\
\frac{(\alpha_{i+1}-\alpha_i)\beta_1}{mn}D_{i,0} & + & \frac{(\alpha_{i+1}-\alpha_i)(\beta_2-\beta_1)}{mn}D_{i,1} & + & \cdots & + & \frac{(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)}{mn}D_{i,j}
\end{array}$$

and also denote the following sum by $\bar{\Lambda}_{m,n}$

$$\begin{array}{ccccccccccc}
 & & & & & & D_{1,n+1} & + & \cdots & + & D_{1,\beta_{j+1}-1} \\
 & & & & & & D_{2,n+1} & + & \cdots & + & D_{2,\beta_{j+1}-1} \\
 & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & D_{m,n+1} & + & \cdots & + & D_{m,\beta_{j+1}-1} \\
D_{m+1,1} & + & \cdots & + & D_{m+1,n} & + & D_{m+1,n+1} & + & \cdots & + & D_{m+1,\beta_{j+1}-1} \\
D_{m+2,1} & + & \cdots & + & D_{m+2,n} & + & D_{m+2,n+1} & + & \cdots & + & D_{m+2,\beta_{j+1}-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
D_{\alpha_{i+1}-1,1} & + & \cdots & + & D_{\alpha_{i+1}-1,n} & + & D_{\alpha_{i+1}-1,n+1} & + & \cdots & + & D_{\alpha_{i+1}-1,\beta_{j+1}-1}
\end{array}.$$

Then we can now rewrite $M_{m,n}$ in the following manner $\bar{\Omega}_{m,n} - \frac{1}{mn}\bar{\Lambda}_{m,n}$. The relation $\bar{\Omega}_{m,n} - \frac{1}{mn}\bar{\Lambda}_{m,n}$ hold for each (m,n) and defines a four-dimensional transformation of the form

$$\sigma_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}s_{k,l}$$

which carries $D_{m,n}$ into $M_{m,n}$. This transformation clearly satisfies RH_1 and RH_2. This transformation satisfies RH_3 and RH_4 only if $\frac{2\alpha_{i+1}-m-2}{m}$ and $\frac{2\beta_{j+1}-n-2}{n}$ are bounded respectively for each (m,n), which is equivalent to $\frac{\alpha_{i+1}}{m}$ and $\frac{\alpha_{i+1}}{m}$ are bounded, which is also equivalent to the boundedness of $\frac{q_m}{m}$ and $\frac{p_n}{n}$ for each (m,n). Condition RH_5 and RH_6 hold only when both $\frac{2\alpha_{i+1}-m-2}{m}$ and $\frac{2\beta_{j+1}-n-2}{n}$ are bounded, and as above the is equivalent to boundedness of $\frac{q_m}{m}$ and $\frac{p_n}{n}$ for each (m,n). Since D is a factorable four-dimensional summability matrix the main theorem in [1] assure us that it has an inverse. Thus the result follows for the Robison-Hamilton characterization of regularity. Q.E.D.

Theorem 3.6. The double Cesáro mean includes $D_{m-1,q_m,n-1,p_n}$ be a Deferred Cesàro mean with $q_m = m$, $p_n = n; m \neq \alpha_1, \alpha_2, \ldots$ and $n \neq \beta_1, \beta_2, \ldots$ with

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \ldots, \alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j + 1, 2, 3, \ldots, \beta_n$$

where $\{q_{\alpha_i}\}$ and $\{p_{\beta_j}\}$ are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$.

Proof. Observe that for each pair (m, n), let

$$i = i_m \text{ and } j = j_n$$

be such that $h_i \leq m < h_{i+1}$ and $t_j \leq n < t_{j+1}$. Let $\{s_{m,n}\}$ be a given double sequence and consider the following four dimensional Cesàro transformation

$$M_{m,n} = \frac{1}{mn}\left[\begin{array}{ccccccccc} s_{1,1} & + & s_{1,2} & + & s_{1,3} & + & \cdots & + & s_{1,n} \\ s_{2,1} & + & s_{2,2} & + & s_{2,3} & + & \cdots & + & s_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{m,1} & + & s_{m,2} & + & s_{m,3} & + & \cdots & + & s_{m,n} \end{array}\right].$$

