Categorical construction of the ring of fractions

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Abstract

It is shown that the ring of fractions of the algebra of all bounded linear operators on a separable infinite dimensional Banach space is isomorphic to the Adams completion of the algebra with respect to a carefully chosen set of morphisms in the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1.

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1 Adams completion

The notion of (generalized) Adams completion arose from a general categorical completion process, suggested by Adams [1,2]. Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [6] where an arbitrary category and an arbitrary set of morphisms of the category are considered.

Let \mathscr{C} be a category and S be a set of morphisms of \mathscr{C} . Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of \mathscr{C} with respect to S and $F:\mathscr{C}\to\mathscr{C}[S^{-1}]$, the canonical functor. Let S denote the category of sets and functions. Then for a given object Y of \mathscr{C} , $\mathscr{C}[S^{-1}](-,Y):\mathscr{C}\to S$ defines a contravariant functor. If this functor is representable by an object Y_S of \mathscr{C} , i.e., $\mathscr{C}[S^{-1}](-,Y)\cong\mathscr{C}(-,Y_S)$, then Y_S is called the *generalized Adams completion of Y* with respect to the set of morphisms S or simply the S-completion of Y. We shall often refer to Y_S as the *completion* of Y [6].

The following theorem shows that under certain assumptions the Adams completion of an object of a category $\mathscr C$ always exists.

- **1.1 Theorem.** ([7], p.32, Theorem 1) Let \mathscr{C} be a cocomplete small \mathscr{U} -category (\mathscr{U} is a fixed Grothendeick universe) and S a set of morphisms of \mathscr{C} that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.
 - (a) If each $s_i: X_i \to Y_i$ $i \in I$, is an element of S, where the index set I is an element of \mathcal{U} , then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \to \bigvee_{i \in I} Y_i$$

is an element of S.

Then every object X of $\mathscr C$ has an Adams completion X_S with respect to the set of morphisms S.

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The following theorem shows that under certain conditions the Adams completion of an object of a category \mathscr{C} always exists.

1.2 Theorem. ([5], p.528, Theorem 1.2) Let S be a set of morphisms of C admitting a calculus of left fractions. Then an object Y_S of C is the S-completion of the object Y with respect to S if and only if there exists a morphism $e: Y \to Y_S$ in \overline{S} which is couniversal with respect to morphisms of S: given a morphism $s: Y \to Z$ in S there exists a unique morphism $t: Z \to Y_S$ in \overline{S} such that ts = e. In other words, the following diagram is commutative:



In order to show the morphism $e: Y \to Y_S$, as stated in Theorem 1.2 is in S, the following result is to be used.

- **1.3 Theorem.** ([5], p.528, Theorem 1.2) Let S be a set of morphisms in a category $\mathscr C$ admitting a calculus of left fractions. Let $e: Y \to Y_S$ be the canonical morphism as defined in Theorem1.2, where Y_S is the S-completion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr C$ which have the following properties:
 - (a) S_1 and S_2 are closed under composition,
 - (b) $fg \in S_1$ implies that $g \in S_1$,
 - (c) $fg \in S_2$ implies that $f \in S_2$,
 - (d) $S = S_1 \cap S_2$.

Then $e \in S$.

2 Ring of fractions

We recall the following definitions from [10].

Let A be a ring and let S be a multiplicatively closed subset of A, i.e., $s,t \in S$ implies $st \in S$ and $1 \in S$. A ring of fractions (right) of A with respect to S [10] is defined as a ring $A[S^{-1}]$ together with a ring homomorphism $u: A \to A[S^{-1}]$ satisfying

- (i) u(s) is inverible for every $s \in S$,
- (ii) every element in $A[S^{-1}]$ has the form $u(a)u(s)^{-1}$ with $s \in S$,
- (iii) u(a) = 0 if and only if as = 0 for some $s \in S$.
- **2.1 Proposition.** [10] Let S be a multiplicatively closed subset of A. Then $A[S^{-1}]$ exists, if and only if S satisfies

- (i) if $s \in S$ and $a \in A$ then there exists $t \in S$ and $b \in A$ such that sb = at,
- (ii) if sa = 0 with $s \in S$, then at = 0 for some $t \in S$.

When $A[S^{-1}]$ exists, it has the form

$$A[S^{-1}] = A \times S / \sim$$

where \sim is the equivalence relation on $A \times S$ defined as $(a,s) \sim (b,t)$ if there exists $c,d \in A$ such that ac = bd and $sc = td \in S$.

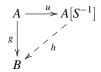
Addition and multiplication of $(a,s),(b,t) \in A[S^{-1}]$ are defined in the obvious way:

$$(a,s)+(b,t)=(ac+bd,u)$$
 for some $c\in A,u$ and $d\in S$ with $u=sc=td$.

$$(a,s) \cdot (b,t) = (ac,tu)$$
 for some $c \in A$ and $u \in S$ with $sc = bu$.

