Categorical construction of the ring of fractions

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Abstract

It is shown that the ring of fractions of the algebra of all bounded linear operators on a separable infinite dimensional Banach space is isomorphic to the Adams completion of the algebra with respect to a carefully chosen set of morphisms in the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1.

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1 Adams completion

The notion of (generalized) Adams completion arose from a general categorical completion process, suggested by Adams [1,2]. Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [6] where an arbitrary category and an arbitrary set of morphisms of the category are considered.

Let $\mathscr{C}$ be a category and S be a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and $F : \mathscr{C} \to \mathscr{C}[S^{-1}]$, the canonical functor. Let S denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$, $\mathscr{C}[S^{-1}](\text{-},Y) : \mathscr{C} \to S$ defines a contravariant functor. If this functor is representable by an object Y_s of $\mathscr{C}$, i.e., $\mathscr{C}[S^{-1}](\text{-},Y) \cong \mathscr{C}(\text{-},Y_S)$, then Y_S is called the *generalized Adams completion of* Y with respect to the set of morphisms S or simply the *S-completion* of Y. We shall often refer to Y_S as the *completion* of Y [6].

The following theorem shows that under certain assumptions the Adams completion of an object of a category $\mathscr{C}$ always exists.

1.1 Theorem. ([7], p.32, Theorem 1) *Let $\mathscr{C}$ be a cocomplete small $\mathscr{U}$-category ($\mathscr{U}$ is a fixed Grothendeick universe) and S a set of morphisms of $\mathscr{C}$ that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.*

(a) *If each $s_i : X_i \to Y_i$ $i \in I$, is an element of S, where the index set I is an element of $\mathscr{U}$, then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} X_i \to \bigvee_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathscr{C}$ has an Adams completion X_S with respect to the set of morphisms S.

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The following theorem shows that under certain conditions the Adams completion of an object of a category $\mathscr{C}$ always exists.

1.2 Theorem. ([5], p.528, Theorem 1.2) *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathscr{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in $\bar{S}$ which is couniversal with respect to morphisms of S : given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in $\bar{S}$ such that $ts = e$. In other words, the following diagram is commutative* :

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

In order to show the morphism $e : Y \to Y_S$, as stated in Theorem 1.2 is in S, the following result is to be used.

1.3 Theorem. ([5], p.528, Theorem 1.2) *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem*1.2, *where Y_S is the S-completion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr{C}$ which have the following properties*:

(a) *S_1 and S_2 are closed under composition*,

(b) *$fg \in S_1$ implies that $g \in S_1$*,

(c) *$fg \in S_2$ implies that $f \in S_2$*,

(d) *$S = S_1 \cap S_2$*.

Then $e \in S$.

2 Ring of fractions

We recall the following definitions from [10].

Let A be a ring and let S be a multiplicatively closed subset of A, i.e., $s,t \in S$ implies $st \in S$ and $1 \in S$. A *ring of fractions* (right) of A with respect to S [10] is defined as a ring $A[S^{-1}]$ together with a ring homomorphism $u : A \to A[S^{-1}]$ satisfying

(i) $u(s)$ is inverible for every $s \in S$,

(ii) every element in $A[S^{-1}]$ has the form $u(a)u(s)^{-1}$ with $s \in S$,

(iii) $u(a) = 0$ if and only if $as = 0$ for some $s \in S$.

2.1 Proposition. [10] *Let S be a multiplicatively closed subset of A.* Then *$A[S^{-1}]$ exists, if and only if S satisfies* *(i) if $s \in S$ and $a \in A$ then there exists $t \in S$ and $b \in A$ such that $sb = at$,*

(ii) if $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.

When $A[S^{-1}]$ exists, it has the form

$$A[S^{-1}] = A \times S/\sim$$

where $\sim$ is the equivalence relation on $A \times S$ defined as $(a,s) \sim (b,t)$ if there exists $c,d \in A$ such that $ac = bd$ and $sc = td \in S$.

Addition and multiplication of $(a,s),(b,t) \in A[S^{-1}]$ are defined in the obvious way:

$$(a,s)+(b,t) = (ac+bd,u) \text{ for some } c \in A, u \text{ and } d \in S \text{ with } u = sc = td.$$
$$(a,s)\cdot(b,t) = (ac,tu) \text{ for some } c \in A \text{ and } u \in S \text{ with } sc = bu.$$

$A[S^{-1}]$ is a ring and that $a \mapsto (a,1)$ is a ring homomorphism $u : A \to A[S^{-1}]$.