Using the definition of double Deferred Cesàro mean we can rewrite $mnM_{m,n}$ using the following, respectively, $A^{i}_{m,n}, A^{i-1}_{m,n}, A^{i-2}_{m,n} \cdots, A^{\alpha}_{m,n}$ and $K_{m,n}$ where $K_{m,n}$ is

$$\begin{array}{ccccccc} s_{1,1} & + & s_{1,2} & + & \cdots & + & s_{1,\beta_\delta-1} \\ s_{2,1} & + & s_{2,2} & + & \cdots & + & s_{2,\beta_\delta-1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots \\ s_{\beta_\Delta-1,1} & + & s_{\beta_\Delta-1,2} & + & \cdots & + & s_{\beta_\Delta-1,\beta_\delta-1} \end{array}$$

with Δ and δ are 2 or 1 depending on weather α and/ or β are odd or even, and the A's are define below, respectively

$$\begin{array}{ccccccc} s_{m,n} & + & s_{m,n-1} & + & \cdots & + & s_{m,t_j+1} \\ s_{m-1,n} & + & s_{m-1,n-1} & + & \cdots & + & s_{m-1,t_j+1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots \\ s_{h_i+1,n} & + & s_{h_i+1,n-1} & + & \cdots & + & s_{h_i+1,t_j+1} \end{array},$$

$$\begin{array}{ccccccccccccc} & & & & & & & & s_{m,t_j} & + & \cdots & + & s_{m,t_{j-1}} \\ & & & & & & & & s_{m-1,t_j} & + & \cdots & + & s_{m-1,t_{j-1}} \\ & & & & & & & & \vdots & + & \cdots & + & \vdots \\ & & & & & & & & s_{hl_i+1,t_j} & + & \cdots & + & s_{h_i+1,t_{j-1}} \\ s_{h_i,n} & + & s_{h_i,n-1} & + & \cdots & + & s_{h_i,t_j+1} & + & s_{\alpha_i,t_j} & + & \cdots & + & s_{h_i,t_{j-1}} \\ \vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\ s_{h_{i-1},n} & + & s_{h_{i-1},n-1} & + & \cdots & + & s_{h_{i-1},t_j+1} & + & s_{h_{i-1},t_j} & + & \cdots & + & s_{h_{i-1},t_{j-1}} \end{array},$$

$$\begin{array}{ccccccccccccc} & & & & & & & & s_{m,t_{j-1}-1} & + & \cdots & + & s_{m,t_{j-2}+1} \\ & & & & & & & & \vdots & + & \cdots & + & \vdots \\ s_{h_{i-1}-1,n} & + & s_{h_{i-1}-1,n-1} & + & \cdots & + & s_{h_{i-1}-1,t_j+1} & + & s_{h_{i-1}-1,t_j} & + & \cdots & + & s_{h_{i-1}-1,t_{j-2}+1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\ s_{h_{i-2}+1,n} & + & s_{h_{i-2}+1,n-1} & + & \cdots & + & s_{h_{i-2}+1,t_j+1} & + & s_{h_{i-2}+1,t_j} & + & \cdots & + & s_{h_{i-2}+1,t_{j-2}+1} \end{array},$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & s_{m,t_{j-2}} & + & \cdots & + & s_{m,t_{j-3}+1} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
s_{h_{i-2},n} & + & s_{h_{i-2},n-1} & + & \cdots & + & s_{h_{i-2},t_j+1} & + & s_{h_{i-2},t_{j-2}} & + & \cdots & + & s_{h_{i-2},t_{j-3}+1} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{h_{i-3}+1,n} & + & s_{h_{i-3}+1,n-1} & + & \cdots & + & s_{h_{i-3}+1,t_j+1} & + & s_{h_{i-3}+1,t_{j-2}} & + & \cdots & + & s_{h_{i-3}+1,t_{j-3}+1}
\end{array},
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & s_{m,t_{\beta+1}-2} & + & \cdots & + & s_{m,t_\beta} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
s_{h_{\alpha+1},n} & + & s_{h_{\alpha+1},n-1} & + & \cdots & + & s_{h_{\alpha+1},t_{\beta+1}-1} & + & s_{h_{\alpha+1},t_{\beta+1}-2} & + & \cdots & + & s_{h_{\alpha+1},t_\beta} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{h_\alpha+1,n} & + & s_{h_\alpha+1,n-1} & + & \cdots & + & s_{h_\alpha+1,t_{\beta+1}-1} & + & s_{h_\alpha+1,t_{\beta+1}-2} & + & \cdots & + & s_{h_\alpha+1,t_\beta}
\end{array}.
$$