 $A[S^{-1}]$ is a ring and that $a \mapsto (a,1)$ is a ring homomorphism $u: A \to A[S^{-1}]$.

2.2 Proposition. [10] When $A[S^{-1}]$ exists, it has the following universal property: for every ring homomorphism $g: A \to B$ such that g(s) is invertible in B for every $s \in S$, then there exists a unique ring homomorphism $h: A[S^{-1}] \to B$ such that g = hu, i.e., the following diagram is commutative:



2.3 Note. The objects of $A[S^{-1}]$ are of the form $u(a)u(s)^{-1}$, $a \in A$, $s \in S$. Every element of B is of the form $g(a)g(s)^{-1}$, $a \in A$, $s \in S$ and $h:A[S^{-1}] \to B$ is an isomorphism and it is unique. Thus $u:A \to A[S^{-1}]$ is surjective.

Let A be the algebra of all bounded linear operators on a separable infinite dimensional Banach space.

We prove the following.

- **2.4 Proposition.** Let A and B be the algebras of all bounded linear operators on a separable infinite dimensional Banach space. Let $g: A \to B$ be a surjective bounded linear homomorphism such that
 - (a) g(s) is a unit in B,
 - (b) g(a) = 0 implies as = 0 for some $s \in S$.

Then there exists a unique ring homorphism $\theta: B \to A[S^{-1}]$ such that $\theta g = u$, i.e., the following diagram is commutative



Proof. By Proposition 2.2 there exists a unique isomomorhism $h: A[S^{-1}] \to B$ such that g = hu:

$$A \xrightarrow{u} A[S^{-1}]$$

$$g \downarrow \qquad \qquad h$$

$$B$$

Let $\theta = h^{-1}$. for any $a \in A$, $\theta g(a) = h^{-1}g(a) = h^{-1}hu(a) = u(a)$ implying $\theta g = u$, i.e., the above diagram is commutative.

We show that θ is unique. Let there exist another $\theta': B \to A[S^{-1}]$ such that $\theta'g = u$. For any $b \in B$, we have $\theta'(b) = \theta'(g(b')) = u(b) = \theta g(b)$, i.e., $\theta = \theta'$. This completes the proof .

3 The category \mathscr{B}

Let \mathscr{U} be a fixed Grothendieck universe [9]. Let \mathscr{B} denote the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1. We assume that the underlying sets of the elements of \mathscr{B} are elements of \mathscr{U} . Let

 $S = \{s \text{ in } \mathcal{B} : s \text{ is a bounded linear surjective norm preserving operator of norm at most } 1\}.$

We prove the following proposites of *S*.

3.1 Proposition. Let $s_i: X_i \to Y_i$ lie in S, for each $i \in I$, where the index set I is an emement of \mathcal{U} . Then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \to \bigvee_{i \in I} Y_i$$

lies in S.

Proof. Coproducts in \mathscr{B} are l_1 sums. Let $X = \underset{i \in I}{\vee} X_i$, where $i \in I$ and $Y = \underset{i \in I}{\vee} Y_i$, where $i \in I$. Define $s = \underset{i \in I}{\vee} s_i : X \to Y$ by the rule $s(x) = (s_i(x_i))_{i \in I}$. Clearly, s is well defined bounded linear homomorhism.

For any $(y_i)_{i \in I} \in Y_i$, we have $(y_i)_{i \in I} = (s_i(x_i))_{i \in I} = s(x)$ since $s_i(X_i) = y_i$. Thus s(X) = Y, i.e., is surjective. That s is a bounded linear norm preserving operator of norm at most 1, can be proved easily. Hence $s = \bigvee_{i \in I} s_i$ lies in S. This competes the proof.

3.2 Proposition. *S admits a calculus of left fractions.*

Proof. Let *A* and *B* be any two objects of category \mathscr{B} . Clearly *S* is a closed family of morphisms of the category \mathscr{B} . We shall verify conditions (i) and (ii) of Theorem 1.3 ([6], p.67). Let $s: M \to N$ and $t: N \to P$ be two morphisms of the category \mathscr{B} . We show that if $ts \in S$ and $s \in S$, then $t \in S$. Clearly $t \in S$. Hence the condition (i) of Theorem 1.3 ([6], p.67) holds.