2.2 Proposition. [10] *When $A[S^{-1}]$ exists, it has the following universal property: for every ring homomorphism $g : A \to B$ such that $g(s)$ is invertible in B for every $s \in S$, then there exists a unique ring homomorphism $h : A[S^{-1}] \to B$ such that $g = hu$, i.e., the following diagram is commutative*:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & A[S^{-1}] \arrow[dl, dashed, "h"] \\ B & \end{tikzcd}$$

2.3 Note. The objects of $A[S^{-1}]$ are of the form $u(a)u(s)^{-1}$, $a \in A$, $s \in S$. Every element of B is of the form $g(a)g(s)^{-1}$, $a \in A$, $s \in S$ and $h : A[S^{-1}] \to B$ is an isomorphism and it is unique. Thus $u : A \to A[S^{-1}]$ is surjective.

Let A be the algebra of all bounded linear operators on a separable infinite dimensional Banach space.

We prove the following.

2.4 Proposition. *Let A and B be the algebras of all bounded linear operators on a separable infinite dimensional Banach space. Let $g : A \to B$ be a surjective bounded linear homomorphism such that*

(a) $g(s)$ is a unit in B,

(b) $g(a) = 0$ implies $as = 0$ for some $s \in S$.

Then there exists a unique ring homorphism $\theta : B \to A[S^{-1}]$ such that $\theta g = u$, i.e., the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[dl, dashed, "\theta"] \\ A[S^{-1}] & \end{tikzcd}$$

Proof. By Proposition 2.2 there exists a unique isomomorhism $h : A[S^{-1}] \to B$ such that $g = hu$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & A[S^{-1}] \arrow[dl, dashed, "h"] \\ B & \end{tikzcd}$$

Let $\theta = h^{-1}$. for any $a \in A$, $\theta g(a) = h^{-1}g(a) = h^{-1}hu(a) = u(a)$ implying $\theta g = u$, i.e., the above diagram is commutative.

We show that θ is unique. Let there exist another $\theta' : B \to A[S^{-1}]$ such that $\theta' g = u$. For any $b \in B$, we have $\theta'(b) = \theta'(g(b')) = u(b) = \theta g(b)$, i.e., $\theta = \theta'$. This completes the proof . ■

3 The category $\mathscr{B}$

Let $\mathscr{U}$ be a fixed Grothendieck universe [9]. Let $\mathscr{B}$ denote the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1. We assume that the underlying sets of the elements of $\mathscr{B}$ are elements of $\mathscr{U}$. Let

$$S = \{s \text{ in } \mathscr{B} : s \text{ is a bounded linear surjective norm preserving operator of norm at most } 1\}.$$

We prove the following proposites of S.

3.1 Proposition. *Let $s_i : X_i \to Y_i$ lie in S, for each $i \in I$, where the index set I is an emement of $\mathscr{U}$. Then*

$$\underset{i\in I}{\vee} s_i : \underset{i\in I}{\vee} X_i \to \underset{i\in I}{\vee} Y_i$$

lies in S.

Proof. Coproducts in $\mathscr{B}$ are l_1 sums. Let $X = \underset{i\in I}{\vee} X_i$, where $i \in I$ and $Y = \underset{i\in I}{\vee} Y_i$, where $i \in I$. Define $s = \underset{i\in I}{\vee} s_i : X \to Y$ by the rule $s(x) = (s_i(x_i))_{i\in I}$. Clearly, s is well defined bounded linear homomorhism.

For any $(y_i)_{i\in I} \in Y_i$, we have $(y_i)_{i\in I} = (s_i(x_i))_{i\in I} = s(x)$ since $s_i(X_i) = y_i$. Thus $s(X) = Y$, i.e., is surjective. That s is a bounded linear norm preserving operator of norm at most 1, can be proved easily. Hence $s = \underset{i\in I}{\vee} s_i$ lies in S. This competes the proof. ■

3.2 Proposition. *S admits a calculus of left fractions.*

Proof. Let A and B be any two objects of category $\mathscr{B}$. Clearly S is a closed family of morphisms of the category $\mathscr{B}$. We shall verify conditions (i) and (ii) of Theorem 1.3 ([6], p.67). Let $s : M \to N$ and $t : N \to P$ be two morphisms of the category $\mathscr{B}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$. Clearly $t \in S$. Hence the condition (i) of Theorem 1.3 ([6], p.67) holds.