It is clear that

$$M_{m,n} = \frac{1}{mn}\left[A^i_{m,n} + A^{i-1}_{m,n} + A^{i-2}_{m,n} + \cdots + A^{\alpha}_{m,n} + K_{m,n}\right].$$

Now using the above identities we can rewrite our equation as follow $T_{m,n} = M_{m,n} - \frac{K_{m,n}}{mn}$ and the D's grants us the following:

$$
T_{m,n} = \frac{1}{mn}\left[\begin{array}{ccccccc}
D_{m,n} & + & D_{m,n-1} & + & \cdots & + & D_{m,\beta_j+1} \\
D_{m-1,n} & + & D_{m-1,n-1} & + & \cdots & + & D_{m-1,\beta_j+1} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
D_{\alpha_i+1,n} & + & D_{\alpha_i+1,n-1} & + & \cdots & + & D_{\alpha_i+1,\beta_j+1}
\end{array}\right] +
$$

$$
\begin{array}{ccc}
\frac{(\alpha_i-\alpha_{i-1}+1)(\beta_j+1-n)}{mn} D_{\alpha_i,n} & + & \frac{(\alpha_i-\alpha_{i-1}+1)(\beta_j-\beta_{j-}+1-n)}{mn} D_{\alpha_{i-1},\beta_{j-1}} \\
 & + & \frac{(\alpha_i-m+1)(\beta_j-n+1)}{mn} D_{m,\beta_j} \\
\vdots & \vdots & \vdots \\
+\frac{(\alpha_{\delta+1}-m+1)(\beta_{\Delta+1}-\beta_\Delta+1)}{mn} D_{m,\beta_\delta} & + & \frac{(\alpha_{\delta+1}-\alpha_\delta+1)(\beta_{\Delta+1}-\beta_\Delta+1)}{mn} D_{\alpha_\delta,\beta_\Delta} \\
 & + & \frac{(\alpha_{\delta+1}-\alpha_\delta+1)(\beta_{\Delta+1}-\beta_\Delta+1)}{mn} D_{\alpha_\delta,n}
\end{array}.
$$

Since the above equalities define four dimensional RH-regular transformation from $\{D_{m,n}\}$ to $\{T_{m,n}\}$ we are granted the if the double sequence $\{D_{m,n}\}$ convergence to L in the Pringsheim sense then $\{T_{m,n}\}$ convergence to L in the Pringsheim sense and since the double sequence $\{K_{m,n}\}$ is bounded then $\{M_{m,n}\}$ convergence to L in the Pringsheim sense. Thus double Cesàro means includes double Deferred Cesàro means. The completes the proof.

Q.E.D.

References

[1] C. R. Adams, *On Summability of Double Series*, Trans. Amer. Math. Soc. **34**, No.2 (1932), 215–230.

[2] R. P. Agnew, *On Deferred Cesàro Means*, Annals of Math., **33** (1932), 413–421.

[3] H. J. Hamilton, *Transformations of Multiple Sequences*, Duke Math. Jour., **2** (1936), 29–60.

[4] H. J. Hamilton, *A Generalization of Multiple Sequences Transformation*, Duke Math. Jour., **4** (1938), 343–358.

[5] H. J. Hamilton, *Change of Dimension in Sequence Transformation*, Duke Math. Jour., **4** (1938), 341 - 342.

[6] H. J. Hamilton, *Preservation of Partial Limits in Multiple Sequence Transformations*, Duke Math. Jour., **5** (1939), 293–297.

[7] G. H. Hardy, *Divergent Series.* Oxford Univ. Press, London. 1949.

[8] K. Knopp, *Zur Theorie der Limitierungsverfahren* (Erste Mitteilung), Math. Zeit. **31** (1930), 115–127.

[9] I. J. Maddox, *Some Analogues of Knopp's Core Theorem*, Internat. J. Math. & Math. Sci. **2**(4) (1979) 604–614. **2** (1970), 63–65.

[10] R. F. Patterson, *Analogues of some Fundamental Theorems of Summability Theory*, Internat. J. Math. & math. Sci. **23(1)**, (2000), 1–9.

[11] R. F. Patterson & F. Nuray, *Inclusion Theorems of Double Cesáro Means*, (under consideration).

[12] A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Mathematische Annalen, **53** (1900) 289-320.

[13] G. M. Robison, *Divergent Double Sequences and Series*, Amer. Math. Soc. trans. **28** (1926) 50–73.