In order to prove condition (ii) of Theorem 1.3 ([6], p.67), consider the diagram

$$A \xrightarrow{f} B$$

$$\downarrow s \downarrow \qquad \qquad \downarrow S$$

$$\downarrow C$$

in \mathcal{B} with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{array}{c|c}
A & \xrightarrow{f} & B \\
\downarrow s & \downarrow t \\
V & \downarrow C - \frac{1}{g} & > D
\end{array}$$

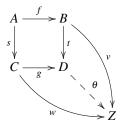
in \mathscr{B} with $t \in S$. Let $D = (B \oplus_{l_1} C)/\bar{\triangle}$ is the quotient of the direct sum $(B \oplus_{l_1} D)$, endowed with the l_1 norm and $\bar{\triangle}$ is the closer of the subspace

$$\triangle = \{ (f(a), -s(a)) : a \in A \}.$$

Define $g: C \to D$ by the rule $g(c) = (0,c) + \bar{\triangle}$ and $t: B \to D$ by the rule $t(b) = (b,0) + \bar{\triangle}$. Clearly, the two maps are well defined and bounded linear homomorphisms. For any $a \in A$, $tf(a) = t(f(a)) = (f(a),0) + \bar{\triangle} = (0,s(a)) + \bar{\triangle} = g(s(a)) = gs(a)$ implying that tf = gs. Hence the diagram is commutative.

In order to show that t is surjective, take an element $(b,c)+\bar{\triangle}\in D+\bar{\triangle}$. Then $(b,c)+\bar{\triangle}=(b,0)+(0,c)+\bar{\triangle}=t(b)+g(c)=t(b)+g(s(a))=t(b)+tf(a)=t(b+f(a))$ (since t is linear) implying t is surjective, i.e., $t\in S$.

Next let $v: B \to Z$ and $w: C \to Z$ be in category \mathscr{B} such that vf = ws



Define $\theta: D \to Z$, by the rule $\theta((b,c) + \bar{\triangle}) = u(b) + v(c)$. It is easy to show that θ is well defined and bounded linear homomorphism (since $\|\theta\| \le \max\{\|u\|, \|v\|\}$). Next for any $b \in B$, $\theta t(b) = \theta(b,0) = u(b)$, showing that $\theta t = v$; simillarly $\theta g = w$. Thus the two triangles are commutative. This completes the proof.

The following result is well known.

3.3 Proposition. The category \mathcal{B} is cocomplete.

From Theorems 3.1, 3.2 and 3.3 we see that all the conditions of the Theorem 1.1 and Theorem 1.2 are satisfied and hence we have the following result.

3.4 Theorem. Every object A of the category \mathcal{B} has an Adams completion A_S with respect to the set S of homorphisms. Furthermore, there exists a homorphism $e:A\to A_S$ in \bar{S} which is couniversal with respect to the homomorphisms in S: given a homorphism $s:A\to B$ in S there exists a unique homorphism $t:B\to A_S$ in \bar{S} such that ts=e. In other words the following diagram is commutative:



3.5 Theorem. The homorphism $e: A \rightarrow A_S$ is in S.

Proof. Let $S_1 = \{s : X \to Y \text{ in } \mathcal{B} \mid s \text{ is a surjective homorphism } \}$ and $S_2 = \{X \to Y \text{ in } \mathcal{B} \mid s \text{ is a homomorphism } \}$. For S_1 and S_2 , it easily follows that all the conditions of Theorem 1.3 are satisfied. Therefore, $e \in S$. This completes the proof.

4 The result

We show that the ring of fractions $A[S^{-1}]$ of the algebra A of all bounded linear operators on a separable infinite dimensional Banach space, is precisely the Adams completion A_S of A.

4.1 Theorem. $A[S^{-1}] \cong A_S$.

Proof. Consider the diagram:

$$\begin{array}{c|c}
A & \xrightarrow{u} & A[S^{-1}] \\
\downarrow & & \swarrow \\
A_S & & & & \\
\end{array}$$

By Theorem 2.4, there exists a unique homorphism $\varphi: A_S \to A[S^{-1}]$ in S such that $\varphi e = u$. Next consider the diagram

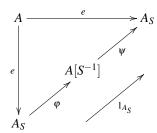
$$A \xrightarrow{e} A_{S}$$

$$\downarrow u \qquad \qquad \downarrow \psi$$

$$A[S^{-1}]$$

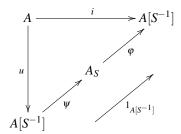
By Theorem 3.4, there exists a unique homorphism $\psi: A[S^{-1}] \to A_S$ in S such that $\psi u = e$.

In the following diagram



we have $\psi \varphi e = \psi u = e$. By the uniqueness condition of the couniversal property of e, we conclude that $\psi \varphi = 1_{A_S}$.

Also in the diagram



we have $\varphi \psi u = \varphi e = u$. By the uniqueness condition of the couniversal property of u, we conclude that $\varphi \psi = 1_{A[S^{-1}]}$. Thus $A[S^{-1}] \cong A_S$. This completes the proof.

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