In order to prove condition (ii) of Theorem 1.3 ([6], p.67), consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{B}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{B}$ with $t \in S$. Let $D = (B \oplus_{l_1} C)/\bar{\triangle}$ is the quotient of the direct sum $(B \oplus_{l_1} D)$, endowed with the l_1 norm and $\bar{\triangle}$ is the closer of the subspace

$$\triangle = \{(f(a), -s(a)) : a \in A\}.$$

Define $g : C \to D$ by the rule $g(c) = (0,c) + \bar{\triangle}$ and $t : B \to D$ by the rule $t(b) = (b,0) + \bar{\triangle}$. Clearly, the two maps are well defined and bounded linear homomorphisms. For any $a \in A, tf(a) = t(f(a)) = (f(a),0) + \bar{\triangle} = (0,s(a)) + \bar{\triangle} = g(s(a)) = gs(a)$ implying that $tf = gs$. Hence the diagram is commutative.

In order to show that t is surjective, take an element $(b,c) + \bar{\triangle} \in D + \bar{\triangle}$. Then $(b,c) + \bar{\triangle} = (b,0) + (0,c) + \bar{\triangle} = t(b) + g(c) = t(b) + g(s(a)) = t(b) + tf(a) = t(b + f(a))$ (since t is linear) implying t is surjective, i.e., $t \in S$.

Next let $v : B \to Z$ and $w : C \to Z$ be in category $\mathscr{B}$ such that $vf = ws$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "v"] & \\ C \arrow[r, "g"'] \arrow[drr, bend right, "w"'] & D \arrow[dr, dashed, "\theta"] & \\ & & Z \end{tikzcd}$$

Define $\theta : D \to Z$, by the rule $\theta((b,c) + \bar{\triangle}) = u(b) + v(c)$. It is easy to show that θ is well defined and bounded linear homomorphism (since $\|\theta\| \leq \max\{\|u\|, \|v\|\}$). Next for any $b \in B, \theta t(b) = \theta(b,0) = u(b)$, showing that $\theta t = v$; simillarly $\theta g = w$. Thus the two triangles are commutative. This completes the proof. ■

The following result is well known.

3.3 Proposition. *The category $\mathscr{B}$ is cocomplete.*

From Theorems 3.1, 3.2 and 3.3 we see that all the conditions of the Theorem 1.1 and Theorem 1.2 are satisfied and hence we have the following result.

3.4 Theorem. *Every object A of the category $\mathscr{B}$ has an Adams completion A_S with respect to the set S of homorphisms. Furthermore, there exists a homorphism $e : A \to A_S$ in $\bar{S}$ which is couniversal with respect to the homomorphisms in S : given a homorphism $s : A \to B$ in S there exists a unique homorphism $t : B \to A_S$ in $\bar{S}$ such that $ts = e$. In other words the following diagram is commutative* :

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "s"'] & A_S \\ B \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

3.5 Theorem. *The homorphism $e : A \to A_S$ is in S.*

Proof. Let $S_1 = \{s : X \to Y \text{ in } \mathscr{B} \mid s \text{ is a surjective homorphism}\}$ and $S_2 = \{X \to Y \text{ in } \mathscr{B} \mid s \text{ is a homomorphism}\}$. For S_1 and S_2, it easily follows that all the conditions of Theorem 1.3 are satisfied. Therefore, $e \in S$. This completes the proof. ■

4 The result

We show that the ring of fractions $A[S^{-1}]$ of the algebra A of all bounded linear operators on a separable infinite dimensional Banach space, is precisely the Adams completion A_S of A.

4.1 Theorem. $A[S^{-1}] \cong A_S$.

Proof. Consider the diagram :

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & A[S^{-1}] \\ A_S \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

By Theorem 2.4, there exists a unique homorphism $\varphi : A_S \to A[S^{-1}]$ in S such that $\varphi e = u$.
Next consider the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & A_S \\ A[S^{-1}] \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

By Theorem 3.4, there exists a unique homorphism $\psi : A[S^{-1}] \to A_S$ in S such that $\psi u = e$.

In the following diagram

$$
\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dd, "e"'] & & A_S \\
& A[S^{-1}] \arrow[ur, "\psi"'] & \\
A_S \arrow[ur, "\varphi"'] \arrow[uurr, bend right, "1_{A_S}"'] & &
\end{tikzcd}
$$

we have $\psi\varphi e = \psi u = e$. By the uniqueness condition of the couniversal property of e, we conclude that $\psi\varphi = 1_{A_S}$.

Also in the diagram

$$
\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dd, "u"'] & & A[S^{-1}] \\
& A_S \arrow[ur, "\varphi"'] & \\
A[S^{-1}] \arrow[ur, "\psi"'] \arrow[uurr, bend right, "1_{A[S^{-1}]}"'] & &
\end{tikzcd}
$$

we have $\varphi\psi u = \varphi e = u$. By the uniqueness condition of the couniversal property of u, we conclude that $\varphi\psi = 1_{A[S^{-1}]}$. Thus $A[S^{-1}] \cong A_S$. This completes the proof . ■

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